Mass scales in grand unified theories

R. W. Robinett
Department of Physics, University of Wisconsin—Madison, Madison, Wisconsin 53706

Jonathan L. Rosner*
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
(Received 6 May 1982)

Explicit expressions for the upper and lower bounds for all unification mass scales in allowed breakdowns of SU(5) and SO(10) are reviewed. All breakdown patterns of E_6 through maximal subgroups are catalogued and bounds on the mass scales in such chains are derived. In all cases but one, the mass scale characterizing proton decay is at least as large as in SU(5). The dependence of the bounds on $\sin^2\theta_W$ is examined and all bounds are found to be members of a single-parameter family whose significance is discussed. A weak-$SU_L(2)$ scale parameter Λ_2 is introduced in analogy with the strong scale Λ_3 and the unification masses are found to depend simply on Λ_2 and Λ_3 in a certain approximation.

I. INTRODUCTION

Grand unified theories (GUT's) (Ref. 1) of the strong, weak, and electromagnetic interactions make predictions of new physics at mass scales larger than that of the massive electroweak bosons. The group theory involved in the determination of the spontaneous-symmetry-breakdown patterns, fermion mass relations, etc., however, only determines the relative order of any new mass scales. Georgi, Quinn, and Weinberg[2] showed how to apply the renormalization group to the running coupling constants of the constituent subgroups to derive expressions for the mass scales of GUT's in terms of the breakdown chain, the fermion content, and the low-energy values of the couplings. Application of their method to the SU(5) GUT of Georgi and Glashow[3] not only determines the single unification scale M_x but also yields a prediction for $\sin^2\theta_W$ as a consistency condition. Because of the restrictive nature of this minimal model, no new parameters (apart from Higgs-boson masses and the *t*-quark mass) are needed and (reasonably) precise predictions for all new processes, e.g., proton decay, are calculable in principle. Thus, the mass scale relevant to any new physics (M_x) and the electroweak unification ($\sin^2\theta_W$) are inextricably related.

Application of the renormalization techniques, using the currently accepted low-energy inputs (especially $\sin^2\theta_W$), to the basic subgroup coupling constants without regard for the actual pattern of symmetry breakdown, however, allows for the existence of two disparate mass scales. The $SU_c(3)$ and $SU_L(2)$ couplings become equal at a different scale than do the $SU_L(2)$ and U(1) couplings. The unique SU(5) superheavy mass is recovered only by using the SU(5) prediction for $\sin^2\theta_W$.

Larger unification groups such as SO(10) and E_6 naturally exhibit two or more mass scales as the result of partial unifications and may allow a less restrictive relation between parameters; how rigidly are unification masses and $\sin^2\theta_W$ connected in these theories? More than one mass scale may also allow us to confirm grand unification structure at energy scales below superheavy values; how are these mass scales constrained? As we will see, precise predictions for mass scales (and thus process rates) cannot in principle be made, but allowed ranges for such quantities are calculable.

If we restrict ourselves to a one-loop approximation to the β functions of the running couplings, explicit formulas for the upper and lower bounds of the partial-unification scales can often be obtained. If we also neglect the contributions of Higgs scalars to the β function (in general different for different groups and breakdown patterns), we can more easily compare the natural mass scales in models based on different groups. We will use these approximations throughout this paper.

In Sec. II we review the familiar SU(5) grand unification model and how the coupling-constant-renormalization analysis gives information on the unification mass scale. In Secs. III and IV we re-

 ©1982 The American Physical Society

view the structure of unified models based on SO(10) and E_6 and use the renormalization arguments to derive explicit bounds on many of the new mass scales present in the allowed breakdown patterns of these groups. We also comment on the approximate values for new process rates, especially proton decay, in these schemes. We then collect in Sec. V the mass bounds found previously and find that they are all members of a one-parameter family. We discuss the significance of this fact in relation to the possible existence of partial-unification mass scales different from M_x.

In Sec. VI we investigate the relationship between the unification masses and the natural dimensional parameters of non-Abelian gauge theories, the scale parameters Λ. Defining a weak-SU(2) scale parameter Λ_2 in analogy with the strong scale Λ_3, we find simple relations between Λ_2, Λ_3, and the various unification scales in an approximation where fermions can be ignored.

II. SU(5) MODEL

Considering the minimal rank-4 simple or powers of simple groups containing $SU_c(3)\times SU(2)\times U(1)$, Georgi and Glashow[3] showed that only SU(5) is consistent with grand unification. It has the simple spontaneous-symmetry-breakdown pattern

$$SU(5)\underset{2M_x}{\rightarrow} SU_c(3)\times SU(2)\times U(1) \underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1) \tag{2.1}$$

(see Fig. 1) and only one new mass scale M_x.

As Georgi, Quinn, and Weinberg[2] pointed out,

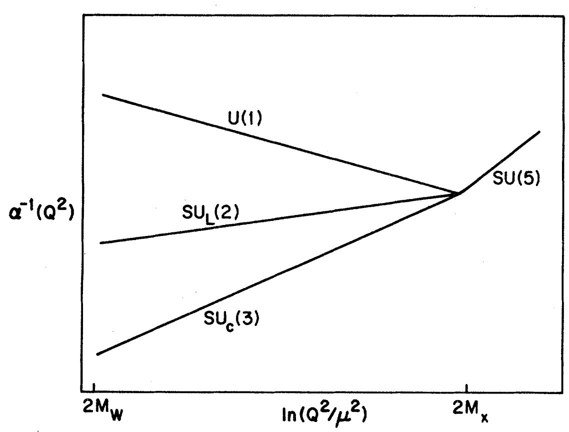

FIG. 1. Running couplings $\alpha_i^{-1}(Q^2)\equiv(g_i^2/4\pi)^{-1}$ versus $t=\ln(Q^2/\mu^2)$ in one-loop approximation for $SU(5)\rightarrow SU_c(3)\times SU_L(2)\times U(1)$ breakdown.

the coupling constants of the constituent subgroups, all equal at unification, are different at momentum transfers corresponding to current laboratory energies because they are renormalized differently. As pointed out in the Introduction, we will explicitly consider only a one-loop approximation to the β function, take all thresholds to be θ functions, and will neglect the contributions of Higgs scalars. The running coupling constants in this approximation can be written[4]

$$\alpha_i^{-1}(Q^2)=\alpha_i^{-1}(\mu^2)+\frac{11C_2(G_i)-4T(R_i)}{12\pi}\ln\left[\frac{Q^2}{\mu^2}\right]=\alpha_i^{-1}(\mu^2)+\frac{\beta_0^i}{4\pi}t \tag{2.2}$$

with $\alpha_i\equiv g_i^2/4\pi$, $\delta_{ab}C_2(G_i)=f^i_{acd}f^i_{bcd}$, $\delta_{ab}T(R_i)=\mathrm{Tr}(t^i_a t^i_b)$ (where t^i are the matrices which represent the generators in the relevant fermion representations), $\beta_0^i=\frac{1}{3}[11C_2(G_i)-4T(R_i)]$, and $t=\ln(Q^2/\mu^2)$.

The fermions of the theory are contained in N $\underline{5}^*+\underline{10}$ representations of SU(5), i.e., N copies of the lowest (u,d,e,ν_e) generation. Calculation then gives

$$\alpha_1^{-1}=\alpha_{\rm GUM}^{-1}+\frac{N}{3\pi}\ln\left[\frac{M_x^2}{M_W^2}\right], \tag{2.3}$$

$$\alpha_2^{-1}=\alpha_{\rm GUM}^{-1}-\left[\frac{22-4N}{12\pi}\right]\ln\left[\frac{M_x^2}{M_W^2}\right], \tag{2.4}$$

$$\alpha_3^{-1}=\alpha_{\rm GUM}^{-1}-\left[\frac{33-4N}{12\pi}\right]\ln\left[\frac{M_x^2}{M_W^2}\right], \tag{2.5}$$

where $\mu=2M_W$ is the renormalization point, unification occurs at $Q^2=4M_x^2$, and $\alpha_{\rm GUM}^{-1}$ is the coupling at unification. We introduce the notation

$$\alpha_i^{-1}\equiv\alpha_i^{-1}(4M_W^2)\ , \tag{2.6}$$

that is, any inverse coupling is to be understood as being evaluated at $\mu=2M_W$ unless stated otherwise.

The Weinberg-Salam (hereafter WS) model[5,6] and SU(5) normalization give

$$\alpha_2^{-1}=\alpha_{\rm EM}^{-1}\sin^2\theta_W\equiv\alpha^{-1}x \tag{2.7}$$

and

$$\alpha^{-1}\equiv\alpha_{\rm EM}^{-1}=\alpha_2^{-1}+\tfrac{5}{3}\alpha_1^{-1}\ , \tag{2.8}$$

where we have defined

$$\alpha^{-1}\equiv\alpha_{\rm EM}{}^{-1}=\alpha_{\rm EM}{}^{-1}(4M_W{}^2) \tag{2.9}$$

and

$$x\equiv\sin^2\theta_W=\sin^2\theta_W(4M_W{}^2) \tag{2.10}$$

as above.

Using the running couplings, Eqs. (2.3)–(2.5) and Eqs. (2.7) and (2.8), there are seen to be two useful combinations of the low-energy input data

$$\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1}=\frac{11}{2\pi}\ln\left[\frac{M_x{}^2}{M_W{}^2}\right], \tag{2.11}$$

$$\alpha^{-1}(3-8x)=\frac{55}{6\pi}\ln\left[\frac{M_x{}^2}{M_W{}^2}\right]. \tag{2.12}$$

These two combinations are seen not to depend on N or $\alpha_{\rm GUM}{}^{-1}$ and provide constraints on the relevant new mass scale. These two combinations will always have these properties whenever $T(R_i)$ is the same for all constituent subgroups. A fermion assignment for which all particles in a single representation remain massless down to the WS breakdown will always satisfy this requirement.[2] [Suitably generalized to larger groups, Eqs. (2.11) and (2.12) will be used to set upper and lower bounds on the unification scales that appear in the SO(10) and E_6 models we will consider in Secs. III and IV.]

Combining (2.11) and (2.12), we find the well-known SU(5) one-loop predictions

$$M_x=M_W\exp\left[\frac{\pi}{11}(\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1})\right], \tag{2.13}$$

$$x=x_5\equiv\frac{1}{6}+\frac{5}{9}\frac{\alpha_3{}^{-1}}{\alpha^{-1}}. \tag{2.14}$$

Using one-loop β functions and θ-function thresholds for the three generations of observed fermions, we can estimate the strong and electromagnetic couplings at the renormalization point. Using $m_u=m_d=300$ MeV, $m_s=500$ MeV, $m_c=1.5$ GeV, $m_b=5$ GeV, $m_t=25$ GeV, and a nominal value of $M_W=80$ GeV, we find that

$$\alpha^{-1}=\alpha_{\rm EM}{}^{-1}(0)-\frac{1}{3\pi}\sum_f Q_f{}^2\ln\left[\frac{M_W{}^2}{m_f{}^2}\right]$$
$$\simeq 128.3\ . \tag{2.15}$$

Similarly, we estimate the strong coupling to be

$$\alpha_3{}^{-1}=8.7-\frac{25}{12\pi}\ln\left[\frac{\Lambda}{200\text{ MeV}}\right]^2, \tag{2.16}$$

where we have taken a value of $\Lambda=0.2$ GeV (defined with four quark flavors[7-10]). Ellis *et al.*[8] reviewed the deep-inelastic-scattering data and use a value of $\Lambda=400$ MeV (with a possible factor-of-2 uncertainty) in their analysis of the proton lifetime. Recent analyses of the gluonic width of the Υ including radiative corrections by MacKenzie and Lepage[11] give a value of

$$\Lambda=100^{+34}_{-25}\text{ MeV}\ . \tag{2.17}$$

We choose to straddle the fence and use a nominal value of $\Lambda=200$ MeV for numerical estimates but we intend to keep track of the Λ dependence in all mass scales we find.

Using these values we find

$$M_x=8.6\times10^{14}\left[\frac{\Lambda}{200\text{ MeV}}\right]^{100/99}\text{ GeV}\ ,$$
$$x_5=\sin^2\theta_W \tag{2.18}$$
$$=0.204-0.0057\ln\left[\frac{\Lambda}{200\text{ MeV}}\right].$$

The coupling at unification is then

$$\alpha_{\rm GUM}{}^{-1}=\left[\frac{33-4N}{66}\right]\alpha^{-1}+\left[\frac{16N-33}{99}\right]\alpha_3{}^{-1} \tag{2.19}$$

and with $N=3$ generations

$$\alpha_{\rm GUM}{}^{-1}=42.1\ . \tag{2.20}$$

The total rate for proton decay has been estimated by many authors and a mean of some results[1] gives

$$\tau_p^{\rm SU(5)}\simeq4.5\times10^{30\pm1.3}\left[\frac{M_x(\text{GeV})}{6\times10^{14}}\right]^4\text{ yr}$$
$$\simeq2\times10^{31\pm1.3}\left[\frac{\Lambda}{200\text{ MeV}}\right]^{4.04}\text{ yr} \tag{2.21}$$

in our one-loop approximation.

Kim *et al.*,[12] by analyzing neutral-current data, found that

$$\sin^2\theta_W(\exp)\equiv x_0=0.23\pm0.01 \tag{2.22}$$

in fair agreement with the one-loop SU(5) prediction. Higher-order effects[7-10,13] [one-loop effects in the extraction of $\sin^2\theta_W$ from experiment and two-loop effects and the contributions of Higgs scalars in the SU(5) renormalization-group analyses] serve to enhance agreement further. Marciano and Sirlin[9] find, for example,

$$\sin^2\theta_W(\mathrm{exp})=0.215\pm0.012\ ,\quad \sin^2\theta_W(\mathrm{SU}(5))=0.212 \tag{2.23}$$

for $\Lambda=0.2$ GeV, remarkably successful agreement.

Such effects also reduce the value of M_x and the same authors find a reduction by a factor of 3.2 in the superheavy mass scale, leading to a proton lifetime some $(3.2)^4\simeq100$ times shorter. As mentioned previously, we ignore such effects in order to better compare the patterns of mass scales in theories based on different groups. These more refined results then give some estimate of the numerical effects of such an approximation.

If we simplemindedly ask, however, where the renormalized coupling constants become equal (pairwise) independent of any definite spontaneous-symmetry-breakdown pattern, we obtain

$$\alpha_1^{-1}(4M_{12}^2)=\alpha_2^{-1}(4M_{12}^2)\Longrightarrow M_{12}=M_W\exp\left[\frac{3\pi}{55}\alpha^{-1}(3-8x)\right], \tag{2.24}$$

$$\alpha_2^{-1}(4M_{23}^2)=\alpha_3^{-1}(4M_{23}^2)\Longrightarrow M_{23}=M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3^{-1})\right]. \tag{2.25}$$

We find two different scales, both explicitly dependent on $x=\sin^2\theta_W$, reducing to the unique SU(5) M_x only when the SU(5) prediction for $\sin^2\theta_W$, $x_5=\frac{1}{6}+(5\alpha_3^{-1})/(9\alpha^{-1})$, is used. If the experimental value of $x_0=0.23$ is used instead, we find $M_{12}=9.5\times10^{12}$ GeV and $M_{23}=2.4\times10^{17}$ GeV and a large range of unification mass scales are present when the restrictiveness of the SU(5) GUT is relaxed. Equivalently, we see how sensitively the unification scale depends on x.

In groups larger than SU(5) where various partial unifications can occur before the ultimate grand unification such different scales, depending explicitly on the value of x, arise naturally. Using our one-loop renormalization approximation, we can find explicit expressions for these natural mass scales and analyze their dependence on the inputs, especially x.

III. SO(10) MODELS

In searching for GUT's based on groups larger than SU(5), one can analyze all (or many) simple groups (or powers of simple groups) for various desired properties (e.g., the correct low-energy symmetry and representations,[14] cancellation of anomalies,[15] etc.) and investigate those (presumably of low rank) with any combination of the relevant features.

Of the next higher rank 5 candidates for grand unification, SU(6), Sp(10), SO(10), SO(11), and $[\mathrm{SU}(2)]^5$, only SO(10) seems to be a viable candidate while satisfying many of the desirable criteria mentioned above.[16] Any SO(n) group ($n\neq6$) is automatically anomaly-free while any SO($2n$) group admits complex representations. Each observed generation of fermions can be placed in a single irreducible representation, a 16-dimensional spinor with the SU(5) decomposition

$$\underline{16}=\underline{10}+\underline{5}^*+\underline{1} \tag{3.1}$$

with the additional field, a neutral lepton $\bar{N}_L$, naturally allowing for the possibility of a finite (and naturally small) neutrino mass.[17] The gauge bosons, as always, are assigned to the adjoint representation, in this case a $\underline{45}$.

Since SO(10) contains SU(5) as a maximal subgroup, it is natural to think of it as an extension of the minimal GUT model, but SO(10) also has a completely different breakdown scheme through maximal subgroups not containing an SU(5) subgroup but still leading to an acceptable low-energy phenomenology, i.e.,

$$\mathrm{SO}(10)\rightarrow\mathrm{SU}_L(2)\times\mathrm{SU}_R(2)\times\mathrm{SU}(4)\ . \tag{3.2}$$

The eventual restoration of parity invariance $[\mathrm{SU}_R(2)]$ and enlargement of the color group [SU(4)] imply the existence of two separate mass scales below the ultimate grand unification. The determination of the magnitude of these scales is important to the discussion of predictions of new allowed processes (right-handed weak interactions, $K_L^0\rightarrow\mu^+e^-$, neutron oscillations,[18] etc.).

The question of which breakdown is favored in a minimal Higgs scheme (which direction in group space gives an absolute minimum in the Higgs potential) has been examined.[19] Both patterns are allowed for a wide range of the undetermined Higgs-boson couplings. We will examine the mass scales in both.

A. Breakdown through SU(5)

The symmetry-breakdown pattern for this case is

$$\mathrm{SO}(10)\underset{2M_u}{\rightarrow}\mathrm{SU}(5)\times \mathrm{U}_\chi(1)$$
$$\underset{2M_x}{\rightarrow}\mathrm{SU}_c(3)\times\mathrm{SU}_L(2)\times\mathrm{U}(1)\times\mathrm{U}_\chi(1)$$
$$\underset{\mu=2M_W}{\rightarrow}\mathrm{SU}_c(3)\times\mathrm{U}_{\mathrm{EM}}(1)\ . \quad (3.3)$$

Under the restriction to SU(5), the adjoint $\underline{45}$ reduces to

$$\underline{45}=\underline{24}+\underline{10}+\underline{10}^*+\underline{1} \quad (3.4)$$

and the $\underline{10}+\underline{10}^*$ contain new superheavy ($\sim M_u$) proton-decay-inducing bosons X_D, Y' (with charges $\frac{2}{3},-\frac{1}{3}$), heavier (since $M_u \geq M_x$) than the X,Y of the SU(5) $\underline{24}$. The SO(10) symmetry can be broken down at $2M_u$ to SU(5) directly via a $\underline{16}$ or $\underline{126}$ of Higgs scalars giving masses to the members of the $\underline{10}+\underline{10}^*$ while also giving mass to the neutral boson in the additional $\mathrm{U}_\chi(1)$. If broken down to $\mathrm{SU}(5)\times\mathrm{U}_\chi(1)$ via an adjoint $\underline{45}$ of Higgs scalars, the $\mathrm{U}_\chi(1)$ symmetry can remain unbroken down to near the WS scale with its associated boson gaining its mass via an SO(10) $\underline{16}'$ which has nonzero Q_χ quantum numbers. [For a discussion of the low-energy electroweak phenomenology including the extra $\mathrm{U}_\chi(1)$, see Masiero[20] and Robinett and Rosner.[21]]

If the fields of the $\underline{16}$ are represented by

$$\tfrac{1}{2}\,|\,\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle \quad (3.5)$$

with $\epsilon_i=\pm 1$, $\prod_{i=1}^{5}\epsilon_i=-1$, then the unnormalized χ-charge operator is given by[22]

$$Q_\chi \propto \sum_{i=1}^{5}\epsilon_i \quad (3.6)$$

so that

$$\underline{1}=|-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}\rangle\ ,\quad Q_\chi\propto -5$$
$$\underline{10}=|-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}\rangle+\text{permutations}\ ,\qquad Q_\chi\propto -1 \quad (3.7)$$
$$\underline{5}^*=|-\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}\rangle+\text{permutations}\ ,\qquad Q_\chi\propto +3\ .$$

Normalizing the Q_χ in the same way as the other constituent subgroups of SU(5), $T(R_\chi)=\mathrm{Tr}(T_\chi T_\chi)=1$ per SO(10) generation, gives

$$Q_\chi(\underline{1})=\frac{-5}{\sqrt{40}},\quad Q_\chi(\underline{10})=\frac{-1}{\sqrt{40}}\ ,\qquad Q_\chi(\underline{5}^*)=\frac{+3}{\sqrt{40}}\ . \quad (3.8)$$

To apply the renormalization analysis of Georgi, Quinn, and Weinberg, we need to be able to express α^{-1} in terms of the coupling constants of the constituent subgroups. Georgi and Weinberg[23] have shown that the generalized Gell-Mann—Nishijima relation

$$Q=\sum_i \beta_i D_i \quad (3.9)$$

[where D_i are the diagonal operators combining to form the $\mathrm{U}_{\mathrm{EM}}(1)$] implies that

$$\alpha^{-1}=\sum_i \beta_i^2\alpha_i^{-1}\ . \quad (3.10)$$

Thus, charge assignments suffice to determine (3.10) in each case. Note also that at unification

$$\sin^2\theta_W=\left[\sum_i \beta_i^2\right]^{-1}\ . \quad (3.11)$$

Application of (3.9) to members of the $\underline{16}$ shows that $\beta_\chi=0$ so that

$$\alpha^{-1}=\alpha_2^{-1}+\tfrac{5}{3}\alpha_1^{-1}\ , \quad (3.12)$$

the same coupling-constant relation as in the SU(5) case. Thus, the $\mathrm{U}_\chi(1)$ (even if unbroken down to near the WS scale) effectively decouples from the renormalization-group analysis and since the renormalized couplings (2.3)—(2.5) are unchanged, the predictions for M_x and $\sin^2\theta_W$ obtained previously still hold. Only the natural ordering $M_u\geq M_x$ then constrains M_u and no new intrinsic mass scale appears in this breakdown.

Since the gauge bosons whose mass is generated at $2M_u$ can also contribute to proton decay, the lifetime in this scheme, $\tau_p^{\mathrm{SO}(10),\mathrm{A}}$, may be somewhat smaller than $\tau_p^{\mathrm{SU}(5)}$ if $M_u\simeq M_x$ (Ref. 24) while if $M_u \gg M_x$, we obviously have $\tau_p^{\mathrm{SO}(10),\mathrm{A}}=\tau_p^{\mathrm{SU}(5)}$. Because of the perhaps inevitable uncertainties in any total rate calculation (due to the present intractability of the strong-interaction processes inside the proton), possibly the only way to distinguish between these two cases will be to measure ratios of partial decay rates[25] or final lepton polarizations[26] which are sensitive to the operator structure of the interaction Hamiltonians, different for the SU(5) and SO(10) models. In fact, Weinberg[27] has compared the present situation to the experimental determination of the operator structure of the weak interactions in the 1950's. Accurate measurements of such quantities, however, would have to wait for a third- or fourth-generation detection experiment.

B. Breakdown through $SU_L(2)\times SU_R(2)\times SU(4)$

The other spontaneous-symmetry-breakdown chain of SO(10) that leads to an acceptable low-energy flavor and color group is

$$\begin{aligned} SO(10) &\underset{2M_u}{\longrightarrow} SU_L(2)\times SU_R(2)\times SU(4) \\ &\underset{2m_{41}}{\longrightarrow} SU_L(2)\times SU_R(2)\times SU_c(3)\times U_{41}(1) \\ &\underset{2m_{21}}{\longrightarrow} SU_c(3)\times SU_L(2)\times U_R(1)\times U_{41}(1) \\ &\underset{\mu=2M_W}{\longrightarrow} SU_c(3)\times U_{EM}(1) \end{aligned} \tag{3.13}$$

(illustrated in Fig. 2).

Various Higgs structures[19] allow either the SU(4) or $SU_R(2)$ to be broken first and the results of a one-loop renormalization analysis are identical in either case so the only *a priori* constraint on the mass scales is $M_x \geq m_{21}, m_{41} \geq M_W$.

The decomposition of the fermion representation

$$\underline{16}=(\underline{2};\underline{1};\underline{4})+(\underline{1};\underline{2};\underline{4}^*) \tag{3.14}$$

illustrates the *L-R* symmetry with the assignment

$$(\underline{2};\underline{1};\underline{4})=\begin{bmatrix} u_1 & d_1 \\ u_2 & d_2 \\ u_3 & d_3 \\ \nu & e^- \end{bmatrix}_L \tag{3.15}$$

for the first generation. The decomposition of the adjoint representation is

$$\begin{aligned} \underline{45} = &(\underline{2};\underline{2};\underline{6})+(\underline{3};\underline{1};\underline{1}) \\ &+(\underline{1};\underline{3};\underline{1})+(\underline{1};\underline{1};\underline{15}) \end{aligned} \tag{3.16}$$

with the bosons in $SO(10)/SU_L(2)\times SU_R(2)\times SU(4)$, i.e., in the $(\underline{2};\underline{2};\underline{6})$ mediating proton decay and having $\sim M_u$ masses. The bosons in the

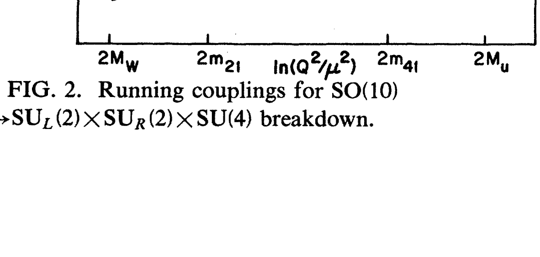

FIG. 2. Running couplings for SO(10) $\to SU_L(2)\times SU_R(2)\times SU(4)$ breakdown.

$(\underline{1};\underline{3};\underline{1})$ are a triplet of right-handed $W^{\pm 0}$ fields which mediate weak $(V+A)$ interactions and acquire their mass at $2m_{21}\simeq 2M_{W_R}$. The $(\underline{1};\underline{1};\underline{15})$ contains the gluons and a gauged $B-L$ (Ref. 28) symmetry, while its off-diagonal, fractionally charged, leptoquark bosons, $B^{\pm 2/3}$, acquire mass at $2m_{41}\simeq 2M_B$.

Many authors[29,30] have considered models with parity restoration (not necessarily in the context of grand unification) and derived limits on the masses of the right-handed analogs of the $W_L^{\pm}, Z_L^0$ by consideration of such electroweak processes as neutrino neutral currents, parity nonconservation in atoms, polarized *ed* scattering, forward-backward asymmetry in $e^+e^-\to\mu^+\mu^-$, μ decay,[31] and the K_L-K_S mass difference.[32] Most of these analyses[29–31] find that the right-handed bosons need only be 3–4 times heavier than their left-handed counterparts. Considering the last process mentioned, however, Beall *et al.*[32] claim that $M_{W_R} > 1.6$ TeV. The B leptoquarks can mediate unobserved lepton-number-violating processes such as $K_L^0\to\mu^+e^-$. Lower bounds on the order of 100 TeV have been set on $m_{41}\simeq M_B$ by considering that process.[33] The corresponding limits on the partial-unification mass scales obtained through renormalization arguments will be seen to be much larger than these.

We can write expressions for the low-energy values of the coupling constants (again using one-loop β functions, θ-function thresholds, and ignoring Higgs scalars) renormalized from their common value at unification:

$$\alpha_L^{-1}=\alpha_{GUM}^{-1}-\left[\frac{22-4N}{12\pi}\right]\ln\left[\frac{M_u^2}{M_W^2}\right], \tag{3.17}$$

$$\alpha_R^{-1}=\alpha_{GUM}^{-1}-\left[\frac{22-4N}{12\pi}\right]\ln\left[\frac{M_u^2}{m_{21}^2}\right] +\frac{N}{3\pi}\ln\left[\frac{m_{21}^2}{M_W^2}\right], \tag{3.18}$$

$$\alpha_3^{-1}=\alpha_{GUM}^{-1}-\left[\frac{44-4N}{12\pi}\right]\ln\left[\frac{M_u^2}{m_{41}^2}\right] -\left[\frac{33-4N}{12\pi}\right]\ln\left[\frac{m_{41}^2}{M_W^2}\right], \tag{3.19}$$

$$\alpha_{41}^{-1}=\alpha_{GUM}^{-1}-\left[\frac{44-4N}{12\pi}\right]\ln\left[\frac{M_u^2}{m_{41}^2}\right] +\frac{N}{3\pi}\ln\left[\frac{m_{41}^2}{M_W^2}\right], \tag{3.20}$$

where N is the number of $\underline{16}$'s used to accommodate the fermions. As before, the contributions of fermions to all β_0^i are equal, ensuring no N dependence in the mass constraint relations.

The diagonal operators that combine to generate the $U_{EM}(1)$ are T_L^3, T_R^3, and

$$T_{41}=\frac{1}{\sqrt{24}}\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3\end{bmatrix}$$

[with T_{41} in an SU(4) basis] where a $(\underline{1};\underline{1};\underline{15})\in\underline{45}$ must develop a nonzero vacuum expectation value (VEV) in the T_{41} direction in order to break SU(4) down to $SU_c(3)$ and generate the $U_{41}(1)$. Fermion charge assigments then give

$$Q=T_L^3+T_R^3+(\tfrac{2}{3})^{1/2}T_{41} \tag{3.21}$$

so that by Eq. (3.10) we have

$$\alpha^{-1}=\alpha_L{}^{-1}+\alpha_R{}^{-1}+\tfrac{2}{3}\alpha_{41}{}^{-1} \tag{3.22}$$

[giving $\sin^2\theta_W=\frac{3}{8}$ at unification, by Eq. (3.11), as it must].

The useful combinations of low-energy inputs are then given by

$$\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1}=\frac{11}{3\pi}\ln\left[\frac{M_u{}^2m_{21}}{M_W{}^3}\right], \tag{3.23}$$

$$\alpha^{-1}(3-8x)=\frac{11}{3\pi}\ln\left[\frac{m_{21}{}^3m_{41}{}^4}{M_u{}^2M_W{}^5}\right], \tag{3.24}$$

or combining (3.23) and (3.24)

$$\alpha^{-1}x-\alpha_3{}^{-1}=\frac{11}{6\pi}\ln\left[\frac{M_u{}^2}{M_Wm_{41}}\right]. \tag{3.25}$$

The constraints are not sufficient to specify exact predictions for the three masses, but the relations (3.23) and (3.25) along with the ordering constraints, $M_u\geq m_{41},m_{21}$, *do* yield definite upper and lower bounds on all three mass scales,

$$M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right]\geq M_u\geq M_W\exp\left[\frac{\pi}{11}(\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1})\right], \tag{3.26}$$

$$M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right]\geq m_{41}\geq M_W\exp\left[\frac{2\pi}{11}\left[\alpha^{-1}(1-3x)+\frac{\alpha_3{}^{-1}}{3}\right]\right], \tag{3.27}$$

$$M_W\exp\left[\frac{\pi}{11}(\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1})\right]\geq m_{21}\geq M_W\exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x)+\tfrac{4}{3}\alpha_3{}^{-1}]\right]. \tag{3.28}$$

(Similar results have been obtained by other authors.[34]) One must remember that these upper and lower bounds are not independent, e.g., M_u and m_{41} attain their maximum values when m_{21} is a minimum. We find that three new scales appear in this breakdown, all depending explicitly on $x=\sin^2\theta_W$. Because there are four outputs to be determined, M_x, m_{21}, m_{41}, and $\alpha_{\rm GUM}{}^{-1}$, a value of $\sin^2\theta_W$ must be used as input along with our estimates for α^{-1} and $\alpha_3{}^{-1}$. It is important to note that if the one-loop SU(5) prediction for $\sin^2\theta_W$ is used,

$$x=x_5=\frac{1}{6}+\frac{5}{9}\frac{\alpha_3{}^{-1}}{\alpha^{-1}},$$

then all of the bounds in (3.26)–(3.28) converge to the SU(5) M_x. It is only if we choose a value of $x>x_5$ that we have any chance of having a partial unification in the "desert" between M_W and M_u. If we use the lowest-order, experimentally determined value, $x=x_0=0.23$, then we obtain the numerical estimates

$$2.4\times10^{17}\tilde{\Lambda}^{25/11}10^{+95.5\Delta x}\ \text{GeV}\geq M_u\geq 8.6\times10^{14}\tilde{\Lambda}^{100/99}\ \text{GeV}, \tag{3.29}$$

$$2.4\times10^{17}\tilde{\Lambda}^{25/11}10^{+95.5\Delta x}\ \text{GeV}\geq m_{41}\geq 3.1\times10^{12}\tilde{\Lambda}^{-25/99}10^{-95.5\Delta x}\ \text{GeV}, \tag{3.30}$$

$$8.6\times10^{14}\tilde{\Lambda}^{100/99}\ \text{GeV}\geq m_{21}\geq 1.1\times10^{10}\tilde{\Lambda}^{-50/33}10^{-191\Delta x}\ \text{GeV}, \tag{3.31}$$

where we explicitly display the Λ and x dependence of each scale with $\tilde{\Lambda}=\Lambda/200$ MeV and $\Delta x=x-x_0$.

The overall grand unification scale M_u is then restricted to lie between the usual SU(5) value and a mass at most some three orders of magnitude larger. The partial-unification mass scales are both far above the best empirical limits and direct evidence for their existence will be difficult to obtain. In fact, using these bounds, we can estimate the right-handed contributions to charged-current weak processes to be

$$\left[\frac{\alpha_L{}^{-1}M_{W_L}{}^2}{\alpha_R{}^{-1}M_{W_R}{}^2}\right]^2 \lesssim 10^{-33}\tilde{\Lambda}^{200/33}10^{+764\Delta x} \tag{3.32}$$

and the rate for $K_L^0 \rightarrow \mu^+ e^-$ due to B-boson exchange at

$$\frac{\Gamma(K_L^0 \rightarrow \mu^+ e^-)}{\Gamma(K_L^0 \rightarrow \text{all})} \lesssim 4\times 10^{-40}\tilde{\Lambda}^{100/99}10^{+382\Delta x} . \tag{3.33}$$

The value of $\alpha_{\text{GUM}}{}^{-1}$, the coupling relevant for proton decay, can also be bounded so that from Eqs. (3.26) and (3.17) we find

$$\left[\frac{33-4N}{11}\right]\alpha^{-1}x - \left[\frac{22-4N}{11}\right]\alpha_3{}^{-1} \geq \alpha_{\text{GUM}}{}^{-1} \geq \left[\frac{33-4N}{66}\right]\alpha^{-1} + \left[\frac{16N-33}{99}\right]\alpha_3{}^{-1} \tag{3.34}$$

and with $N=3$ generations

$$48.4 \geq \alpha_{\text{GUM}}{}^{-1} \geq 42.1 . \tag{3.35}$$

While an exact calculation of the proton-decay rate has not been performed in this case, the main dependence comes from the coupling and propagator factors as always, $\Gamma \propto (\alpha_{\text{GUM}}/M_u{}^2)^2$, so that using SU(5) matrix elements we can estimate that

$$1 \leq \frac{\tau_p^{\text{SO}(10),B}}{\tau_p^{\text{SU}(5)}} \leq 8\times 10^9 \tilde{\Lambda}^{500/99}10^{+382\Delta x} . \tag{3.36}$$

Both here and in all but one of the E_6 models we will examine, the SU(5) prediction sets a lower bound on the proton-decay lifetime, with lifetimes allowed to be much longer.

IV. E_6 MODELS

Of the 12 (up to isomorphism) rank-6 simple or power of simple groups, E_6, SU(7), Sp(12), SO(12), SO(13), $[\text{SU}(4)]^2$, $[\text{SO}(7)]^2$, $[\text{Sp}(6)]^2$, $[\text{G}_2]^3$, $[\text{SU}(3)]^3$, $[\text{SO}(5)]^3$, and $[\text{SU}(2)]^6$, the groups SU(7) (Ref. 35), SO(12) (Ref. 36), and E_6 (Ref. 37) have been considered as candidates for grand unification. If we associate SU(5) with "E_4" and SO(10) with "E_5" (as suggested by Dynkin-diagram language), then E_6 is a natural extension of the sequence and we will consider only it and its allowed breakdown schemes.

E_6 is the only exceptional group that is anomaly-free in all representations and that allows complex representations. The extra rank of E_6 can lead to up to four symmetry breakdowns and we might expect a wider range of allowed values for the associated mass scales.

The gauge bosons are placed in the adjoint 78-dimensional representation while the lowest-dimensional E_6 representation, the $\underline{27}$, contains the fermions. The restriction to SO(10),

$$\underline{27} = \underline{16} + \underline{10} + \underline{1} , \tag{4.1}$$

shows that the usual SO(10) generation is accommodated, while a further restriction [to $\text{SU}_c(3) \times \text{SU}_L(2) \times \text{SU}_R(2)$] gives

$$\underline{10} = (\underline{3}_c;\underline{1};\underline{1}) + (\underline{3}_c^*;\underline{1};\underline{1}) + (\underline{1}_c;\underline{2};\underline{2}) . \tag{4.2}$$

The fermions in the $\underline{10}$ are assigned the quantum numbers of a $Q=-\frac{1}{3}$ quark, h (and $\bar{h}$), a charged lepton, E^- (and E^+), and two corresponding neutral leptons, ν_E and $\bar{N}_E$.[37] The SO(10) singlet consists of one component of a neutral lepton n^0.

If the fermions in the $\underline{10}+\underline{1}$ somehow acquire superheavy ($\sim M_u$) masses, the relevant fermion spectrum is the usual one, N SO(10) generations. If, however, the full E_6 $\underline{27}$ remains massless down to the WS breakdown, then two such $\underline{27}$'s suffice to give a six-quark model. The extra quarks in each $\underline{27}$ are then associated with the b quark and an as yet undiscovered b', giving a model with no top quark. Such models, with the b quark in an $\text{SU}_L(2)$ singlet, make clear predictions for B decays[38]; fewer nonleptonic decays and more $B \rightarrow X l^+ l^-$ than the usual assignment. Experimental determination of the branching ratios for $B \rightarrow X l^- \nu$ and $B \rightarrow X l^+ l^-$,[39] however, almost completely rule out such an assignment. Thus we need three complete E_6 $\underline{27}$'s to accommodate the known fermions.

Since the spontaneous-symmetry-breakdown patterns of E_6 (Refs. 40 and 41) have not been as extensively discussed as either those of SU(5) or SO(10), we will tabulate all breakdown chains through maximal subgroups that lead to an acceptable low-energy color and flavor group $[SU_c(3)\times SU_L(2)$ plus U(1) factors] and acceptable fermion content. We will then analyze the constraints placed on the resulting mass scales by the familiar coupling-constant renormalization arguments.

In these larger groups, the Higgs scalars are assigned, in general, to representations of larger dimension than in SU(5). Their numerical effects on the renormalization-group analysis can then, in principle, be much more important than those mentioned in Sec. II. Similarly, the additional fermions beyond those in the SO(10) $\underline{16}$ can also increase the importance of two-loop effects. Ellis *et al.*[8] have discussed the effect of additional $\underline{5}+\underline{5}^*$ fermions (such as those in the E_6 $\underline{27}$) in SU(5) and find that each such additional set (three of which are needed in E_6) introduces an uncertainty of ~ 3 in M_x.

A. Breakdowns through SO(10)

There are two acceptable schemes beginning with

$$E_6 \underset{2M_u}{\rightarrow} SO(10)\times U_\psi(1) , \tag{4.3}$$

where the SO(10) symmetry is subsequently broken down as in Secs. III A and III B. In both cases it can be shown[40] that the fermions not in the $\underline{16}$ of SO(10) can acquire superheavy masses and effectively decouple from any one-loop renormalization analysis. In this case N E_6 $\underline{27}$'s give the usual N SO(10) generations and the renormalization-group equations for constituent subgroups are the same as in Secs. II and III B. If it is chosen not to give the $\underline{10}+\underline{1}$ fermions superheavy masses, calculation shows that $T(R_i)=3N_6/2$ (N_6 the number of E_6 $\underline{27}$'s) for all subgroups appearing in these breakdowns, so that this assignment is also natural, i.e., it leads to no N dependence in the mass constraint equations.

Slansky[42] has given the relative value of the ψ quantum numbers for many E_6 representations. Normalizing Q_ψ in the same way as all other U(1) charges in E_6 then gives

$$Q_\psi(\underline{16})=\frac{1}{\sqrt{24}},\quad Q_\psi(\underline{10})=\frac{-2}{\sqrt{24}}, \quad Q_\psi(\underline{1})=\frac{4}{\sqrt{24}} \tag{4.4}$$

for members of the $\underline{27}$. The ψ operator is not in the linear combination of neutral generators which make up the electric charge operator. Thus the coupling-constant relations (2.8) and (3.22) for the two SO(10) breakdown schemes are unchanged by the presence of the $U_\psi(1)$. It therefore decouples from the renormalization analysis and all the mass bounds of Secs. III A and III B still apply. No new mass bounds beyond those discussed already appear.

B. Breakdown through $[SU(3)]^3$

The next acceptable breakdown is

$$\begin{aligned} E_6 &\underset{2M_u}{\rightarrow} SU_c(3)\times SU_L(3)\times SU_R(3) \\ &\underset{2m_1}{\rightarrow} SU_c(3)\times SU_L(2)\times SU_R(3)\times U_{L_1}(1) \\ &\underset{2m_2}{\rightarrow} SU_c(3)\times SU_L(2)\times SU_R(2) \times U_{L_1}(1)\times U_{R_1}(1) \\ &\underset{2m_3}{\rightarrow} SU_c(3)\times SU_L(2)\times U_R(1) \times U_{L_1}(1)\times U_{R_1}(1) \\ &\underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1) . \end{aligned} \tag{4.5}$$

(See Fig. 3 for a coupling-constant diagram.)

The order in which the $SU_{L,R}(3)$ break into $SU_{L,R}(2)\times U_{L_1,R_1}(1)$ is irrelevant in a one-loop calculation, so the only *a priori* constraints on the masses are $M_u \geq m_1 \geq M_W$ and $M_u \geq m_2 \geq m_3 \geq M_W$. Economy in the use of Higgs scalars would suggest that both break together, i.e., $m_1=m_2$, which might be accomplished

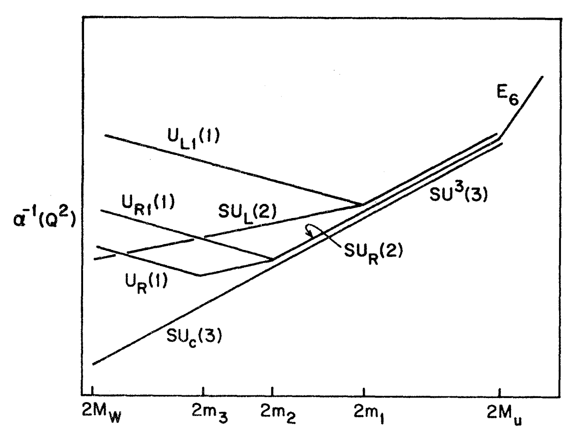

FIG. 3. Couplings for $E_6\rightarrow[SU(3)]^3$ breakdown.

by giving the $(\underline{1}_c;\underline{1}_L;\underline{8}_R)$ and $(\underline{1}_c;\underline{8}_L;\underline{1}_R)$ of an E_6 $\underline{78}$ the same VEV. We consider the more general case described above.

The decomposition of the adjoint in this case

$$\underline{78}=(\underline{8}_c;\underline{1}_L;\underline{1}_R)+(\underline{1}_c;\underline{8}_L;\underline{1}_R)+(\underline{1}_c;\underline{1}_L;\underline{8}_R)+(\underline{3}_c;\underline{3}^*_L;\underline{3}^*_R)+(\underline{3}^*_c;\underline{3}_L;\underline{3}_R) \quad (4.6)$$

shows that the gauge bosons in $E_6/[SU(3)]^3$, the $(\underline{3}_c;\underline{3}^*_L;\underline{3}^*_R)$ and $(\underline{3}^*_c;\underline{3}_L;\underline{3}_R)$, which are responsible for nucleon decay have their masses generated at $2M_u$. The fermions can be assigned [using the decomposition to $SU_c(3)\times SU_L(2)\times SU_R(2)$] to the $\underline{27}$ as follows:

$$\underline{27}=(\underline{3}_c;\underline{3}_L;\underline{1}_R)+(\underline{1}_c;\underline{3}^*_L;\underline{3}_R)+(\underline{3}^*_c;\underline{1}_L;\underline{3}^*_R)$$

$$=\underset{\begin{bmatrix}u\\d\end{bmatrix}_L}{(\underline{3}_c;\underline{2};\underline{1})}+\underset{h_L}{(\underline{3}_c;\underline{1};\underline{1})}+\underset{\begin{bmatrix}\nu_E & E^+\\E^- & \bar{N}_E\end{bmatrix}_L}{(\underline{1}_c;\underline{2};\underline{2})}+\underset{\begin{bmatrix}\nu\\e^-\end{bmatrix}_L}{(\underline{1}_c;\underline{2};\underline{1})}+\underset{(\bar{N},e^+)_L}{(\underline{1}_c;\underline{1};\underline{2})}+\underset{n_L^0}{(\underline{1}_c;\underline{1};\underline{1})}+\underset{(\bar{u},\bar{d})_L}{(\underline{3}^*_c;\underline{1};\underline{2})}+\underset{\bar{h}_L}{(\underline{3}^*_c;\underline{1};\underline{1})} \,. \quad (4.7)$$

In contrast to the two cases in Sec. IV A, only full E_6 $\underline{27}$'s give equal contributions to all subgroup β functions in this chain, in which case $T(R_i)=3N_6/2$ for all subgroups. The coupling-constant diagram (Fig. 3) shows that M_u will only be bounded from below by m_1 and that $\alpha_{GUM}{}^{-1}\geq\alpha_i{}^{-1}(4m_1{}^2)$.

The renormalized coupling constants [with $N=N_6$ and $\alpha_{GUM}{}^{-1}=\alpha_i{}^{-1}(4m_1{}^2)$] are then given by

$$\alpha_{L_1}{}^{-1}=\alpha_{GUM}{}^{-1}+\frac{N}{2\pi}\ln\left[\frac{m_1{}^2}{M_W{}^2}\right],$$

$$\alpha_{R_1}{}^{-1}=\alpha_{GUM}{}^{-1}-\left[\frac{33-6N}{12\pi}\right]\ln\left[\frac{m_1{}^2}{m_2{}^2}\right]+\frac{N}{2\pi}\ln\left[\frac{m_2{}^2}{M_W{}^2}\right],$$

$$\alpha_L{}^{-1}=\alpha_{GUM}{}^{-1}-\left[\frac{22-6N}{12\pi}\right]\ln\left[\frac{m_1{}^2}{M_W{}^2}\right], \quad (4.8)$$

$$\alpha_R{}^{-1}=\alpha_{GUM}{}^{-1}-\left[\frac{33-6N}{12\pi}\right]\ln\left[\frac{m_1{}^2}{m_2{}^2}\right]-\left[\frac{22-6N}{12\pi}\right]\ln\left[\frac{m_2{}^2}{m_3{}^2}\right]+\frac{N}{2\pi}\ln\left[\frac{m_3{}^2}{M_W{}^2}\right],$$

$$\alpha_3{}^{-1}=\alpha_{GUM}{}^{-1}-\left[\frac{33-6N}{12\pi}\right]\ln\left[\frac{m_1{}^2}{M_W{}^2}\right].$$

We have assumed that $m_1>m_2$ for definiteness in deriving these couplings. If $m_2>m_1$, we have different expressions for the couplings, but the resulting constraints on the various masses are identical.

Charge assignments give the coupling relation

$$\alpha^{-1}=\alpha_L{}^{-1}+\alpha_R{}^{-1}+\tfrac{1}{3}\alpha_{L_1}{}^{-1}+\tfrac{1}{3}\alpha_{R_1}{}^{-1} \quad (4.9)$$

and the usual procedure gives the constraints

$$\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1}=\frac{11}{3\pi}\ln\left[\frac{m_1m_2m_3}{M_W{}^3}\right], \quad (4.10)$$

$$\alpha^{-1}(3-8x)=\frac{11}{3\pi}\ln\left[\frac{m_2{}^3m_3{}^3}{m_1M_W{}^5}\right]. \quad (4.11)$$

Rewriting (4.10) and (4.11) gives

$$m_1=M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right] \quad (4.12)$$

and

$$m_2m_3=M_W{}^2\exp\left[\frac{3\pi}{11}(\alpha^{-1}(1-2x)-\tfrac{2}{3}\alpha_3{}^{-1})\right]. \quad (4.13)$$

We then have the constraints

$$M_u\geq m_1=M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right], \quad (4.14)$$

$$M_u\geq m_2\geq M_W\exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x)-\tfrac{2}{3}\alpha_3{}^{-1}]\right]\geq m_3\,. \quad (4.15)$$

We find one new bound in this chain, the lower bound for m_2 and upper bound for m_3, which reduces to the canonical SU(5) M_x if $x=x_5$ is used as an input. Using our estimates for the inputs (including $x=x_0=0.23$), we find the numerical estimate

$$M_W \exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x)-\tfrac{2}{3}\alpha_3{}^{-1}]\right] = 5.1\times 10^{13}\tilde{\Lambda}^{25/66}10^{-47.7\Delta x}\ \text{GeV}\ . \quad (4.16)$$

We note that if the special case with $m_1=m_2$ is chosen, we find

$$M_u \geq m_1 = m_2 = M_W \exp\left[\frac{6\pi}{11}(\alpha^{-1}x - \alpha_3{}^{-1})\right], \quad (4.17)$$

$$m_3 = M_W \exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x)+\tfrac{4}{3}\alpha_3{}^{-1}]\right]. \quad (4.18)$$

Approximating the proton lifetime as before, we find that

$$\frac{\tau_p^{E_6,B}}{\tau_p^{SU(5)}} \gtrsim 8\times 10^{9}\tilde{\Lambda}^{500/99}10^{382\Delta x}\ . \quad (4.19)$$

This is the first case we find where the proton lifetime is not, at least in principle, bounded from above.

C. Breakdown through SU(2)×SU(6)

The only remaining maximal subgroup of E_6 which leads to an acceptable low-energy color and flavor group is SU(2)×SU(6).[42] We find ten allowed breakdown chains in three classes depending on the embedding of the SU(2)×SU(6).

In class I, we identify the SU(2) factor with the weak gauge group $SU_L(2)$. The corresponding reduction of the fermion $\underline{27}$ is

$$\underline{27} = (\underline{2}_L;\underline{6}) + (\underline{1}_L;15^*) \quad (4.20)$$

with

$$(\underline{2}_L;\underline{6}) = \begin{pmatrix} u_1 & d_1 \\ u_2 & d_2 \\ u_3 & d_3 \\ \nu & e^- \\ \nu_E & E^- \\ E^+ & \bar{N}_E \end{pmatrix}_L \quad (4.21)$$

and

$$(\underline{1}_L;\underline{15}) = \begin{pmatrix} 0 & h_3 & h_2 & \bar{h}_1 & \bar{d}_1 & \bar{u}_1 \\ & 0 & h_1 & \bar{h}_2 & \bar{d}_2 & \bar{u}_2 \\ & & 0 & \bar{h}_3 & \bar{d}_3 & \bar{u}_3 \\ & & & 0 & e^+ & \bar{N} \\ & & & & 0 & n^0 \\ & & & & & 0 \end{pmatrix}_L . \quad (4.22)$$

The four allowed breakdown patterns in this class are

1. $$E_6 \xrightarrow[2M_u]{} SU_L(2)\times SU(6) \xrightarrow[2m_1]{} SU_L(2)\times SU(5)\times U_{51}(1) \xrightarrow[2m_2]{} SU_L(2)\times SU_R(2)\times SU_c(3)\times U_{32}(1)\times U_{51}(1) \xrightarrow[2m_3]{} SU_L(2)\times U_R(1)\times SU_c(3)\times U_{32}(1)\times U_{51}(1) \xrightarrow[\mu=2M_W]{} SU_c(3)\times U_{EM}(1)\ , \quad (4.23)$$

2. $$E_6 \xrightarrow[2M_u]{} SU_L(2)\times SU(6) \xrightarrow[2m_1]{} SU_L(2)\times SU(5)\times U_{51}(1) \xrightarrow[2m_2]{} SU_L(2)\times SU(4)\times U_{41}(1)\times U_{51}(1) \xrightarrow[2m_3]{} SU_L(2)\times SU_c(3)\times U_{31}(1)\times U_{41}(1)\times U_{51}(1) \xrightarrow[\mu=2M_W]{} SU_c(3)\times U_{EM}(1)\ , \quad (4.24)$$

3. $$E_6 \xrightarrow[2M_u]{} SU_L(2)\times SU(6) \xrightarrow[2m_1]{} SU_L(2)\times SU_R(2)\times SU(4)\times U_{42}(1) \xrightarrow[2m_2]{} SU_L(2)\times SU_R(2)\times SU_c(3)\times U_{31}(1)\times U_{42}(1) \xrightarrow[2m_3]{} SU_L(2)\times U_R(1)\times SU_c(3)\times U_{31}(1)\times U_{42}(1) \xrightarrow[\mu=2M_W]{} SU_c(3)\times U_{EM}(1)\ , \quad (4.25)$$

4. $$E_6 \xrightarrow[2M_u]{} SU_L(2)\times SU(6)$$

$$\underset{2m_1}{\rightarrow} SU_L(2)\times SU(3)\times SU_c(3)\times U_{33}(1)$$
$$\underset{2m_2}{\rightarrow} SU_L(2)\times SU_R(2)\times SU_c(3)\times U_{21}(1)\times U_{33}(1)$$
$$\underset{2m_3}{\rightarrow} SU_L(2)\times U_R(1)\times SU_c(3)\times U_{21}(1)\times U_{33}(1)$$
$$\underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\ . \tag{4.26}$$

In class II the initial SU(2) is identified with $SU_R(2)$ and the reduction of the $\underline{27}$ is

$$\underline{27}=(\underline{2}_R;\underline{6}^*)+(\underline{1}_R;\underline{15})\ . \tag{4.27}$$

The left-handed fermion assignments are obtained from (4.21) and (4.22) by interchanging particle and antiparticle. We find three allowed breakdowns in this class:

5. $$E_6 \underset{2M_u}{\rightarrow} SU_R(2)\times SU(6)$$
$$\underset{2m_1}{\rightarrow} SU_R(2)\times SU(5)\times U_{51}(1)$$
$$\underset{2m_2}{\rightarrow} SU_R(2)\times SU_L(2)\times SU_c(3)\times U_{32}(1)\times U_{51}(1)$$
$$\underset{2m_3}{\rightarrow} U_R(1)\times SU_L(2)\times SU_c(3)\times U_{32}(1)\times U_{51}(1)$$
$$\underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\ , \tag{4.28}$$

6. $$E_6 \underset{2M_u}{\rightarrow} SU_R(2)\times SU(6)$$
$$\underset{2m_1}{\rightarrow} SU_R(2)\times SU_L(2)\times SU(4)\times U_{42}(1)$$
$$\underset{2m_2}{\rightarrow} SU_R(2)\times SU_L(2)\times SU_c(3)\times U_{31}(1)\times U_{42}(1)$$
$$\underset{2m_3}{\rightarrow} U_R(1)\times SU_L(2)\times SU_c(3)\times U_{31}(1)\times U_{42}(1)$$
$$\underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\ , \tag{4.29}$$

7. $$E_6 \underset{2M_u}{\rightarrow} SU_R(2)\times SU(6)$$
$$\underset{2m_1}{\rightarrow} SU_R(2)\times SU(3)\times SU_c(3)\times U_{33}(1)$$
$$\underset{2m_2}{\rightarrow} SU_R(2)\times SU_L(2)\times SU_c(3)\times U_{21}(1)\times U_{33}(1)$$
$$\underset{2m_3}{\rightarrow} U_R(1)\times SU_L(2)\times SU_c(3)\times U_{21}(1)\times U_{33}(1)$$
$$\underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\ . \tag{4.30}$$

In the class III embedding the SU(2) factor has no definite handedness and we will see that its effect on the renormalization analysis of mass scales is minimal; we therefore call it $SU_I(2)$ because of its inert character. The fermion assignments in this embedding are

$$\underline{27}=(\underline{2}_I;\underline{6}^*)+(\underline{1}_I;\underline{15})\ , \tag{4.31}$$

where

$$(2_I;6^*)=\begin{bmatrix} \bar h_1 & \bar d_1 \\ \bar h_2 & \bar d_2 \\ \bar h_3 & \bar d_3 \\ \nu_E & \nu \\ E^- & e^- \\ \bar N & n^0 \end{bmatrix}_L , \tag{4.32}$$

$$(1_I;15)=\begin{bmatrix} 0 & \bar u_3 & \bar u_2 & d_1 & u_1 & h_1 \\ & 0 & \bar u_1 & d_2 & u_2 & h_2 \\ & & 0 & d_3 & u_3 & h_3 \\ & & & 0 & e^+ & \bar N_E \\ & & & & 0 & E^+ \\ & & & & & 0 \end{bmatrix}_L . \tag{4.33}$$

The three allowed breakdown chains in this class are as follows:

8. $$E_6 \underset{2M_u}{\rightarrow} SU_I(2)\times SU(6)$$
$$\underset{2m_1}{\rightarrow} SU_I(2)\times SU(5)\times U_{51}(1)$$
$$\underset{2M_x}{\rightarrow} SU_I(2)\times SU_c(3)\times SU_L(2)\times U_{32}(1)\times U_{51}(1)$$
$$\underset{2m_3}{\rightarrow} U_I(1)\times SU_c(3)\times SU_L(2)\times U_{32}(1)\times U_{51}(1)$$
$$\underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\ . \tag{4.34}$$

[In this chain, the intermediate SU(5) symmetry is the Georgi-Glashow SU(5). The scale at which it breaks is therefore denoted by M_x.]

$$\text{9. } E_6 \underset{2M_u}{\rightarrow} SU_I(2)\times SU(6) \underset{2m_1}{\rightarrow} SU_I(2)\times SU_c(3)\times SU(3)\times U_{33}(1) \underset{2m_2}{\rightarrow} SU_I(2)\times SU_c(3)\times SU_L(2)\times U_{21}(1)\times U_{33}(1) \underset{2m_3}{\rightarrow} U_I(1)\times SU_c(3)\times SU_L(2)\times U_{21}(1)\times U_{33}(1) \underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\,, \quad (4.35)$$

$$\text{10. } E_6 \underset{2M_u}{\rightarrow} SU_I(2)\times SU(6) \underset{2m_1}{\rightarrow} SU_I(2)\times SU(4)\times SU_L(2)\times U_{42}(1) \underset{2m_2}{\rightarrow} SU_I(2)\times SU_c(3)\times SU_L(2)\times U_{31}(1)\times U_{42}(1) \underset{2m_3}{\rightarrow} U_I(1)\times SU_L(3)\times SU_L(2)\times U_{31}(1)\times U_{42}(1) \underset{\mu=2M_W}{\rightarrow} SU_c(3)\times U_{EM}(1)\,. \quad (4.36)$$

In these ten cases, just as in the $[SU(3)]^3$ case, only the assignment where full E_6 27's remain massless down to low energies is natural in leading to no N dependence in the mass constraints. The fermion contributions to the various β functions are then given by $T(R_i)=3N_6/2$.

We begin by deriving the mass constraints for the four cases in class I.

For case 1 we have the usual renormalized couplings and charge assignments give the coupling-constant relation

$$\alpha^{-1}=\alpha_L{}^{-1}+\alpha_R{}^{-1}+\tfrac{1}{15}\alpha_{32}{}^{-1}+\tfrac{3}{5}\alpha_{51}{}^{-1}\,. \quad (4.37)$$

The familiar combinations of low-energy inputs then give the constraints

$$\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1}=\frac{11}{3\pi}\ln\left[\frac{M_u{}^2m_1m_3}{m_2M_W{}^3}\right], \quad (4.38)$$

$$\alpha^{-1}(3-8x)=\frac{11}{3\pi}\ln\left[\frac{m_1{}^7m_2{}^5m_3{}^3}{M_u{}^{10}M_W{}^5}\right] \quad (4.39)$$

or

$$\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}=\frac{11}{6\pi}\ln\left[\frac{m_1{}^3m_3{}^2}{M_W{}^5}\right], \quad (4.40)$$

$$\alpha^{-1}x-\alpha_3{}^{-1}=\frac{11}{6\pi}\ln\left[\frac{M_u{}^4}{m_1m_2{}^2M_W}\right], \quad (4.41)$$

and, of course, the ordering constraint

$$M_u\geq m_1\geq m_2\geq m_3\geq M_W\,. \quad (4.42)$$

These constraints lead to upper and lower bounds for all mass scales. We find

$$\begin{aligned}
&M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right]\geq M_u\geq M_W\exp\left[\frac{\pi}{11}(\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1})\right],\\
&M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right]\geq m_1\geq M_W\exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}]\right],\\
&M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right]\geq m_2\geq M_W\exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x)-\tfrac{2}{3}\alpha_3{}^{-1}]\right],\\
&M_W\exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}]\right]\geq m_3\geq M_W\exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x)+\tfrac{4}{3}\alpha_3{}^{-1}]\right].
\end{aligned} \quad (4.43)$$

One new mass bound m_1^-, not previously seen, appears in this chain. As expected, it has the property that it reduces to the SU(5) M_x if $x=x_5$. Using our inputs we estimate its magnitude at

$$M_W\exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}]\right]=2.8\times10^{14}\tilde{\Lambda}^{25/33}10^{-19.1\Delta x}\text{ GeV}\,. \quad (4.44)$$

We find an estimated nucleon lifetime as in Sec. III B,

$$1 \lesssim \frac{\tau_p^{E_6,C1}}{\tau_p^{SU(5)}} \lesssim 8\times 10^9 \tilde{\Lambda}^{500/99} 10^{382\Delta x} . \quad (4.45)$$

For case 2 we have

$$\alpha^{-1} = \alpha_L^{-1} + \tfrac{2}{3}\alpha_{31}^{-1} + \tfrac{2}{5}\alpha_{41}^{-1} + \tfrac{3}{5}\alpha_{51}^{-1} \quad (4.46)$$

and the constraints

$$\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1} = \frac{11}{3\pi}\ln\left[\frac{M_u^2 m_1}{M_W^3}\right], \quad (4.47)$$

$$\alpha^{-1}(3-8x) = \frac{11}{3\pi}\ln\left[\frac{m_1^7 m_2^4 m_3^4}{M_u^{10} M_W^5}\right] \quad (4.48)$$

or

$$\alpha^{-1}(1-x) - \tfrac{5}{3}\alpha_3^{-1} = \frac{11}{6\pi}\ln\left[\frac{m_1^3 m_2 m_3}{M_W^5}\right], \quad (4.49)$$

$$\alpha^{-1}(1+2x) - \tfrac{14}{3}\alpha_3^{-1} = \frac{11}{3\pi}\ln\left[\frac{M_u^6}{M_W^4 m_2 m_3}\right] \quad (4.50)$$

and the ordering

$$M_u \ge m_1 \ge m_2 \ge m_3 \ge M_W . \quad (4.51)$$

These give the bounds

$$\begin{aligned}
&M_W \exp\left[\frac{3\pi}{110}[\alpha^{-1}(3+2x) - 10\alpha_3^{-1}]\right] \ge M_u \ge M_W \exp\left[\frac{\pi}{11}(\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1})\right], \\
&M_W \exp\left[\frac{\pi}{11}(\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1})\right] \ge m_1 \ge M_W \exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x) - \tfrac{5}{3}\alpha_3^{-1}]\right], \\
&M_W \exp\left[\frac{\pi}{11}(\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1})\right] \ge m_2 \ge M_W \exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x) - \tfrac{2}{3}\alpha_3^{-1}]\right], \\
&M_W \exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x) - \tfrac{5}{3}\alpha_3^{-1}]\right] \ge m_3 \ge M_W \exp\left[\frac{2\pi}{11}[\alpha^{-1}(1-3x) + \tfrac{1}{3}\alpha_3^{-1}]\right].
\end{aligned} \quad (4.52)$$

We again find one new scale M_u^+, which as usual reduces to M_x when $x = x_5$. We estimate that

$$M_W \exp\left\{\frac{3\pi}{110}[\alpha^{-1}(3+2x) - 10\alpha_3^{-1}]\right\} = 1.5\times 10^{15}\tilde{\Lambda}^{25/22} 10^{9.5\Delta x}\ \text{GeV} \quad (4.53)$$

with an estimated proton-decay lifetime

$$1 \lesssim \frac{\tau_p^{E_6,C2}}{\tau_p^{SU(5)}} \lesssim 9\tilde{\Lambda}^{50/99} 10^{38\Delta x} , \quad (4.54)$$

a much narrower range than the other non-SU(5) cases we have examined.

For case 3 we have

$$\alpha^{-1} = \alpha_L^{-1} + \alpha_R^{-1} + \tfrac{2}{3}\alpha_{31}^{-1} \quad (4.55)$$

and the constraints

$$\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1} = \frac{11}{3\pi}\ln\left[\frac{M_u^2 m_3}{M_W^3}\right], \quad (4.56)$$

$$\alpha^{-1}(3-8x) = \frac{11}{3\pi}\ln\left[\frac{m_1^8 m_2^4 m_3^3}{M_W^5 M_u^{10}}\right] \quad (4.57)$$

or

$$\alpha^{-1}(1-x) - \tfrac{5}{3}\alpha_3^{-1} = \frac{11}{6\pi}\ln\left[\frac{m_3^2 m_1^2 m_2}{M_W^5}\right], \quad (4.58)$$

$$\alpha^{-1}x-\alpha_3{}^{-1}=\frac{11}{6\pi}\ln\left[\frac{M_u{}^4}{M_W m_1{}^2 m_2}\right], \tag{4.59}$$

along with

$$M_u \geq m_1 \geq m_2, m_3 \geq M_W . \tag{4.60}$$

These give the bounds

$$\begin{aligned}
&M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right] \geq M_u \geq M_W\exp\left[\frac{\pi}{11}(\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1})\right],\\
&M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right] \geq m_1 \geq M_W\exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}]\right],\\
&M_W\exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right] \geq m_2 \geq M_W\exp\left[\frac{2\pi}{11}[\alpha^{-1}(1-3x)+\tfrac{1}{3}\alpha_3{}^{-1}]\right],\\
&M_W\exp\left[\frac{\pi}{11}(\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1})\right] \geq m_3 \geq M_W\exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x)+\tfrac{4}{3}\alpha_3{}^{-1}]\right]
\end{aligned} \tag{4.61}$$

with no new bound appearing. The proton-decay rate is bounded by

$$1 \lesssim \frac{\tau_p^{E_6,C3}}{\tau_p^{SU(5)}} \lesssim 8\times 10^9 \tilde{\Lambda}^{500/99} 10^{382\Delta x} . \tag{4.62}$$

For case 4 we have

$$\alpha^{-1}=\alpha_L{}^{-1}+\alpha_R{}^{-1}+\tfrac{1}{3}\alpha_{21}{}^{-1}+\tfrac{1}{3}\alpha_{33}{}^{-1} \tag{4.63}$$

and the constraints

$$\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1}=\frac{11}{3\pi}\ln\left[\frac{M_u{}^2 m_2 m_3}{m_1 M_W{}^3}\right], \tag{4.64}$$

$$\alpha^{-1}(3-8x)=\frac{11}{3\pi}\ln\left[\frac{m_1{}^9 m_2{}^3 m_3{}^3}{M_u{}^{10} M_W{}^5}\right] \tag{4.65}$$

or

$$\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}=\frac{11}{6\pi}\ln\left[\frac{m_1 m_2{}^2 m_3{}^2}{M_W{}^5}\right], \tag{4.66}$$

$$\alpha^{-1}x-\alpha_3{}^{-1}=\frac{11}{6\pi}\ln\left[\frac{M_u{}^4}{m_1{}^3 M_W}\right] \tag{4.67}$$

and

$$M_u \geq m_1 \geq m_2 \geq m_3 \geq M_W . \tag{4.68}$$

These give the same bounds as for case 3 except that

$$\begin{aligned}
&M_u \geq M_W\exp\left[\frac{3\pi}{110}[\alpha^{-1}(3+2x)-10\alpha_3{}^{-1}]\right],\\
&m_2 \geq M_W\exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x)-\tfrac{2}{3}\alpha_3{}^{-1}]\right],\\
&m_3 \leq M_W\exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x)-\tfrac{5}{3}\alpha_3{}^{-1}]\right],
\end{aligned} \tag{4.69}$$

while the limits on the nucleon-decay lifetime are

$$9\tilde{\Lambda}^{50/99}10^{38\Delta x} \lesssim \frac{\tau_p^{E_6,C4}}{\tau_p^{SU(5)}} \lesssim 8\times 10^9\tilde{\Lambda}^{500/99}10^{382\Delta x} . \tag{4.70}$$

Thus, in all four cases in class I, we find definite upper and lower bounds on all unification mass scales. This is not the case in the three breakdown chains in class II which we now examine.

In case 5 we have

$$\alpha^{-1}=\alpha_L{}^{-1}+\alpha_R{}^{-1}+\tfrac{1}{15}\alpha_{32}{}^{-1}+\tfrac{3}{5}\alpha_{51}{}^{-1} \tag{4.71}$$

and the constraints are

$$\alpha^{-1}-\tfrac{8}{3}\alpha_3{}^{-1}=\frac{11}{3\pi}\ln\left[\frac{M_u{}^2 m_1 m_3}{m_2 M_W{}^3}\right], \tag{4.72}$$

$$\alpha^{-1}(3-8x)=\frac{11}{3\pi}\ln\left[\frac{M_u{}^6 m_1{}^3 m_3{}^3}{m_2{}^7 M_W{}^5}\right], \tag{4.73}$$

along with the orderings

$$M_u \geq m_1 \geq m_2, \quad M_u \geq m_3 \, . \tag{4.74}$$

These lead to the bounds

$$M_u \geq m_1 \geq m_2 \geq M_W \exp\left[\frac{6\pi}{11}(\alpha^{-1}x - \alpha_3{}^{-1})\right], \tag{4.75}$$

$$M_W \exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x) + \tfrac{4}{3}\alpha_3{}^{-1}]\right] \geq m_3 \, . \tag{4.76}$$

The proton-decay lifetime satisfies

$$\frac{\tau_p^{E_6,C5}}{\tau_p^{SU(5)}} \gtrsim 8\times 10^9 \tilde{\Lambda}^{500/99} 10^{382\Delta x} \, . \tag{4.77}$$

In case 6 we find

$$\alpha^{-1} = \alpha_L{}^{-1} + \alpha_R{}^{-1} + \tfrac{2}{3}\alpha_{31}{}^{-1} \tag{4.78}$$

and the constraints

$$\alpha^{-1} - \tfrac{8}{3}\alpha_3{}^{-1} = \frac{11}{3\pi}\ln\left[\frac{M_u{}^2 m_3}{M_W{}^3}\right], \tag{4.79}$$

$$\alpha^{-1}(3-8x) = \frac{11}{3\pi}\ln\left[\frac{M_u{}^6 m_2{}^4 m_3{}^3}{M_W{}^5 m_1{}^8}\right] \tag{4.80}$$

or

$$\alpha^{-1}x - \alpha_3{}^{-1} = \frac{11}{6\pi}\ln\left[\frac{m_1{}^2}{M_W m_2}\right]. \tag{4.81}$$

These give

$$M_u \geq M_W \exp\left[\frac{\pi}{11}(\alpha^{-1} - \tfrac{8}{3}\alpha_3{}^{-1})\right] \geq m_3 \, , \tag{4.82}$$

$$M_W \exp\left[\frac{6\pi}{11}(\alpha^{-1}x - \alpha_3{}^{-1})\right] \geq m_1 \geq m_2 \tag{4.83}$$

(subject to $M_u \geq m_1 \geq m_2$) and

$$\frac{\tau_p^{E_6,C6}}{\tau_p^{SU(5)}} \gtrsim 1 \, . \tag{4.84}$$

For case 7 we have

$$\alpha^{-1} = \alpha_L{}^{-1} + \alpha_R{}^{-1} + \tfrac{1}{3}\alpha_{21}{}^{-1} + \tfrac{1}{3}\alpha_{33}{}^{-1} \, , \tag{4.85}$$

$$\alpha^{-1} - \tfrac{8}{3}\alpha_3{}^{-1} = \frac{11}{3\pi}\ln\left[\frac{M_u{}^2 m_2 m_3}{m_1 M_W{}^3}\right], \tag{4.86}$$

$$\alpha^{-1}(3-8x) = \frac{11}{3\pi}\ln\left[\frac{M_u{}^6 m_3{}^3}{m_1{}^3 m_2 M_W{}^5}\right], \tag{4.87}$$

and the same mass bounds and proton lifetime limits as in case 5.

The three cases of class II are similar to the breakdown through $[SU(3)]^3$ in not having any upper limits placed on M_u or lower limits placed on m_3 by the renormalization analysis. If we demand, however, that M_u lie below the Planck mass, we find that m_3 must be larger than 9×10^4 GeV in case 5 and 4×10^6 GeV in cases 6 and 7. In case 6 no lower bounds on m_1 or m_2 are imposed by the mass constraints, just the relation (4.81). Since the leptoquarks whose mass is generated at m_2 must be heavier than 100 TeV (from the limits on $K_L^0 \to \mu^+ e^-$ discussed in Sec. III), the scale m_1 is forced to be larger than 1.5×10^{11} GeV.

The three cases in class III to which we now turn are special cases. In case 8 the presence of the Georgi-Glashow SU(5) as an intermediate stage ensures that the canonical SU(5) scale M_x will appear with $M_u \geq m_1 \geq M_x$ and

$$\tau_p^{E_6,C8} \simeq \tau_p^{SU(5)} \, . \tag{4.88}$$

The SU(5) prediction for $\sin^2\theta_W$ is also obtained in this case. These results can also be obtained in case 9. In that case we have

$$\alpha^{-1} = \alpha_L{}^{-1} + \tfrac{4}{3}\alpha_{33}{}^{-1} + \tfrac{1}{3}\alpha_{21}{}^{-1} \tag{4.89}$$

and the constraints

$$\alpha^{-1} - \tfrac{8}{3}\alpha_3{}^{-1} = \frac{11}{3\pi}\ln\left[\frac{m_1{}^2 m_2}{M_W{}^3}\right], \tag{4.90}$$

$$\alpha^{-1}(3-8x) = \frac{11}{3\pi}\ln\left[\frac{m_1{}^6}{M_W{}^5 m_2}\right] \tag{4.91}$$

with the order $M_u \geq m_1 \geq m_2$ required. Equations (4.90) and (4.91) yield

$$m_1 = M_W \exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x) - \tfrac{2}{3}\alpha_3{}^{-1}]\right], \tag{4.92}$$

$$m_2 = M_W \exp\left[\frac{6\pi}{11}(\alpha^{-1}x - \alpha_3{}^{-1})\right]. \tag{4.93}$$

In order for (4.92) and (4.93) to be consistent with $m_1 \geq m_2$, we must have $x \leq \frac{1}{6} + (5\alpha_3{}^{-1})/(9\alpha^{-1}) \equiv x_5$. If we only allow $x \geq x_5$ (as will be discussed in Sec. V), then we must have $x = x_5$. The bounds m_1 and m_2 then reduce to M_x. Thus, both cases 8 and 9 reproduce the SU(5) predictions for M_x and $\sin^2\theta_W$ to one-loop order.

For case 10 we have

$$\alpha^{-1} = \alpha_L^{-1} + \tfrac{3}{2}\alpha_{42}^{-1} + \tfrac{1}{6}\alpha_{31}^{-1} , \tag{4.94}$$

and the constraints

$$\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1} = \frac{11}{3\pi}\ln\left[\frac{m_1^4}{M_W^3 m_2}\right] , \tag{4.95}$$

$$\alpha^{-1}(3-8x) = \frac{11}{3\pi}\ln\left[\frac{m_1^4 m_2}{M_W^5}\right] . \tag{4.96}$$

These give the results

$$m_1 = M_W \exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x) - \tfrac{2}{3}\alpha_3^{-1}]\right] , \tag{4.97}$$

$$m_2 = M_W \exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x) + \tfrac{4}{3}\alpha_3^{-1}]\right] . \tag{4.98}$$

Some of the gauge bosons whose masses are generated at m_1 can mediate proton decay in this scheme so we have

$$\frac{\tau_p^{E_6,C10}}{\tau_p^{SU(5)}} \simeq 8.7\times 10^{-6}\tilde{\Lambda}^{-250/99}10^{-190.8\Delta x} . \tag{4.99}$$

Such a short proton lifetime is already ruled out by experiment. This is the *only* case we find where the mass scale which characterizes proton decay is not automatically equal to or larger than the SU(5) scale M_x.

We find, then, that in all but one of the breakdown patterns that we consider, the proton lifetime is at least as long as in the simplest SU(5) model. There is an important distinction, however, between chains in which the proton lifetime is, in principle, bounded from above and those, such as in Sec. IV B and class II of Sec. IV C, where it is not. In the former case the upper limit on the lifetime invariably depends on x in such a way that if $x = x_5$ is used as an input, the proton lifetime reduces to the SU(5) value. In these cases the SU(5) prediction for $\sin^2\theta_W$ and an observable proton lifetime are inextricably related. We have in fact seen that all the mass bounds which appear in the breakdown chains of SO(10) and E_6 reduce to the unique SU(5) scale, M_x, when $x = x_5$. This regularity will be examined further in the next section.

V. GENERAL UNIFICATION MASS-SCALE FORMULA

We can now collect, in Table I, all of the unification scales that have appeared in the last three sections in order of decreasing magnitude. The labels n in the table reflect the fact that all of the mass scales encountered so far can be simply parametrized by the formula

$$M_n = M_W \exp\left\{\frac{6\pi}{11(6-n)}\left[\alpha^{-1}(1-nx) + \left[\frac{-8+3n}{3}\right]\alpha_3^{-1}\right]\right\} \tag{5.1}$$

$$= M_W \exp\left[\frac{\pi}{11}(\alpha^{-1} - \tfrac{8}{3}\alpha_3^{-1})\left[\frac{6}{6-n}\right] + \frac{6\pi}{11}(\alpha^{-1}x - \alpha_3^{-1})\left[\frac{-n}{6-n}\right]\right] \tag{5.2}$$

$$= M_0^{6/(6-n)} M_{-\infty}^{-n/(6-n)} . \tag{5.3}$$

Any mass of the form (5.1) can be seen to reduce to the canonical SU(5) M_x when the SU(5) one-loop prediction for $\sin^2\theta_W$,

$$x = \tfrac{1}{6} + (5\alpha_3^{-1})/(9\alpha^{-1}) ,$$

is used as an input. In fact, demanding that all unification mass scales be of the form of an exponential with argument linear in α^{-1}, $\alpha_2^{-1} = \alpha^{-1}x$, and α_3^{-1}, e.g.,

$$M = M_W \exp(a\alpha^{-1} + b\alpha^{-1}x + c\alpha_3^{-1}) , \tag{5.4}$$

and which also reduces to M_x when $x = x_5$, just gives Eq. (5.1) as the result.

TABLE I. Collected mass-scale formulas with numerical values using $\Lambda=200$ MeV and $\sin^2\theta_W=0.23$ along with corresponding values of general formula [Eq. (5.1)] parameter n.

M_n	M (GeV)	n
$M_W \exp\left[\frac{6\pi}{11}(\alpha^{-1}x-\alpha_3{}^{-1})\right]$	2.4×10^{17}	$-\infty$
$M_W \exp\left[\frac{3\pi}{110}[\alpha^{-1}(3+2x)-10\alpha_3{}^{-1}]\right]$	1.5×10^{15}	$-\frac{2}{3}$
$M_x=M_W \exp\left[\frac{\pi}{11}(\alpha^{-1}-\frac{8}{3}\alpha_3{}^{-1})\right]$	8.6×10^{14}	0
$M_W \exp\left[\frac{6\pi}{55}[\alpha^{-1}(1-x)-\frac{5}{3}\alpha_3{}^{-1}]\right]$	2.8×10^{14}	1
$M_W \exp\left[\frac{3\pi}{22}[\alpha^{-1}(1-2x)-\frac{2}{3}\alpha_3{}^{-1}]\right]$	5.1×10^{13}	2
$M_W \exp\left[\frac{2\pi}{11}[\alpha^{-1}(1-3x)+\frac{1}{3}\alpha_3{}^{-1}]\right]$	3.1×10^{12}	3
$M_W \exp\left[\frac{3\pi}{11}[\alpha^{-1}(1-4x)+\frac{4}{3}\alpha_3{}^{-1}]\right]$	1.1×10^{10}	4

With the particular inputs we have chosen, we have $M_j \geq M_k$ whenever $j \leq k$. We can ask what conditions are imposed on the inputs if this is to hold generally. Thus if $j \leq k$, we require that

$$\frac{6\pi}{11(6-j)}\left[\alpha^{-1}(1-jx)+\left[\frac{-8+3j}{3}\right]\alpha_3{}^{-1}\right]-\frac{6\pi}{11(6-k)}\left[\alpha^{-1}(1-kx)+\left[\frac{-8+3k}{3}\right]\alpha_3{}^{-1}\right]\geq 0\,, \quad (5.5)$$

$$\frac{(j-k)}{(6-j)(6-k)}\left[\alpha^{-1}(1-6x)+\frac{10\alpha_3{}^{-1}}{3}\right]\geq 0 \quad (5.6)$$

or (if $j,\,k<6$)

$$x \geq x^- = \frac{1}{6}+\frac{5}{9}\frac{\alpha_3{}^{-1}}{\alpha^{-1}} = x_5\ . \quad (5.7)$$

Further, demanding that each M_n be larger than the "starting point" scale M_W requires that

$$\alpha^{-1}(1-nx)+\left[\frac{-8+3n}{3}\right]\alpha_3{}^{-1}\geq 0 \quad (5.8)$$

or

$$x \leq x^+ = \frac{1}{n}+\left[\frac{-8+3n}{3n}\right]\frac{\alpha_3{}^{-1}}{\alpha^{-1}} \quad (5.9)$$

for $0<n<6$. For $n<0$ this constraint on x is a lower bound,

$$x \geq \frac{\alpha_3{}^{-1}}{\alpha^{-1}} \geq \frac{1}{n}+\left[\frac{-8+3n}{n}\right]\frac{\alpha_3{}^{-1}}{\alpha^{-1}}\,, \quad (5.10)$$

weaker than that of (5.7). For $n=0$ we have no constraint on x, while for $n=6$ we would demand that $x=x_5$ for M_6 to even be defined. In Table II we tabulate, using our nominal inputs, the constraints placed on x by (5.7) and (5.9) for various n.

We note that at $n\simeq 5$ we begin to saturate the upper bound, implying a mass scale at or below

TABLE II. Constraints on allowed input values of $x=\sin^2\theta_W$ from Eqs. (5.7) and (5.9).

n	x^+	x^-
$n \le 0$		0.204
1	0.887	0.204
2	0.477	0.204
3	0.341	0.204
4	0.273	0.204
5	0.232	0.204
6	0.204	0.204

M_W, when we use the currently accepted experimentally determined value of $\sin^2\theta_W$. Whether this has any connection to the fact that all the mass scales that we have found have $n \le 4$ and whether groups larger than E_6 might have scales corresponding to n's of 5 or 6 is uncertain. We remind the reader that the $SU_R(2)$ breakdown is often associated with M_4 and that restoration of parity is, in some sense, the simplest partial unification that could take place before an overall grand unification. We will present arguments in the next section which suggest, in fact, that the scale M_5 is most naturally associated with the SU(3) scale parameter Λ_3.

VI. UNIFICATION MASS SCALES AND THE Λ PARAMETERS

In this section we examine a somewhat unorthodox way of viewing the coupling-constant renormalization arguments presented above in the context of a simplified model of grand unification. We find simple relations between the grand unification mass scale and the natural mass scales of non-Abelian gauge theories, i.e., the scale parameters Λ. In this picture we also find that an intermediate mass scale, presumably related to a partial unification, arises as one step in a simple hierarchy when we use as input a specific value of $x \ne x_5$.

We begin by considering a pure unbroken gauge theory based on the usual product group $SU_c(3)\times SU_L(2)\times U(1)$; that is, we consider the real world but ignore all fermions and the Weinberg-Salam breakdown at $2M_W$. In this approximation the strong-coupling constant has the energy dependence

$$\alpha_3{}^{-1}(Q^2)=\frac{33}{12\pi}\ln\left[\frac{Q^2}{\Lambda_3{}^2}\right] \tag{6.1}$$

at the one-loop level, where we employ the usual strong scale parameter Λ_3. Without the contributions of fermions, the U(1) coupling constant is just that, constant, so that

$$\alpha_1{}^{-1}(Q^2)=\alpha_1{}^{-1}\ . \tag{6.2}$$

The running coupling of the weak SU(2) can then be written as

$$\alpha_2{}^{-1}(Q^2)=\frac{22}{12\pi}\ln\left[\frac{Q^2}{\Lambda_2{}^2}\right], \tag{6.3}$$

where we have defined an $SU_L(2)$ scale parameter Λ_2, the energy scale at which the weak SU(2) would become strong (if it were not broken, of course, at the Weinberg-Salam scale). Even though Λ_2 seems to have no physical significance in our world (we certainly do not see a weak confinement scale), it can be thought of as a remnant; the scale to which the $SU_L(2)$ is "pointing" at the time it is broken. We can define it through the relation (valid in this approximation where we neglect fermions)

$$\begin{aligned}\frac{22}{12\pi}\ln\left[\frac{Q^2}{\Lambda_2{}^2}\right]&=\alpha_2{}^{-1}(Q^2)\\ &=\alpha_2{}^{-1}(\mu^2)+\frac{22}{12\pi}\ln\left[\frac{Q^2}{\mu^2}\right]\\ &=\alpha^{-1}x+\frac{22}{12\pi}\ln\left[\frac{Q^2}{4M_W{}^2}\right]\end{aligned} \tag{6.4}$$

giving

$$\begin{aligned}\Lambda_2&=2M_W\exp\left[-\frac{3\pi}{11}\alpha^{-1}x\right]\\ &\simeq 10^{-8}-10^{-9}\ \text{GeV}\ ,\end{aligned} \tag{6.5}$$

where the range in Λ_2 is due to the uncertainty in $x=\sin^2\theta_W$. (Just how seriously we are to take numerical values in this approximation is uncertain.) Such a Λ would imply a weak confinement distance of $0.2-0.02\ \mu$m. That such small energy scales and large confinement radii might be interesting in the context of gauge theories has been examined previously in another context.[43]

The behavior of the running couplings in this approximation is illustrated in Fig. 4. Equating the SU(3) and SU(2) coupling at unification gives

$$\alpha_3{}^{-1}(4M_u{}^2)=\alpha_2{}^{-1}(4M_u{}^2) \tag{6.6}$$

or the simple relation

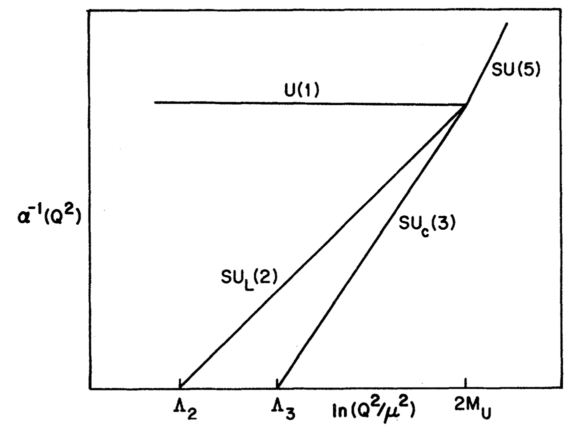

FIG. 4. Couplings for $SU(5)\rightarrow SU_c(3)\times SU_L(2)\times U(1)$ in the no-fermion, no-Weinberg-Salam-breakdown approximation considered in Sec. VI illustrating the scale parameters Λ_2, Λ_3.

$$2M_u=\frac{\Lambda_3{}^3}{\Lambda_2{}^2}\ , \tag{6.7}$$

a natural combination of the only two dimensional quantities in the pure SU(3)×SU(2)×U(1) gauge theory and the unification scale $2M_u$. Using $\Lambda_3=0.2$ GeV and Eq. (6.5) gives $M_u\simeq 4\times 10^{15-13}$ GeV.

Similarly, equating either the SU(3) or SU(2) coupling with the constant $\alpha_1{}^{-1}$ at unification and use of Eq. (6.7) gives

$$\frac{\Lambda_3}{\Lambda_2}=\exp\left[\frac{\pi}{11}\alpha_1{}^{-1}\right]\ , \tag{6.8}$$

so that there is a natural hierarchy between the two scale parameters, sensitively dependent on the value of the U(1) coupling.

We can relate the expressions (6.7) and (6.8) to the ones derived in Sec. II for the single unification scale in the SU(5) case by remembering that we had in that case

$$\alpha^{-1}(\mu^2)=\alpha_2{}^{-1}(\mu^2)+\tfrac{5}{3}\alpha_1{}^{-1}(\mu^2)\ , \tag{6.9}$$

where $\mu=2M_W$. Then Eq. (2.13) becomes, in our no-fermion approximation,

$$\begin{aligned}2M_x&=2M_W\exp\left[\frac{\pi}{11}[\alpha^{-1}(4M_W{}^2)-\tfrac{8}{3}\alpha_3{}^{-1}(4M_W{}^2)]\right]\\&=2M_W\exp\left[\frac{\pi}{11}\left\{\left[\frac{22}{12\pi}\ln\left[\frac{4M_W{}^2}{\Lambda_2{}^2}\right]+\tfrac{5}{3}\alpha_1{}^{-1}\right]-\frac{8}{3}\left[\frac{33}{12\pi}\ln\left[\frac{4M_W{}^2}{\Lambda_3{}^2}\right]\right]\right\}\right]\\&=\exp\left[\frac{5\pi}{33}\alpha_1{}^{-1}\right]\frac{\Lambda_3{}^{4/3}}{\Lambda_2^{1/3}}\ .\end{aligned} \tag{6.10}$$

Similarly, the equation for the renormalized value of $\sin^2\theta_W$, $x=\frac{1}{6}+(5\alpha_3{}^{-1})/(9\alpha^{-1})$, can be written as

$$\alpha_2{}^{-1}=\alpha^{-1}x=\frac{\alpha^{-1}}{6}+\tfrac{5}{9}\alpha_3{}^{-1} \tag{6.11}$$

or

$$\frac{\Lambda_3}{\Lambda_2}=\exp\left[\frac{\pi}{11}\alpha_1{}^{-1}\right]\ , \tag{6.12}$$

which is just Eq. (6.8). We see then that the specific hierarchy of Eq. (6.8) is equivalent to specifying the SU(5) $\sin^2\theta_W$. Combining this last relation with Eq. (6.10) then gives the simple combination in Eq. (6.7). We note that the dependence of the SU(5) predictions on the Weinberg-Salam scale, $2M_W$, disappears when viewed in this way.

We see that our assumption of an unbroken $SU_L(2)\times U(1)$ symmetry is not important in deriving the simple relations (6.7) and (6.8); they can be reproduced from the familiar renormalization equations of Sec. II even if Λ_2 is only defined through Eqs. (6.3) and (6.4). It is the simplifying approximation that no fermions are present that makes the relations between the gauge-group scale parameters and the unification mass almost obvious.

Thus, the two scale parameters Λ_2 and Λ_3 seem to define a *single* grand unification mass via (6.7) with the specific hierarchy obtained from Eq. (6.8). We can then ask how the formula for the general mass scale described in Sec. V appears in this simplified language. We find that

$$M_n=\left[\exp\left[\frac{5\pi}{33}\alpha_1{}^{-1}\right]\frac{\Lambda^{4/3}}{\Lambda_2{}^{1/3}}\right]^{6/(6-n)}\times\left[\frac{\Lambda_3{}^3}{\Lambda_2{}^2}\right]^{-n/(6-n)}\ . \tag{6.13}$$

If we assume the relation (6.8) [equivalent to using the canonical SU(5) x as an input] we have

$$M_n = \frac{\Lambda_3{}^3}{\Lambda_2{}^2} \tag{6.14}$$

and all the mass scales reduce to the SU(5) M_x as expected.

To reproduce the hierarchy of *different* masses in Table I we must use an x larger than x_5, i.e., we use as an independent input

$$x = \frac{1}{6} + \frac{5}{9}\frac{\alpha_3{}^{-1}}{\alpha^{-1}} + \epsilon \ , \tag{6.15}$$

so that in our language

$$\frac{\Lambda_3}{\Lambda_2} = \exp\left[\frac{\pi}{11}(\alpha_1{}^{-1} + \tfrac{18}{5}\alpha^{-1}\epsilon)\right] \tag{6.16}$$

or

$$\left[\frac{\Lambda_3}{\Lambda_2}\right]^{1-\delta} = \exp\left[\frac{\pi}{11}\alpha_1{}^{-1}\right] , \tag{6.17}$$

where

$$\delta = \left[\frac{18\pi\alpha^{-1}}{55\ln(\Lambda_3/\Lambda_2)}\right]\epsilon \ . \tag{6.18}$$

Then (6.13) becomes

$$M_n = \frac{\Lambda_3{}^3}{\Lambda_2{}^2}\left[\frac{\Lambda_3}{\Lambda_2}\right]^{-10\delta/(6-n)} . \tag{6.19}$$

The question then is if there are any especially plausible values of $\delta \neq 0$.

If we examine the ratios of the dimensional parameters $2M_u$, Λ_3, and Λ_2 at our disposal, we find that

$$2M_u : \Lambda_3 : \Lambda_2 = \frac{\Lambda_3{}^3}{\Lambda_2{}^2} : \Lambda_3 : \Lambda_2$$
$$= \frac{\Lambda_3{}^3}{\Lambda_2{}^3} : \frac{\Lambda_3}{\Lambda_2} : 1 \ . \tag{6.20}$$

It seems that another scale would naturally fit in the "desert" between Λ_3 and $2M_u$ at a value of Λ_3/Λ_2 times Λ_3. If we identify this scale with a partial unification, we can define

$$2M_{\rm PU} = \frac{\Lambda_3{}^2}{\Lambda_2} \ . \tag{6.21}$$

We can compare these four scales quantitatively with the ones derived in previous sections (keeping in mind that our assumption of no fermions possibly makes for a bad numerical approximation).

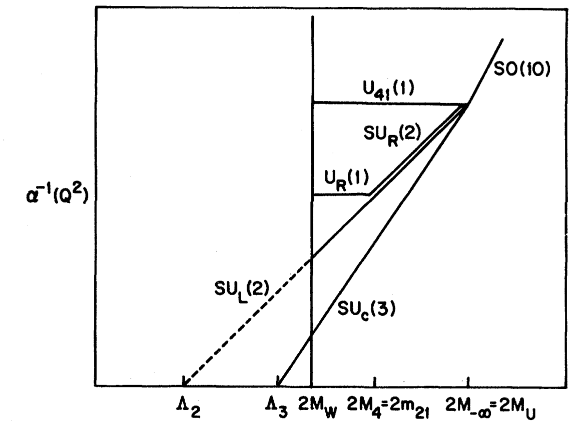

FIG. 5. Couplings in a possible SO(10) breakdown with the evenly spaced scales Λ_2, Λ_3, $\Lambda_3{}^2/\Lambda_2$, and $\Lambda_3{}^3/\Lambda_2{}^2$ superimposed (see Sec. VI).

If we select $\Lambda_3 = 0.2$ GeV and choose extreme limits for $2M_u$, 2×10^{15} to 4×10^{17} GeV, we find that

$$\begin{aligned} 2M_u &= (2\times10^{15}) - (4\times10^{17}) \text{ GeV} , \\ 2M_{\rm PU} &= (2\times10^{7}) - (3\times10^{8}) \text{ GeV} , \\ \Lambda_3 &= 0.2 - 0.2 \text{ GeV} , \\ \Lambda_2 &= (2\times10^{-9}) - (1\times10^{-10}) \text{ GeV} . \end{aligned} \tag{6.22}$$

We see that in order to have $\Lambda_3{}^2/\Lambda_2$ be large enough to be acceptable as a partial-unification scale, we must choose $2M_u$ to be at the high end of its allowed range as in one of the extreme cases in the left-right-symmetric breakdown of SO(10); we might then identify the scale $2M_{\rm PU}$ with the restoration of parity scale, $2m_{21}$. We illustrate the even spacing of these four scales in Fig. 5 superimposed on a possible breakdown pattern of SO(10). The parity breakdown scale is associated with M_4, while the overall grand unification scale is given by $M_{-\infty}$ in this case. Thus if we require that

$$\frac{\Lambda_3{}^2}{\Lambda_2} \equiv M_{\rm PU} = M_4 = \frac{\Lambda_3{}^3}{\Lambda_2{}^2}\left[\frac{\Lambda_3}{\Lambda_2}\right]^{-5\delta} , \tag{6.23}$$

we find that $\delta = \frac{1}{5}$ is dictated. With this δ the scale associated with $n=0$, the SU(5) scale, is now given by

$$M_x = M_0 = \Lambda_3\left[\frac{\Lambda_3}{\Lambda_2}\right]^{5/3} . \tag{6.24}$$

Furthermore, the scale corresponding to $n=5$, logically the next member of the sequence of mass scales (but not seen in any breakdown), is given by

$$M_5=\Lambda_3 \,. \tag{6.25}$$

Finally, we note that $M_{16/3}=\Lambda_2$ and list in Table III the Λ_2, Λ_3 dependence of all masses discussed so far when $\delta=\frac{1}{5}$.

The specific value of $\delta=\frac{1}{5}$ that yields these results corresponds to using an x larger than the SU(5) value by an amount

$$\epsilon=\left[\frac{55\ln(\Lambda_3/\Lambda_2)}{18\pi\alpha^{-1}}\right]\delta$$

$$\simeq 0.030-0.025 \tag{6.26}$$

so that $x\simeq 0.234-0.229$, consistent with the nominal value used previously.

We conclude that an SU(3)$\times$SU(2)$\times$U(1) gauge theory (broken or unbroken), in the limit where fermions can be neglected, exhibits a very simple relation between the two natural scales Λ_2, Λ_3 of the non-Abelian product groups and the overall grand unification mass. Moreover, the presence of an intermediate mass scale which can be associated with a partial unification can only arise, as before, if a value of $x\neq x_5$ is used as input. A specific value of x leading to $\delta=\frac{1}{5}$ allows for the possibility of an evenly spaced (in logarithm) hierarchy of scales with $\Lambda_2=M_{16/3}$, $\Lambda_3=M_5$, $\Lambda_3{}^2/\Lambda_2=M_4$, and $\Lambda_3{}^3/\Lambda_2{}^2=M_{-\infty}$.

TABLE III. Collected mass scales in terms of Λ_3, Λ_2, Eq. (6.19), with the choice $\delta=\frac{1}{5}$ (see Sec. VI).

M_n	n
$\Lambda_3(\Lambda_3/\Lambda_2)^2$	$-\infty$
$\Lambda_3(\Lambda_3/\Lambda_2)^{17/10}$	$-\frac{2}{3}$
$\Lambda_3(\Lambda_3/\Lambda_2)^{5/3}$	0
$\Lambda_3(\Lambda_3/\Lambda_2)^{8/5}$	1
$\Lambda_3(\Lambda_3/\Lambda_2)^{3/2}$	2
$\Lambda_3(\Lambda_3/\Lambda_2)^{4/3}$	3
$\Lambda_3(\Lambda_3/\Lambda_2)$	4
Λ_3	5
Λ_2	$\frac{16}{3}$

VII. CONCLUSIONS AND DISCUSSION

We have reviewed the one-loop coupling-constant renormalization analysis of Georgi, Quinn, and Weinberg and its predictions for the unique SU(5) unification mass scale M_x and $\sin^2\theta_W$. Applying the analysis to the allowed breakdown patterns of SO(10) and E_6, we have found explicit formulas for upper and lower bounds for many of the new mass scales appearing in these chains. In all cases but one, the mass scale characterizing nucleon decay is at least as large as M_x.

All of the bounds are found to belong to a single one-parameter family whose members reduce to M_x if the SU(5) prediction for $\sin^2\theta_W$, x_5, is used as input. We have noted the excellent agreement between the two-loop SU(5) $\sin^2\theta_W$ and the one-loop value extracted from experiment. If this success is taken to imply that the one-loop SU(5) value, Eq. (2.14), should always be used in the one-loop renormalization analysis of larger groups, then we lose any hierarchy of partial-unification scales and the "desert" of the minimal SU(5) GUT remains. In many, but not all, cases, setting $x=x_5$ also forces the proton lifetime to be essentially the same as its SU(5) predictions.

Seemingly, however, there is no compelling theoretical reason for choosing this specific value for GUT's larger than SU(5). A value of $\sin^2\theta_W$ larger than x_5 then forces at least one partial-unification scale to lie below the canonical M_x while allowing the scale characterizing proton decay to increase beyond the limit that could be detected with foreseeable technology.[44] It will be left then to experiment to confirm any expected or discover any suspected unification mass scales.

Accelerator measurements of electroweak physics in the $W^{\pm}$, Z^0 energy range and beyond will allow us to verify further one of the fundamental assumptions of the SU(5) GUT, the Weinberg-Salam model. Sensitive enough measurements can even test the radiative corrections necessary for a precise comparison with the predicted SU(5) $\sin^2\theta_W$. More structure than expected (extra Z's, W's, etc.), however, would be evidence for an extended electroweak symmetry [extra U(1)'s, SU(2)'s, etc.] which in turn may signal grand unification groups larger than SU(5).

Experiments to improve limits on rare processes (e.g., $K_L^0\to e^+\mu^-$) would further constrain any superweak force attributable to massive leptoquarks and extend the scale of a possible color-group enlargement, e.g., SU(4). Limits on neutrino masses can set bounds on the parity- restoration mass scale so that precise measurements of neutrino oscillations and forthcoming improvements of β- and muon-decay experiments will be important tests.

Of course, nucleon-decay experiments finding a lifetime near the predicted SU(5) value would be a stunning confirmation of something very like the simplest model. Measurements of sufficient precision to differentiate among models with the same grand unification scale will require accurate particle identification and high statistics and may only be possible in a third- or fourth-generation detector. Nonobservation of proton decay need not rule out grand unification, however, as we have seen that M_u can often be pushed two orders of magnitude beyond M_x giving a lifetime 10^8 times longer.

ACKNOWLEDGMENTS

We would like to thank our experimentalist colleagues at Minnesota for providing the initial stimulus for this work. One of us (J.L.R.) would like to thank Professor Z. Maki and Professor K. Igi for warm hospitality at the Research Institute for Fundamental Physics (Kyoto) and the University of Tokyo, where part of this work was performed with partial support from the Ministry of Education, Science, and Culture (Japan). Part of this work forms the Ph.D. thesis of one of us (R.R.) submitted to the University of Minnesota. This research was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Department of Energy under Contracts Nos. EY-76-C-02-1764 and DE-AC02-76ER00881.

---

*Present address: The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637.

[1]For a review of grand unified theories, see P. Langacker, Phys. Rep. 72C, 185 (1981).

[2]H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).

[3]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).

[4]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); Phys. Rev. D 8, 3633 (1973); 9, 980 (1974); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); Phys. Rep. 14C, 129 (1974).

[5]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[6]S. L. Glashow, Nucl. Phys. 22, 579 (1961); S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D 2, 1285 (1970).

[7]T. Goldman and D. A. Ross, Nucl. Phys. B171, 273 (1980).

[8]J. Ellis *et al.*, Nucl. Phys. B176, 61 (1980).

[9]W. J. Marciano and A. Sirlin, in *Proceedings of the Second Workshop on Grand Unification, Ann Arbor, Michigan, 1981,* edited by J. P. Leveille, L. R. Sulak, and D. G. Unger (Birkhauser, Boston, 1981), p. 151; C. H. Llewellyn Smith and J. F. Wheater, Phys. Lett. 105B, 486 (1981).

[10]L. Hall, Nucl. Phys. B178, 75 (1981).

[11]Paul B. Mackenzie and G. P. Lepage, Phys. Rev. Lett. 47, 1244 (1981).

[12]J. E. Kim, P. Langacker, M. Levine, and H. H. Williams, Rev. Mod. Phys. 53, 211 (1981).

[13]S. Dawson, J. S. Hagelin, and L. Hall, Phys. Rev. D 23, 2666 (1981).

[14]M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. 50, 721 (1978); A. Hacinliyan and C. Saclioglu, Phys. Rev. D 19, 2440 (1979).

[15]H. Georgi, Nucl. Phys. B156, 126 (1979); P. H. Frampton, Phys. Lett. 88B, 2991 (1979).

[16]H. Georgi, in *Particles and Fields—1974,* proceedings of the meeting of the Division of Particles and Fields of the American Physical Society, Williamsburg, Virginia, edited by C. E. Carlson (AIP, New York, 1974), p. 575; H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975).

[17]E. Witten, Phys. Lett. 91B, 81 (1980); M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity,* edited by D. Z. Freeman and P. van Nieuwenhuizen (North-Holland, Amsterdam, 1979), p. 315.

[18]R. N. Mohapatra and R. E. Marshak, Phys. Lett. 94B, 183 (1980).

[19]M. Yasué, Phys. Lett. 103B, 33 (1981); Phys. Rev. D 24, 1005 (1981); G. Lazarides, M. Magg, and Q. Shafi, Phys. Lett. 97B, 87 (1980).

[20]A. Masiero, Phys. Lett. 93B, 295 (1980).

[21]R. W. Robinett and J. L. Rosner, Phys. Rev. D 25, 3036 (1982).

[22]F. Wilczek and A. Zee, Phys. Lett. 88B, 311 (1979); Phys. Rev. D 25, 553 (1982); M. Yasué, Phys. Lett. 91B, 85 (1980); F. Mansouri, *ibid.* 100B, 25 (1981).

[23]H. Georgi and S. Weinberg, Phys. Rev. D 17, 275 (1978).

[24]T. Matsuki and N. Yamamoto, Report No. 81-0838, KEK-Tsukuba (unpublished); Y. Hara, T. Matsuki, and N. Yamamoto, Report No. UTHEP-89 (unpublished).

[25]F. Wilczek and A. Zee, Phys. Rev. Lett. 43, 1571 (1979); A. De Rújula, H. Georgi, and S. L. Glashow, *ibid.* 45, 413 (1980).

[26]S. Weinberg, Phys. Rev. Lett. 43, 1566 (1979).

[27]S. Weinberg, in *Proceedings of the Second Workshop on Grand Unification, Ann Arbor, Michigan, 1981,* edited by J. P. Leveille, L. Sulak, and D. G. Unger (Birkhauser, Boston, 1981), p. 297.

[28]R. E. Marshak and R. N. Mohapatra, Phys. Lett. 91B, 222 (1980).

[29]J. C. Pati and A. Salam, Phys. Rev. Lett. 31, 661 (1973); Phys. Rev. D 10, 275 (1974); R. N. Mohapatra

and J. C. Pati, *ibid.* 11, 566 (1975); 11, 2558 (1975); G. Senjanović and R. N. Mohapatra, *ibid.* 12, 1502 (1975); R. N. Mohapatra and D. P. Sidhu, *ibid.* 18, 856 (1978); D. P. Sidhu, *ibid.* 22, 1158 (1980).

[30]T. G. Rizzo and G. Senjanović, Phys. Rev. Lett. 46, 1315 (1981); Phys. Rev. D 24, 704 (1981); 25, 235 (1982); and Brookhaven National Laboratory Report No. BNL-29706 (unpublished).

[31]M. A. B. Bég, R. V. Budny, R. Mohapatra, and A. Sirlin, Phys. Rev. Lett. 38, 1252 (1977); B. Holstein and S. B. Treiman, Phys. Rev. D 16, 2369 (1977).

[32]G. Beall, M. Bander, and A. Soni, Phys. Rev. Lett. 48, 848 (1982).

[33]R. W. Robinett, Dissertation, University of Minnesota, 1981 (unpublished); S. Dimopoulos, S. Raby, and G. L. Kane, Michigan Report No. UMHE 80-22 (unpublished).

[34]H. Georgi and D. V. Nanopoulos, Nucl. Phys. B159, 16 (1979); Q. Shafi, M. Sondermann, and Ch. Wetterich, Phys. Lett. 92B, 304 (1980); T. Goldman and D. A. Ross, Nucl. Phys. B162, 102 (1980); T. Goldman, in *Proceedings of the First Workshop on Grand Unification,* Durham, New Hampshire, 1980, edited by P. H. Frampton, S. L. Glashow, and A. Yildiz (Math Sci Press, Brookline, Massachusetts, 1980), p. 91; Riazuddin and Fayyazuddin, Phys. Rev. D 24, 2490 (1981); M. Yasué, Prog. Theor. Phys. 65, 708 (1981); T. G. Rizzo and G. Senjanović, Phys. Rev. D 25, 235 (1982).

[35]M. Claudson, A. Yildiz, and P. M. Cox, Phys. Lett. 97B, 224 (1980); P. M. Cox, P. H. Frampton, and A. Yildiz, Phys. Rev. Lett. 46, 1051 (1981).

[36]S. Rajpoot and P. Sithikang, Phys. Rev. D 23, 1649 (1981).

[37]F. Gürsey and P. Sikivie, Phys. Rev. Lett. 36, 775 (1976); Phys. Rev. D 16, 816 (1977); F. Gürsey and M. Serdaroğlu, Lett. Nuovo Cimento 21, 28 (1978); Y. Achiman and B. Stech, Phys. Lett. 77B, 389 (1978); Q. Shafi, *ibid.* 79B, 301 (1979); F. Gürsey and M. Serdaroğlu, Nuovo Cimento 65A, 337 (1981).

[38]V. Barger and S. Pakvasa, Phys. Lett. 81B, 195 (1979); G. Branco and H. P. Nilles, Nucl. Phys. B151, 529 (1979); G. L. Kane and M. Peskin, *ibid.* B195, 29 (1982); M. Gorn and E. A. Paschos, Phys. Lett. 101B, 255 (1981).

[39]C. Bebek *et al.*, Phys. Rev. Lett. 46, 84 (1981); K. Chadwick *et al.*, *ibid.* 46, 88 (1981).

[40]R. Barbieri and D. V. Nanopoulos, Phys. Lett. 91B, 369 (1980); B. Stech, in *Unification of the Fundamental Particle Interactions,* proceedings of the Europhysics Study Conference, Erice, Italy, 1980, edited by S. Ferrara, J. Ellis, and P. van Nieuwenhuizen (Plenum, New York, 1980), p. 23.

[41]J. A. Harvey, Nucl. Phys. B163, 254 (1980).

[42]R. Slansky, Phys. Rep. 79C, 1 (1981).

[43]L. B. Okun, Zh. Eksp. Teor. Fiz. 31, 156 (1980) [JETP Lett. 31, 144 (1980)]; Nucl. Phys. B173, 1 (1980).

[44]J. G. Learned, in *Proceedings of the Second Workshop on Grand Unification, Ann Arbor, Michigan, 1981,* edited by J. P. Leveille, L. R. Sulak, and D. G. Unger (Birkhauser, Boston, 1981), p. 125.