PHYSICAL REVIEW D
PARTICLES AND FIELDS
THIRD SERIES, VOLUME 26, NUMBER 10 15 NOVEMBER 1982

Second-order contributions to gravitational deflection of light in the parametrized post-Newtonian formalism. II. Photon orbits and deflections in three dimensions

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(Received 27 May 1982)

Using a parametrized expansion of the solar metric to second order in the Newtonian potential we calculate the trajectories of photons scattered by the Sun. In general the photon is scattered in two directions orthogonal to its initial momentum because of the anisotropic features of the solar field. We calculate both components of the deflection as measured by an observer within the solar field.

I. INTRODUCTION

In two earlier articles[1,2] (hereafter referred to as papers I and II) we discussed light propagation in the solar environment and introduced a parametrized post-linear (PPL) form of the solar metric. The usual parametrized post-Newtonian (PPN) form of the metric can be used to compute gravitational effects on particle motion in the solar system to second order in $GM/c^2R \equiv \epsilon^2 \approx 2\times 10^{-6}$ (i.e., to order ϵ^4), where M and R are the mass and radius of the Sun. The PPL metric is simply an extension of the PPN metric with which the relativistic effects on light and radio wave propagation in the solar system can be computed to this same order (ϵ^4). In paper II we used the PPL metric to calculate the deflection of light to order ϵ^4 for photon orbits in the equatorial plane of the Sun. Here we complete our development of the light deflection by repeating these calculations for the most general photon trajectory.

In paper I we demonstrated that knowledge of light propagation to order ϵ^n requires knowledge of every term in the metric to that same order. The first nonvanishing contribution to light deflection arises from those terms in the metric which are linear in the Newtonian potential U. To this order (ϵ^2) the metric (sometimes referred to as the linearized metric) has the Cartesian components

$$g_{00} = -1+2\frac{M}{r} , \tag{1.1}$$

$$g_{0i} = 0 , \tag{1.2}$$

$$g_{ij} = \delta_{ij}\left[1+2\gamma\frac{M}{r}\right] , \tag{1.3}$$

where r is related to the usual PPN coordinates by $r=(x^2+y^2+z^2)^{1/2}$. Then, the deflection is given to first order (i.e., to order ϵ^2) by

$$\delta\alpha = (1+\gamma)\frac{M}{r}\left[\frac{1+\cos\alpha}{1-\cos\alpha}\right]^{1/2} \tag{1.4}$$

for an observer with radial coordinate r when the observed angular separation of the Sun and star is α.[3] (Note that here and henceforth we use geometrized units.)

Although the metric contains higher-order terms which are formally treated as though they are of order ϵ^3 in the usual PPN categorization of terms, these terms are actually only of order ϵ^4 for any realistic solar model (paper I). Thus, any proper treatment of light propagation to the next nonvanishing order beyond the terms linear in U will require a systematic expansion of the solar metric to second order in U. Such an expansion was presented in paper II. The resulting parametrized post-linear (PPL) metric has, in the gauge we use (see paper II), the following Cartesian components when the solar gravitational field is assumed to be stationary and all terms smaller than ϵ^4 are neglected:

$$g_{00}=-1+2\frac{M}{r}\left[1-J_2\frac{R^2}{r^2}\frac{3\cos^2\theta-1}{2}\right]-2\beta\frac{M^2}{r^2}\ , \tag{1.5}$$

$$g_{0i}=-\frac{7\Delta_1+\Delta_2}{4}\frac{\epsilon_{ijk}J_jx_k}{r^3}\ , \tag{1.6}$$

$$g_{ij}=\delta_{ij}\left[1+2\gamma\frac{M}{r}\left[1-J_2\frac{R^2}{r^2}\frac{3\cos^2\theta-1}{2}\right]+\tfrac{3}{2}\Lambda\frac{M^2}{r^2}+2\frac{A}{r}\right]\ . \tag{1.7}$$

Each new term in the metric is a term of order ϵ^4. Here J_2 is the dimensionless quadrupole moment parameter of the Sun and $\vec{J}$ is its total angular momentum. A is a quantity with units of centimeters which vanishes identically in general relativity, given by

$$A\equiv\tfrac{1}{2}\left[\Upsilon_1\int\rho_0(r)\Pi(r)d^3x+\Upsilon_2\int\rho_0(r)U(r)d^3x+\Upsilon_3\int P(r)d^3x+\Upsilon_4\int\rho_0(r)\frac{m(r)}{r}d^3x\right]\ , \tag{1.8}$$

where

$$m(r)\equiv\int_0^r\rho_0(r)4\pi r^2dr\ , \tag{1.9}$$

Υ_1, Υ_2, Υ_3, and Υ_4 are constants, and where ρ_0, Π, and P are the baryon mass density of the Sun, specific internal energy density of the Sun, and pressure within the Sun, respectively.[3] Because A appears in g_{ij} but not in g_{00} it cannot be combined with the mass M if both terms are to be correct to order ϵ^4. The angle θ appearing in the quadrupole term of the Newtonian potential is the angle between the symmetry axis of the Sun and the position vector $\vec{r}$ of the photon. If we assume the Sun is symmetric about its angular momentum vector $\vec{J}$ then we have

$$\cos\theta=\frac{\vec{J}\cdot\vec{r}}{|\vec{J}|\,|\vec{r}|}\ . \tag{1.10}$$

(Since we are only keeping terms in the metric to order ϵ^4 and since $\cos^2\theta$ appears only in terms of order ϵ^4, we need no more than the lowest-order form of $\cos\theta$. Thus, we calculate $\cos\theta$ in terms of vector components exactly as in Euclidean space.)

The arbitrary parameters in the metric are γ, β, Δ_1, and Δ_2 (from the usual PPN formalism) and Λ, Υ_1, Υ_2, Υ_3, and Υ_4 (from the extension of the PPN metric to post-linear order). In general relativity these take the values

$$\gamma=\beta=\Delta_1=\Delta_2=\Lambda=1\ , \tag{1.11}$$

$$\Upsilon_1=\Upsilon_2=\Upsilon_3=\Upsilon_4=0\ . \tag{1.12}$$

To obtain the expressions for g_{00} and g_{ij} given in Eqs. (1.5) and (1.7) we assumed the Sun was axially symmetric and truncated the expansion of the Newtonian potential,

$$U=\frac{M}{r}\left[1-\frac{R^2}{r^2}J_2P_2(\cos\theta)-\frac{R^3}{r^3}J_3P_3(\cos\theta)-\cdots\right]\ , \tag{1.13}$$

at the quadrupole term since J_2M/R is less than or on the order of ϵ^4 for any realistic model of the Sun (even if the Sun had a rapidly rotating inner core as suggested by Dicke[4]). If we wish to consider light deflection by any object other than the Sun we must be careful to include enough terms in the expansion of U so that each component of the metric is expanded to the same order. In particular, for Jupiter we find that J_2M/R is larger than $\epsilon^4\approx4\times10^{-12}$ by a factor of about 80. Thus, to study Jovian deflection to order ϵ^4 we might want to expand U further. However, it does not seem unreasonable to assume that the terms beyond the quadrupole will be significantly smaller than ϵ^4. (For example, for the Earth we know that J_4 is on the order of $J_2{}^2$ and that J_3 is even smaller.[5]) Since every other post-linear term in the metric is less than or on the order of ϵ^4 for Jovian deflection, any results we derive using Eqs. (1.5)–(1.7) can be expected to be valid to order ϵ^4 for Jovian deflection in addition to being valid to order ϵ^4 for solar deflection.

II. PHOTON TRAJECTORIES

To find the equations of motion for a photon we begin with the dynamical form of the Lagrangian,

$$L=\tfrac{1}{2}g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\ , \tag{2.1}$$

where λ is an affine parametrization of the trajectory. Since the field is time independent we immediately have the condition

$$p_0=\frac{\partial L}{\partial(dx^0/d\lambda)}=\text{const}\equiv E\ . \tag{2.2}$$

In addition, we must impose the requirement that

$$0=g^{\alpha\beta}p_\alpha p_\beta\ . \tag{2.3}$$

At this point we choose a set of spatial coordinates in which the Euler-Lagrange equations will be the simplest. In the equatorial plane (papers I and II) we worked with polar coordinates. However, in

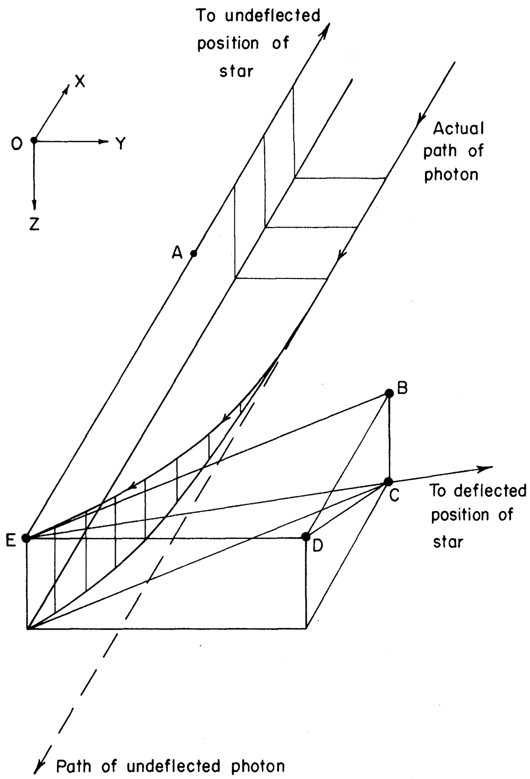

FIG. 1. Geometry of a general photon trajectory: The AEB plane is parallel to the xy plane. Point C lies in the xy plane. The observer is at point E and the Sun is at the origin 0 of the coordinate system.

three dimensions it turns out to be most convenient to use the ordinary Cartesian form of the PPN coordinates,

$$x = r \sin\theta \cos\phi \;, \tag{2.4}$$

$$y = r \sin\theta \sin\phi \;, \tag{2.5}$$

$$z = r \cos\theta \;, \tag{2.6}$$

if we orient them so as to put the star in the xy plane at $x = +\infty$ and such that the initial photon path is parallel to the x axis with impact parameter $y = b$ (see Fig. 1). This expedites the analysis for two reasons. First, note that x is approximately an affine parametrization of the trajectory. In fact, we would just have

$$\frac{dx}{d\lambda} = E \tag{2.7}$$

if there were no deflection. Furthermore, first derivatives with respect to x of y and z will be small (because dy/dx and dz/dx are essentially the two deflection angles; see Sec. III). These facts can be used to great advantage in simplifying the equations of motion.

When we use the first integrals (2.2) and (2.3) the equations of motion in these coordinates take the following form to order ϵ^4:

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left[\frac{1}{g_{xx}}\frac{\partial g_{xx}}{\partial x} - \frac{1}{g_{00}}\frac{\partial g_{00}}{\partial x}\right]\frac{dy}{dx} = \frac{\partial g_{0y}}{\partial x} - \frac{\partial g_{0x}}{\partial y} + \frac{1}{2}\left[\frac{1}{g_{xx}}\frac{\partial g_{xx}}{\partial y} - \frac{1}{g_{00}}\frac{\partial g_{00}}{\partial y}\right] , \tag{2.8}$$

$$\frac{d^2z}{dx^2} = \frac{\partial g_{0z}}{\partial x} - \frac{\partial g_{0x}}{\partial z} + \frac{1}{2}\left[\frac{1}{g_{xx}}\frac{\partial g_{xx}}{\partial z} - \frac{1}{g_{00}}\frac{\partial g_{00}}{\partial z}\right] . \tag{2.9}$$

We then substitute the PPL metric [Eqs. (1.5)–(1.7)] and the equations of motion become (to order ϵ^4)

$$\frac{d^2y}{dx^2} - (1+\gamma)M\frac{x}{r^3}\frac{dy}{dx} + (1+\gamma)M\frac{1}{r^3}y = 2(\gamma^2 - 1 + \beta - \tfrac{3}{4}\Lambda)M^2\frac{b}{r^4} - A\frac{b}{r^3} + \frac{7\Delta_1+\Delta_2}{4}J_z\frac{1}{r^3} + \tfrac{3}{2}(1+\gamma)J_2MR^2\left[-\frac{b}{r^5} + \left[\frac{J_x}{J}\right]^2\frac{5bx^2}{r^7} + \left[\frac{J_y}{J}\right]^2\frac{b(3b^2-2x^2)}{r^7} + \left[\frac{J_x}{J}\right]\left[\frac{J_y}{J}\right]\frac{2x(4b^2-x^2)}{r^7}\right] , \tag{2.10}$$

$$\frac{d^2z}{dx^2} = \frac{7\Delta_1+\Delta_2}{4}\left[J_x\frac{3bx}{r^5} + J_y\frac{2b^2-x^2}{r^5}\right] - 3(1+\gamma)J_2MR^2\frac{J_z}{J}\left[\frac{J_x}{J}\frac{x}{r^5} + \frac{J_y}{J}\frac{b}{r^5}\right] . \tag{2.11}$$

The solutions of these equations to order ϵ^4 which satisfy the boundary conditions

$$y \to b \quad \text{as } x \to \infty \;, \tag{2.12}$$

$$\frac{dy}{dx} \to 0 \quad \text{as } x \to \infty \ , \tag{2.13}$$

$$z \to 0 \quad \text{as } x \to \infty \ , \tag{2.14}$$

$$\frac{dz}{dx} \to 0 \quad \text{as } x \to \infty \tag{2.15}$$

are

$$\begin{aligned} y = b &- \left[(1+\gamma)\frac{M}{b}+\frac{A}{b}\right][(x^2+b^2)^{1/2}-x]+b\left[(1+\gamma)\frac{M}{b}\right]^2\left[1-\frac{x}{(x^2+b^2)^{1/2}}\right] \\ &-[2(1+\gamma)-\beta+\tfrac{3}{4}\Lambda]\frac{M^2}{b}\left[1-\frac{x}{b}\text{Arccot}\frac{x}{b}\right]+\frac{7\Delta_1+\Delta_2}{4}\frac{J_z}{b^2}[(x^2+b^2)^{1/2}-x] \\ &+\tfrac{1}{2}(1+\gamma)MJ_2\frac{R^2}{b^2}\left\{\frac{b}{(x^2+b^2)^{1/2}}-\frac{2}{b}[(x^2+b^2)^{1/2}-x]+\left[\frac{J_x}{J}\right]^2\left[\frac{2}{b}[(x^2+b^2)^{1/2}-x]-\frac{bx^2}{(x^2+b^2)^{3/2}}\right]\right. \\ &\qquad +\left[\frac{J_y}{J}\right]^2\left[\frac{4}{b}[(x^2+b^2)^{1/2}-x]-\frac{b(2x^2+3b^2)}{(x^2+b^2)^{3/2}}\right] \\ &\qquad \left. +2\frac{J_xJ_y}{J^2}\left[1-\frac{x}{(x^2+b^2)^{1/2}}-\frac{b^2x}{(x^2+b^2)^{3/2}}\right]\right\} , \end{aligned} \tag{2.16}$$

$$\begin{aligned} z = &\frac{7\Delta_1+\Delta_2}{4}\frac{J_x}{b}\left[1-\frac{x}{(x^2+b^2)^{1/2}}\right]+\frac{7\Delta_1+\Delta_2}{4}\frac{J_y}{b^2}\left[\frac{x^2}{(x^2+b^2)^{1/2}}-x\right] \\ &-(1+\gamma)MJ_2\frac{R^2}{b^2}\frac{J_z}{J}\left\{\frac{J_x}{J}\left[1-\frac{x}{(x^2+b^2)^{1/2}}\right]+\frac{J_y}{J}\left[\frac{2}{b}[(x^2+b^2)^{1/2}-x]-\frac{b}{(x^2+b^2)^{1/2}}\right]\right\} . \end{aligned} \tag{2.17}$$

III. DEFLECTION

We now know the trajectory of the photon and may solve for the observed deflection angle $\delta\alpha$. We assume that the observer is at rest with respect to the source of the field (the Sun or Jupiter) and at some realistic distance from it ($r \approx 1$ astronomical unit for solar deflection; $r \approx 4-6$ astronomical units for Jovian deflection). Thus, it is necessary to construct an orthonormal frame which is at rest with respect to the PPN frame in which to compute $\delta\alpha$. Such a frame has the following dual basis (accurate to order ϵ^4):

$$\underline{\omega}^{\tilde{0}} = \sqrt{-g_{00}}\,\underline{d}t - \frac{g_{0i}}{\sqrt{-g_{00}}}\underline{d}x^i \ , \tag{3.1}$$

$$\underline{\omega}^{\tilde{x}} = \sqrt{g_{xx}}\,\underline{d}x \ , \tag{3.2}$$

$$\underline{\omega}^{\tilde{y}} = \sqrt{g_{yy}}\,\underline{d}y = \sqrt{g_{xx}}\,\underline{d}y \ , \tag{3.3}$$

$$\underline{\omega}^{\tilde{z}} = \sqrt{g_{zz}}\,\underline{d}z = \sqrt{g_{xx}}\,\underline{d}z \ . \tag{3.4}$$

(This is clearly an orthonormal basis. Furthermore, since the spatial components of a velocity transform as

$$u^{\tilde{i}} = \sqrt{g_{xx}}\,u^i \ , \tag{3.5}$$

an object at rest in one frame is also at rest in the other.) The corresponding basis vectors are

$$\underline{e}_{\tilde{0}} = \frac{1}{\sqrt{-g_{00}}}\frac{\partial}{\partial t} \ , \tag{3.6}$$

$$\underline{e}_{\tilde{x}} = \frac{1}{\sqrt{g_{xx}}}\frac{\partial}{\partial x} - \frac{g_{0x}}{g_{00}\sqrt{g_{xx}}}\frac{\partial}{\partial t} \ , \tag{3.7}$$

$$\underline{e}_{\tilde{y}} = \frac{1}{\sqrt{g_{xx}}} \frac{\partial}{\partial y} - \frac{g_{0y}}{g_{00}\sqrt{g_{xx}}} \frac{\partial}{\partial t} , \tag{3.8}$$

$$\underline{e}_{\tilde{z}} = \frac{1}{\sqrt{g_{xx}}} \frac{\partial}{\partial z} - \frac{g_{0z}}{g_{00}\sqrt{g_{xx}}} \frac{\partial}{\partial t} . \tag{3.9}$$

Notice that the proper time τ of an observer at rest with respect to the Sun is related to the coordinate time t by

$$-d\tau^2 = g_{00} dt^2 , \tag{3.10}$$

from which it follows that the four-velocity $\underline{u}_E$ of the observer is

$$\underline{u}_E = \frac{dt}{d\tau} \frac{\partial}{\partial t} = \frac{1}{\sqrt{-g_{00}}} \frac{\partial}{\partial t} . \tag{3.11}$$

Thus, we see that $\underline{e}_{\tilde{0}}$ is the four-velocity of the observer and the basis vectors $\underline{e}_{\tilde{x}}$, $\underline{e}_{\tilde{y}}$, and $\underline{e}_{\tilde{z}}$ are an orthonormal basis of the three-surface orthogonal to the observer's world line.

The tangent vector $\vec{u}$ to the photon trajectory has the components

$$u^{\tilde{x}} = \sqrt{g_{xx}} u^x = \sqrt{g_{xx}} \frac{dx}{d\lambda} , \tag{3.12}$$

$$u^{\tilde{y}} = \sqrt{g_{xx}} u^y = \sqrt{g_{xx}} \frac{dy}{d\lambda} , \tag{3.13}$$

$$u^{\tilde{z}} = \sqrt{g_{xx}} u^z = \sqrt{g_{xx}} \frac{dz}{d\lambda} , \tag{3.14}$$

in the orthonormal basis of the three-surface orthogonal to the observer's world line. An undeflected photon would have $u^{\tilde{y}} = u^{\tilde{z}} = 0$. Thus, the deflection angle is fixed by the values

$$\frac{u^{\tilde{y}}}{u^{\tilde{x}}} = \frac{u^y}{u^x} = \frac{dy}{dx} , \tag{3.15}$$

$$\frac{u^{\tilde{z}}}{u^{\tilde{x}}} = \frac{u^z}{u^x} = \frac{dz}{dx} . \tag{3.16}$$

At this point we see that the deflection will be determined to order ϵ^4 only by the trajectory of the photon (we need only calculate dy/dx and dz/dx) and not by the geometry of the three-surface in which the angle is measured. To illustrate the deflection in the simplest way possible we treat x, y, and z as flat-space coordinates and sketch the photon trajectory as a curve in Euclidean space in Fig. 1.

Referring to Fig. 1, the total deflection angle $\delta\alpha$ is the angle $\measuredangle$AEC. It is convenient to split this into two components, $\delta\alpha_{\parallel}$ and $\delta\alpha_{\perp}$, which we define as

$$\delta\alpha_{\parallel} \equiv \measuredangle \text{AEB} , \tag{3.17}$$

$$\delta\alpha_{\perp} \equiv \measuredangle \text{BEC} . \tag{3.18}$$

$\delta\alpha_{\parallel}$ is the amount of radial deflection of the photon. $\delta\alpha_{\perp}$ is the amount of deflection orthogonal to the radial plane and is zero to order ϵ^2. Since

$$y' \equiv \frac{dy}{dx} = \tan \measuredangle \text{AEB} , \tag{3.19}$$

$$z' \equiv \frac{dz}{dx} = \tan \measuredangle \text{BDC} , \tag{3.20}$$

we find that to order ϵ^4

$$\delta\alpha_{\parallel} = y' , \tag{3.21}$$

$$\delta\alpha_{\perp} = z' . \tag{3.22}$$

Using some elementary spherical trigonometry we find that the total deflection angle is given to order ϵ^4 by

$$\delta\alpha = \delta\alpha_{\parallel} . \tag{3.23}$$

(This is essentially just because the expansion of

$$[(\epsilon^2+\eta)^2+\xi^2]^{1/2} \tag{3.24}$$

with η,ξ both of order ϵ^4 is just $\epsilon^2+\eta$ to order ϵ^4.) Note that if we were interested only in the total deflection angle to order ϵ^4 we could neglect the z motion of the photon entirely. We will calculate both components of the deflection to order ϵ^4 by finding y' and z' to that order.

The differentiation of y and z is straightforward. To eliminate x from the results we solve the relations

$$x = \rho \cos\phi , \tag{3.25}$$

$$y = \rho \sin\phi , \tag{3.26}$$

for $x/(x^2+b^2)^{1/2}$ and $b/(x^2+b^2)^{1/2}$ to order ϵ^2. [By examining expressions (2.16) and (2.17) it is clear that y' contains one term of order ϵ^2 and several corrections of order ϵ^4 and that z' contains only terms of order ϵ^4. Thus, any substitutions need only be correct to order ϵ^2 to maintain the ϵ^4 level of accuracy in y' and z'.] To eliminate b from the results we solve the first-order (i.e., ϵ^2) equation of motion,

$$\frac{b}{r} = \sin\phi + (1+\gamma)\frac{M}{b}(1-\cos\phi) , \tag{3.27}$$

for r/b to order ϵ^2. Finally, we make use of the fact that

$$\phi=\pi-\alpha+\delta\alpha \tag{3.28}$$

to order ϵ^2 (see paper II) and of the first-order (i.e., ϵ^2) form of $\delta\alpha$ from Eq. (1.4) to express the result in terms of r and α. We thus obtain the following results to order ϵ^4:

$$\begin{aligned}y'=&(1+\gamma)\frac{M}{r}\frac{1+\cos\alpha}{\sin\alpha}-(1+\gamma)^2\frac{M^2}{r^2}\frac{1+\cos\alpha}{\sin\alpha}+[2(1+\gamma)-\beta+\tfrac{3}{4}\Lambda]\frac{M^2}{r^2}\frac{\pi-\alpha+\sin\alpha\cos\alpha}{\sin^2\alpha}\\&+\frac{A}{r}\frac{1+\cos\alpha}{\sin\alpha}-\frac{7\Delta_1+\Delta_2}{4}\frac{J_z}{r^2}\frac{1+\cos\alpha}{\sin^2\alpha}\\&+\tfrac{1}{2}(1+\gamma)\frac{M}{r}J_2\frac{R^2}{r^2}\left\{2\frac{1+\cos\alpha}{\sin^3\alpha}+\cot\alpha-\left[\frac{J_x}{J}\right]^2\left[2\frac{1+\cos\alpha}{\sin^3\alpha}+\cot\alpha(1-3\sin^2\alpha)\right]\right.\\&\left.-\left[\frac{J_y}{J}\right]^2\left[4\frac{1+\cos\alpha}{\sin^3\alpha}+\cot\alpha(2+3\sin^2\alpha)\right]+2\frac{J_xJ_y}{J^2}(1-3\sin^2\alpha)\right\},\end{aligned} \tag{3.29}$$

$$\begin{aligned}z'=&-\frac{7\Delta_1+\Delta_2}{4}\left[\frac{J_x}{r^2}\sin\alpha+\frac{J_y}{r^2}\frac{1+2\cos\alpha-\cos^3\alpha}{\sin^2\alpha}\right]\\&+(1+\gamma)\frac{M}{r}J_2\frac{R^2}{r^2}\frac{J_z}{J}\left[\frac{J_x}{J}+\frac{J_y}{J}\left[2\frac{1+\cos\alpha}{\sin^3\alpha}+\cot\alpha\right]\right].\end{aligned} \tag{3.30}$$

For a limb-grazing ray we have (paper II)

$$b=R\left[1+(1+\gamma)\frac{M}{R}\right]+O(\epsilon^4), \tag{3.31}$$

$$\alpha=\arcsin\left\{\frac{R}{r}\left[1+(1+\gamma)\frac{M}{R}\left[1-\frac{R}{r}\right]\right]\right\}+O\left[\frac{M^2}{R^2}\right]. \tag{3.32}$$

Substituting these into Eqs. (3.29) and (3.30) gives (to order ϵ^4)

$$\begin{aligned}y'_{r_{\min}=R}=&\,2(1+\gamma)\frac{M}{R}-\frac{1}{2}(1+\gamma)\frac{M}{R}\frac{R^2}{r^2}+\{[2(1+\gamma)-\beta+\tfrac{3}{4}\Lambda]\pi-2(1+\gamma)^2\}\frac{M^2}{R^2}+2\frac{A}{R}-\frac{7\Delta_1+\Delta_2}{2}\frac{J_z}{R^2}\\&+2(1+\gamma)\left[1-\left[\frac{J_x}{J}\right]^2-2\left[\frac{J_y}{J}\right]^2\right]J_2\frac{M}{R},\end{aligned} \tag{3.33}$$

$$z'_{r_{\min}=R}=-\frac{7\Delta_1+\Delta_2}{2}\frac{J_y}{R^2}+4(1+\gamma)\frac{J_yJ_z}{J^2}J_2\frac{M}{R}. \tag{3.34}$$

The term proportional to R^2/r^2 is on the order of $\epsilon^4\sim10^{-12}$ by virtue of the fact that

$$\frac{R}{r}\approx5\times10^{-3} \tag{3.35}$$

for observations of solar light deflection. For Jovian deflection it will be much smaller since M/R is 100 times smaller for Jupiter than it is for the Sun and since

$$\frac{R}{r}\lesssim1\times10^{-4} \tag{3.36}$$

for observations of Jovian deflection. (It is equal to 10^{-4} when the observer is between the Sun and Jupiter so as to eliminate solar effects.)

These results agree with those of Epstein and Shapiro[6] and those of Fischbach and Freeman[7] although in each of these previous papers on the subject the authors have neglected those second-order terms which are zero in general relativity (the Υ_i terms) and have calculated the deflection only for observers at infinity.

IV. CONCLUSIONS

For photon orbits that are restricted to the Sun's equatorial plane ($\theta=\pi/2$) we have $J_z=\pm J$, $J_x=J_y=0$ which gives

$$y'_{r_{\min}=R}=2(1+\gamma)\frac{M}{R}-\frac{1}{2}(1+\gamma)\frac{M}{R}\frac{R^2}{r^2}+\{[2(1+\gamma)-\beta+\tfrac{3}{4}\Lambda]\pi-2(1+\gamma)^2\}\frac{M^2}{R^2}$$
$$+2\frac{A}{R}\pm\frac{7\Delta_1+\Delta_2}{2}\frac{J}{R^2}+2(1+\gamma)J_2\frac{M}{R}\ , \tag{4.1}$$

$$z'=0\ . \tag{4.2}$$

These were the results presented in paper II. Table I lists the magnitude of each term in Eq. (4.1). In the general case, the value of the angular momentum term in $y'_{r_{\min}=R}$ varies by a factor of

$$-1\le\frac{J_z}{J}\le 1 \tag{4.3}$$

from its maximum value in the equatorial plane and the quadrupole moment contribution to $y'_{r_{\min}=R}$ varies by a factor of

$$-1\le\left[\frac{J_z}{J}\right]^2-\left[\frac{J_y}{J}\right]^2\le 1 \tag{4.4}$$

from its value in the equatorial plane.

The z' deflection is nonzero only when the orbit is outside of the equatorial plane, and both of the z' terms have factors which vary with the orientation of the initial photon trajectory with respect to the angular momentum vector of the deflecting object. However, as mentioned in Sec. III, the total deflection $\delta\alpha$ is independent of z' to order ϵ^4. Since the best observations of light deflection involve monitoring the angular separations of two discrete radio sources,[8] only $\delta\alpha$ is determined. It is thus unlikely that $\delta\alpha_\perp$ would be of interest observationally.

ACKNOWLEDGMENT

Supported in part by NSF Grant No. PHY 81-07381, by the SERC, and by the National Geographic Society.

TABLE I. Magnitudes of light-deflecting terms as predicted by general relativity.

Term in expression (4.1)	Value in general relativity for solar deflection (μarcsec)	Value in general relativity for Jovian deflection (μarcsec)
$2(1+\gamma)\frac{M}{R}$	1.75×10^6	1.67×10^4
$-\frac{1}{2}(1+\gamma)\frac{M}{R}\frac{R^2}{r^2}$	-9.5	-5×10^{-5}
$2\frac{A}{R}$	0	0
$\pm\frac{7\Delta_1+\Delta_2}{2}\frac{J}{R^2}$	$\pm(0.7-13)$[a]	±0.2
$2(1+\gamma)J_2\frac{M}{R}$	$0.2-40$[a]	300
$\{[2(1+\gamma)-\beta+\frac{3}{4}\Lambda]\pi$ $-2(1+\gamma)^2\}\frac{M^2}{R^2}$	3.5	3×10^{-4}

[a]Here the first term listed is for a uniformly rotating sun; the second is for Dicke's model of the Sun.

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