# Brans-Dicke cosmology with the cosmological constant

K. Uehara and C. W. Kim
*Department of Physics, The Johns Hopkins University, Baltimore, Maryland 21218*
(Received 28 December 1981)

The Brans-Dicke equations with the cosmological constant are studied. We present exact solutions in the spatially flat Robertson-Walker metric in the matter-dominated universe. A brief discussion on this cosmology is given.

Some years ago Brans and Dicke[1] proposed a tensor-scalar theory of the gravitational field based on Mach's principle. The Brans-Dicke (BD) theory is consistent with experiment as long as the Dicke constant $\omega$ is about equal to or greater than 500.[2] In the limit $\omega \to \infty$, the BD theory reduces to the Einstein theory for a constant BD scalar field $\phi$. In order to determine the solutions of the BD equations for cosmology (when applied to cosmology with the cosmological principle) for a given value of $\omega$, it is always necessary to have one more initial condition than needed for determination of the solutions of the Friedmann equations.[3] The BD equations in the absence of the cosmological constant $\lambda$ can be solved analytically in the case of zero space curvature ($k=0$) and zero pressure ($p=0$).[4] In general, the BD equation for the case of $k\neq 0$ and $\lambda\neq 0$ uniquely determines the scale factor $a(t)$, the matter density $\rho(t)$, and the BD scalar field $\phi(t)$ for all $t$ provided the present values of five variables, e.g., $\rho_0$, $H_0$, $q_0$, $\phi_0$, and $\dot{\phi}_0$, as well as equations of state and the constants $\omega$ and $k$ are given. Hubble's constant $H_0$ and the deceleration parameter $q_0$ are defined by

$$H_0=\left[\frac{\dot{a}}{a}\right]_0,\quad q_0=\left[-\frac{a\ddot{a}}{\dot{a}^2}\right]_0, \tag{1}$$

where the subscript zero denotes the present value and an overdot denotes a derivative with respect to $t$. The solutions, however, may not be expressed in terms of elementary functions (recall that the Friedmann equations with $k\neq 0$ and $\lambda\neq 0$ lead to elliptic integrals). Also, the case $k\neq 0$ and $\lambda=0$ cannot be solved analytically even for $p=0$. In this case the present values of four variables are necessary to determine the solutions.

In this paper we present analytic solutions of the BD equations with the nonvanishing cosmological constant for the case $k=0$ and $p=0$. The solutions for $a(t)$, $\rho(t)$, and $\phi(t)$ are determined by $\rho_0$, $H_0$, $q_0$, and $\phi_0$. Hence, in this case the theory has a predictive power for $\dot{\phi}_0$ which in turn provides the present changing rate of the gravitational "constant" for a given value of $\omega$. As in the case $k=0$ for the Friedmann equations (with or without $\lambda$), the multiplicative factor for $a(t)$ cannot be determined in terms of other observable quantities. A brief discussion on the BD cosmology with $k=0$ and $\lambda\neq 0$ is given.

We start with the following Lagrangian density for the BD theory with the cosmological constant $\lambda$ (Ref. 5):

$$\mathscr{L}=\sqrt{-g}\left[-\phi(R+2\lambda)+\omega\frac{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\phi}\right]+16\pi\mathscr{L}_M\ , \tag{2}$$

where $\mathscr{L}_M$ denotes the Lagrangian density of the matter. In this work we follow the "Landau-Lifshitz timelike convention." The Euler-Lagrange equations of motion for $g_{\mu\nu}$ and $\phi$ are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R-\lambda g_{\mu\nu}=\frac{8\pi}{\phi}T_{M\mu\nu}+\frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\phi_{,\rho}\phi^{,\rho})+\frac{1}{\phi}(\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi)\ , \tag{3}$$

$$\Box\phi-\frac{2\lambda}{3+2\omega}\phi=\frac{8\pi}{3+2\omega}T_M{}^{\mu}{}_{\mu}\ , \tag{4}$$

 

where the energy-momentum tensor of the matter $T_{M\mu\nu}$ is defined by

$$T_{M\mu\nu}=2(-g)^{-1/2}\frac{\delta\mathscr{L}_M}{\delta g^{\mu\nu}} . \tag{5}$$

The equivalence principle ($T_{M\mu\nu}{}^{;\mu}=0$) is satisfied. This can be easily checked by using Eqs. (3) and (4), and the Bianchi identities.

We now apply the theory to cosmology where the universe is smeared out into a homogeneous isotropic distribution of the matter. The metric is then given by Robertson-Walker form

$$d\tau^2=dt^2-a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right] , \tag{6}$$

and the energy-momentum tensor is that of the perfect fluid

$$T_{M\mu\nu}=-p(t)g_{\mu\nu}+[\rho(t)+p(t)]u_\mu u_\nu , \tag{7}$$

where $\rho$ is the mass density, $p$ is the pressure, and

$$u_\mu=g_{\mu\nu}u^\nu=g_{\mu\nu}\frac{dx^\nu}{d\tau} .$$

Then, Eqs. (3) and (4) reduce to the following three equations:

$$\frac{3\dot a^2}{a^2}+\frac{3k}{a^2}-\lambda=\frac{8\pi}{\phi}\rho+\frac{\omega}{2}\frac{\dot\phi^2}{\phi^2}-\frac{3\dot a}{a}\frac{\dot\phi}{\phi} , \tag{8}$$

$$-\frac{2\ddot a}{a}-\frac{\dot a^2}{a^2}-\frac{k}{a^2}+\lambda =\frac{8\pi}{\phi}p+\frac{\omega}{2}\frac{\dot\phi^2}{\phi^2}+\frac{\ddot\phi}{\phi}+\frac{2\dot a}{a}\frac{\dot\phi}{\phi} , \tag{9}$$

$$\frac{\ddot\phi}{\phi}+3\left[\frac{\dot a}{a}\right]\left[\frac{\dot\phi}{\phi}\right]=\frac{2\lambda}{3+2\omega}+\frac{8\pi}{\phi}\frac{\rho-3p}{3+2\omega} . \tag{10}$$

These are the BD equations with the nonvanishing $\lambda$. It can be checked that Eqs. (8)–(10) lead to the continuity equation

$$\dot\rho+(\rho+p)\frac{3\dot a}{a}=0 , \tag{11}$$

which is consistent with the principle of equivalence. The gravitational "constant" is given by

$$G=\left[\frac{2\omega+4}{2\omega+3}\right]\frac{1}{\phi} \tag{12}$$

and $G_0\equiv G_N$ is the Newtonian gravitational constant.

We now set $k=0$ in Eqs. (8)–(10) and consider the matter-dominant universe where the pressure can be neglected. Defining

$$\epsilon=-\frac{\dot\phi}{\phi} \tag{13}$$

and using Eq. (1), we rewrite Eq. (8) as

$$3H^2-\frac{\omega}{2}\epsilon^2-3\epsilon H=8\pi\frac{\rho}{\phi}+\lambda . \tag{14}$$

Eliminating $\ddot\phi$ from Eqs. (9) and (10) and using Eqs. (1) and (13), we obtain

$$3qH^2-\omega\epsilon^2-3\epsilon H=8\pi\frac{\rho}{\phi}\frac{\omega+3}{2\omega+3}-\frac{2\omega}{2\omega+3}\lambda . \tag{15}$$

We eliminate $\epsilon$ from Eqs. (14) and (15) to find a quadratic equation for $\lambda$:

$$\frac{4\omega(\omega+1)^2}{(2\omega+3)^2}\lambda^2+\frac{2}{2\omega+3}\left[2\omega(\omega+1)qH^2-(2\omega+1)(2\omega+3)H^2+\frac{16\pi\omega(\omega+1)^2}{2\omega+3}\frac{\rho}{\phi}\right]\lambda+\omega q^2H^4 +2(2\omega+3)(1-q)H^4+\frac{16\pi\omega}{2\omega+3}[(\omega+1)(q-2)-1]\frac{\rho}{\phi}+\omega\left[\frac{8\pi(\omega+1)}{2\omega+3}\frac{\rho}{\phi}\right]^2=0 . \tag{16}$$

This equation determines the values of $\lambda$ provided we are given the present values of $H_0$, $q_0$, $\rho_0$, and $\phi_0$ as well as $\omega$.

Next, using $\lambda$ obtained by solving Eq. (16) and combining Eqs. (14) and (15), we can express $\epsilon$ in terms of $H$, $q$, $\rho$, and $\phi$. The result is

$$\epsilon=\frac{1}{3H}\left[-3qH^2+6H^2+8\pi\left[\frac{\omega+3}{2\omega+3}-2\right]\frac{\rho}{\phi}-\left[\frac{2\omega}{2\omega+3}-2\right]\lambda\right] , \tag{17}$$

where $\lambda$ is given by Eq. (16). Equation (17) determines $\epsilon_0$ in terms of the observable parameters $H_0$, $\rho_0$, $q_0$, and $\phi_0$ for a given value of $\omega$.

Since both $\lambda$ and $\epsilon_0$ can be expressed in terms of $\rho_0$, $H_0$, $q_0$, and $\phi_0$, the solutions of the BD equations will be written in terms of $\lambda$ and $\epsilon_0$ as well as the other four initial values. This enables us to write down the solutions in a compact way.

We now proceed to solve Eqs. (8)–(10) for $a(t)$. The combination, Eq. (8) $-$ Eq. (9) $-$ $(\frac{1}{3})$ Eq. (10), gives the following equation for $\phi a^3$:

$$\frac{d^2}{dt^2}(\phi a^3)-\eta^2\lambda(\phi a^3)=4\pi\eta^2\rho_0 a_0{}^3 , \tag{18}$$

where

$$\eta^2=\frac{2(4+3\omega)}{3+2\omega}$$

and $\rho_0 a_0{}^3$ is an integral constant of Eq. (11):

$$\rho(t)a^3(t)=\rho_0 a_0{}^3 . \tag{19}$$

Equation (18) can easily be solved. Observing that the left-hand side of Eq. (10) is $(\phi a^3)^{-1}d(\dot{\phi}a^3)/dt$, we can rewrite Eq. (10) as

$$\frac{d}{dt}(\dot{\phi} a^3)=8\pi\frac{\rho_0 a_0{}^3}{3+2\omega}+\frac{2\lambda}{3+2\omega}f(t) , \tag{20}$$

where $f(t)=\phi(t)a^3(t)$ is the solution of Eq. (18), expressed in terms of elementary functions. Defining the solution of Eq. (20) by $h(t)=\dot{\phi}(t)a^3(t)$, and taking the time derivative of $f(t)=\phi(t)a^3(t)$, we obtain

$$\dot{f}(t)=h(t)+3f(t)\left[\frac{\dot{a}}{a}\right] . \tag{21}$$

It is elementary to solve Eq. (21) for $a(t)$. However, since Eqs. (8)–(10) are coupled equations, the consistency of the solution with all equations must be checked. This consistency leads to a constraint among the integral constants and the correct number of the integral constants is recovered. The exact solutions with initial condition $a(t)=0$ at $t=0$ are the following.

(1) Positive $\lambda$:

$$a(t)\propto\left[A\cosh\eta\sqrt{\lambda}(t-t_c)-\frac{4\pi}{\lambda}\right]^{\alpha(\omega)}\left[\frac{\left[\frac{4\pi}{\lambda}+A\right]\tanh\frac{\eta\sqrt{\lambda}(t-t_c)}{2}-[(4\pi/\lambda)^2-A^2]^{1/2}}{\left[\frac{4\pi}{\lambda}+A\right]\tanh\frac{\eta\sqrt{\lambda}(t-t_c)}{2}+[(4\pi/\lambda)^2-A^2]^{1/2}}\right]^{\delta\beta(\omega)} \quad \text{for } B>0 , \tag{22}$$

$$a(t)\propto\left[A'\sinh\eta\sqrt{\lambda}(t-t_c')-\frac{4\pi}{\lambda}\right]^{\alpha(\omega)}\left[\frac{\left[\frac{4\pi}{\lambda}\right]\tanh\frac{\eta\sqrt{\lambda}(t-t_c')}{2}+A'-[(4\pi/\lambda)^2+A'^2]^{1/2}}{\left[\frac{4\pi}{\lambda}\right]\tanh\frac{\eta\sqrt{\lambda}(t-t_c')}{2}+A'[(4\pi/\lambda)^2+A'^2]^{1/2}}\right]^{\delta\beta(\omega)} \quad \text{for } B<0 , \tag{23}$$

where

$$B=\left[\frac{4\pi}{\lambda}\right]^2-\frac{3}{2\lambda}\frac{1}{4+3\omega}\left[\frac{\phi_0}{\rho_0}\right]^2[(1+\omega)\epsilon_0+H_0]^2 . \tag{24}$$

(2) Negative $\lambda$:

$$a(t)\propto\left[A\sin\eta\sqrt{-\lambda}(t-t_c'')-\frac{4\pi}{\lambda}\right]^{\alpha(\omega)}\left[\frac{-\frac{4\pi}{\lambda}\tan\frac{\eta\sqrt{-\lambda}(t-t_c'')}{2}+A-[A^2-(4\pi/\lambda)^2]^{1/2}}{-\frac{4\pi}{\lambda}\tan\frac{\eta\sqrt{-\lambda}(t-t_c'')}{2}+A+[A^2-(4\pi/\lambda)^2]^{1/2}}\right]^{\delta\beta(\omega)} . \tag{25}$$

In Eqs. (22), (23), and (25)

$$A=\sqrt{B}\,,\quad A'=\sqrt{-B}\,,\quad \alpha(\omega)=\frac{1+\omega}{4+3\omega}\,,\quad \beta(\omega)=\frac{1}{4+3\omega}\left[\frac{3+2\omega}{3}\right]^{1/2} \tag{26}$$

and $\delta=\mathrm{sgn}[(1+\omega)\epsilon_0+H_0]$, and $t_c$, $t_c'$, and $t_c''$ are given by

$$\begin{aligned} t_c&=-\frac{2}{\eta\sqrt{\lambda}}\tanh^{-1}\left[\frac{4\pi/\lambda-A}{4\pi/\lambda+A}\right]^{1/2},\\ t_c'&=-\frac{2}{\eta\sqrt{\lambda}}\tanh^{-1}\frac{[(4\pi/\lambda)^2+A'^2]-A'}{(4\pi/\lambda)},\\ t_c''&=-\frac{2}{\eta\sqrt{-\lambda}}\tan^{-1}\frac{[A^2-(4\pi/\lambda)^2]^{1/2}-A}{(-4\pi/\lambda)}\,. \end{aligned} \tag{27}$$

As mentioned earlier, multiplicative factors for $a(t)$ in Eqs. (22), (23), and (25) cannot be determined in terms of $H_0$, $\rho_0$, $q_0$, and $\phi_0$.

In the limits $\epsilon_0=0$ and $\omega\to\infty$, the solutions in Eqs. (22) and (25) become, respectively,

$$a(t)\propto\sinh^{2/3}\left[\frac{\sqrt{3\lambda}}{2}t\right]\quad\text{for }\lambda>0,\quad a(t)\propto\sin^{2/3}\left[\frac{\sqrt{-3\lambda}}{2}t\right]\quad\text{for }\lambda<0\,. \tag{28}$$

These are nothing but the exact solutions to the Friedmann equations with $k=p=0$ and $\lambda\neq0$.

The present age of the universe is

$$t_0=\begin{cases}\dfrac{1}{\eta\sqrt{\lambda}}\left[\cosh^{-1}\left[\dfrac{\phi_0/\rho_0+4\pi/\lambda}{A}\right]-2\tanh^{-1}\left[\dfrac{4\pi/\lambda-A}{4\pi/\lambda+A}\right]^{1/2}\right] & \text{for }\lambda>0,\ B>0,\\[2ex] \dfrac{1}{\eta\sqrt{\lambda}}\left[\sinh^{-1}\left[\dfrac{\phi_0/\rho_0+4\pi/\lambda}{A'}\right]-2\tanh^{-1}\left[\dfrac{-A'+[(4\pi/\lambda)^2+A'^2]^{1/2}}{4\pi/\lambda}\right]\right] & \text{for }\lambda>0,\ B<0\,,\\[2ex] \dfrac{1}{\eta\sqrt{-\lambda}}\left[\sin^{-1}\left[\dfrac{\phi_0/\rho_0+4\pi/\lambda}{A}\right]-2\tan^{-1}\left[\dfrac{-A+[A^2-(4\pi/\lambda)^2]^{1/2}}{-4\pi/\lambda}\right]\right] & \text{for }\lambda<0.\end{cases} \tag{29}$$

The corresponding age of the universe for the Friedmann equations with $k=0$, $\lambda\neq0$ is

$$t_0=\begin{cases}\dfrac{2}{\sqrt{3\lambda}}\coth^{-1}(\sqrt{3/\lambda}H_0) & \text{for }\lambda>0,\\[2ex] \dfrac{2}{\sqrt{-3\lambda}}\cot^{-1}(\sqrt{3/-\lambda}H_0) & \text{for }\lambda<0.\end{cases} \tag{30}$$

For numerical illustration of our results we consider the following two examples.

(a) $H_0=50$ km/sec Mpc, $\rho_0=3.5\times10^{-30}$ g/cm$^3$, $q_0=\frac{1}{2}$, $\omega=500$. In this case the calculated values of $\lambda$ [from Eq. (16)] are both positive; $\lambda=7.6\times10^{-37}/\mathrm{sec}^2$ and $\lambda=9.4\times10^{-37}/\mathrm{sec}^2$. These values are smaller than the experimental upper limit of $10^{-35}/\mathrm{sec}^2$. We also have

$$\epsilon_0 = \left[\frac{\dot{G}}{G}\right]_0 = 0.16\times 10^{-11}/\mathrm{yr},\quad t_0 = 1.0\times 10^{10}\ \mathrm{yr}\ \ \text{for}\ \lambda = 7.6\times 10^{-37}/\mathrm{sec}^2\,, \tag{31a}$$

$$\epsilon_0 = -0.20\times 10^{-11}/\mathrm{yr},\quad t_0 = 1.0\times 10^{10}\ \mathrm{yr}\ \ \text{for}\ \lambda = 9.4\times 10^{-37}/\mathrm{sec}^2\,. \tag{31b}$$

The above present changing rates of the gravitational constant are also smaller than the experimental upper limit of $|\epsilon_0| \simeq 10^{-10}/\mathrm{yr}$.

(b) $H_0 = 50$ km/sec Mpc, $\rho_0 = 5\times 10^{-30}$ g/cm$^3$, $q_0 = 1, \omega = 500$. We have the two values of $\lambda = -1.6\times 10^{-36}/\mathrm{sec}^2$ and $\lambda = -1.8\times 10^{-36}/\mathrm{sec}^2$ and

$$\epsilon_0 = -0.20\times 10^{-11}/\mathrm{yr},\quad t_0 = 0.94\times 10^{10}\ \mathrm{yr}\ \ \text{for}\ \lambda = -1.6\times 10^{-36}/\mathrm{sec}^2\,, \tag{32a}$$

$$\epsilon_0 = +0.16\times 10^{-11}/\mathrm{yr},\quad t_0 = 0.94\times 10^{10}\ \mathrm{yr}\ \ \text{for}\ \lambda = -1.8\times 10^{-36}/\mathrm{sec}^2\,. \tag{32b}$$

Note that in the cases of Eqs. (31a) and (32b), the gravitational constant is an increasing function of $t$.

In conclusion, we have presented analytic expressions of $a(t)$ for the BD equations in the $k = p = 0$ and $\lambda \neq 0$ case. The solutions are determined, apart from multiplicative factors, by four values $H_0$, $\rho_0$, $q_0$, and $\phi_0$ (or $G_N$) for a given value of $\omega$. For some typical values of these initial conditions, the predicted values of $\lambda$ and $\epsilon_0 = (\dot{G}/G)_0$ are well within the observed upper limits. The predicted ages of the universe are also reasonable but they are in general smaller than the corresponding ages obtained from the Friedmann equations with the same initial values. (In the latter case, there is always one less independent initial value.)

Finally, we present the solution of $a(t)$ for the BD equations in the $k = p = \lambda = 0$ case. The solution is well known.[4] However, our present formulation enables us to write it in the following compact way[6]:

$$a(t) \propto [t(t-2t_c)]^{\alpha(\omega)} \left[\frac{t}{t-t_c}\right]^{\delta\beta(\omega)}\,,$$

where

$$t_c = -\frac{1}{4\pi\eta^2}\left[\frac{3}{3+2\omega}\right]^{1/2} \frac{\phi_0}{\rho_0}\,|(1+\omega)\epsilon_0 + H_0|\;.$$

This work was supported in part by Grants from the National Science Foundation.

[1]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961); R. H. Dicke, *ibid.* 125, 2163 (1962).

[2]R. D. Reasenberg, I. I. Shapiro, P. E. MacNell, R. B. Goldstein, J. C. Breidenthal, J. P. Brenkie, D. L. Cain, T. M. Kaufman, T. A. Komarek, and A. I. Zygielbaum, Astrophys. J. Lett. 234, L219 (1979).

[3]For a review, see for example S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[4]S. Weinberg, Ref. 3, Chap. 16, Sec. 4.

[5]As is seen when $\phi$ is transformed to $\chi^2$, this Lagrangian density is nothing but the one which represents a scalar field interacting with gravitation. (However, the scalar field is not supposed to contribute to the matter so that the equivalence principle is satisfied.) In this version, the cosmological term corresponds to a mass term for the scalar field. There is no reason to prohibit this mass term unless some invariance is assumed, such as invariance under conformal transformation for which the Dicke constant is also uniquely determined.

[6]Compare this solution with the solutions in Ref. 4 where the physical meaning of $t_c$ is still unsettled.