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Calculation of the stopping power of very-low-velocity magnetic monopoles

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We calculate the electronic stopping power of slow magnetic monopoles in condensed materials and find that, for $\beta \lesssim 0.01$, dE/dx is proportional to the monopole velocity. For monopoles with $g = \pm 137e$ it is found that the monopole stopping power is at least as large as that for a proton with the same velocity. The results presented here are not directly relevant to the evaluation of the response of excitation- and ionization-sensitive particle detectors to slow, supermassive monopoles. However, it is hoped that the techniques presented will be a useful guide in thinking about such problems.

INTRODUCTION

The predictions of the primordial production of supermassive magnetic monopoles ($Mc^2 \sim 10^{16}$ GeV) by grand unification theories (GUT's),[1] and of the subsequent acceleration of these monopoles to velocities $V = \beta c$ with $\beta \sim 10^{-3} - 10^{-2}$, have caused considerable interest in the nature of the interaction of such monopoles with matter.[2] As is well known from the investigation of the interaction of energetic electrically charged particles with matter, it is much simpler to calculate the stopping power of particles with $\beta \gg \alpha$ (the fine-structure constant) than with $\beta \ll \alpha$. There are two basic reasons for this and both apply to magnetic monopoles as well as electric charges.

(1) The first reason is that at large velocities a projectile is capable of interacting with electrons at large impact parameters. This is due to the fact that as long as the impact parameter b is less than the adiabatic impact parameter $b_{ad} = \gamma V/\omega_0$ (γ is the projectile's Lorentz factor, V is its velocity, and ω_0 is a characteristic electron frequency) the electron can receive energy from the projectile. For larger impact parameters the electron is perturbed adiabatically and is, therefore, not efficiently excited. For dielectrics or gases, $\hbar\omega_0$ is a typical atomic energy and $b_{ad} \sim 137\beta\gamma a_0$ where a_0 is the Bohr radius. For $\beta \gtrsim 0.1$, then, a projectile in a condensed medium can interact with cylinders of material which contain 15 or more atoms across their diameter. This enables one to apply the dipole approximation to a large fraction of interacting atoms or equivalently to apply standard techniques of classical macroscopic electrodynamics in which electric fields are defined as averages over microscopic volumes of material which are composed of many atoms.

(2) The second reason is that for sufficiently large velocities, i.e., $\beta \gg \alpha$, the kinematically limited energy transfer from a heavy projectile to a free electron initially at rest ($2mc^2\beta^2\gamma^2$ where m is the electron mass) is much larger than atomic-electron energies $\sim m\alpha^2c^2$. Thus, for very close, and therefore very violent, collisions between the projectile and the atomic electrons, one can neglect the binding of the electrons to atomic nuclei and regard them as free particles. The impact parameter above which this approximation breaks down is given roughly by $b_0 \sim 1/k_0$ where k_0 is the wave vector of an atomic electron ($k_0 \sim 1/a_0$) so that $b_0 \sim a_0$. For the vast majority of collisions in the high-velocity limit, either the free-electron approximation ($b < a_0$) or the classical macroscopic (i.e., dipole) approximation ($a_0 < b < b_{ad}$) is valid.

Bohr,[3] Bethe,[4] and Bloch[5] used these properties of high-velocity charged-particle interactions with matter to perform their classic stopping-power calculations which are reviewed elsewhere by Fano[6] and by Ahlen.[7] For $\beta > 0.1$ and for nuclear projectiles with atomic numbers from 1 to 26 the theory of Bethe, as modified by well understood corrections (see Ahlen[8] for a summary of these), is capable of predicting particle stopping power (energy loss per unit length) to better than 1%. Ahlen[9] has applied similar techniques to evaluate a reliable expression for the stopping power of magnetic monopoles with $\beta > 0.1$.

For projectile velocities $< 0.1c$ the above simplifications fail. With the exception of conducting absorbers, the instantaneous interaction of the projectile is essentially limited to the atom through which it passes. Furthermore, the average energy transferred to atomic electrons is so small that it is impossible to disregard their binding to matter. Since energy transfers to atoms can exceed the

kinematic limit to free electrons, the stopping power at low velocities is enhanced over that which one might expect. In addition, atomic electrons are actually not at rest, so it is kinematically possible for them to receive more energy than the limit for stationary electron targets. Thus, for example, K-shell ionization is standardly observed[10] for proton projectile velocities below that for which $2mV^2\gamma^2$ equals the K absorption edge.

To this date, there is no satisfactory theory which covers the stopping power of protons for $0.01<\beta<0.05$. For nuclei heavier than protons the problem is even more difficult due to the complications of electron capture and loss. For $\beta>0.05$, the Bethe theory accounts well for proton stopping power when augmented by the shell corrections, which take into account the fact that the excitation of inner-shell atomic electrons is strongly modified due to the large velocity of these electrons (the standard Bethe theory assumes that projectile velocities are much larger than electron velocities). Heavier ions can be handled at larger velocities when the capture and loss effects cease to dominate the interaction between the projectile and the stopping medium. For $\beta<0.01$, Lindhard[11] has been quite successful in accounting for the features of the electronic stopping power of charged particles. (Electronic stopping refers to the energy lost in collisions with electrons within the medium which lead to either atomic excitation or ionization. This is to be opposed to nuclear stopping, where energy is lost in collisions with the nuclei of the medium; for proton energies less than several hundred electron volts, nuclear stopping dominates while the opposite is true at higher energies.) He has done so by using a model in which the properties of the absorbing material are simulated by a degenerate Fermi gas of noninteracting electrons. Such a model accounts for many of the features of conduction electrons in metals, and the Thomas-Fermi approximation has proven successful for the description of static properties of large atoms. It is not surprising, therefore, that this approximation is also successful in describing dynamic problems such as those involving the stopping of charged particles. An appealing feature of Lindhard's technique is that it is formally quite similar to that used for high-velocity calculations such as that performed by Landau.[12] The principal difference is in the use of different dielectric permeabilities. For large velocities, the distant collisions transfer energy from the projectile to impact parameters large compared to interatomic spacing. Thus a model in which the stopping material is composed of discrete, bound atoms is appropriate and the familiar frequency-dependent dielectric constant $\epsilon(\omega)$ is adequate to describe the response of the medium. Static screening in dielectrics is reflected by the reduction of the electric potential by the factor $1/\epsilon(0)$. Thus, a point charge q appears many atoms away as a charge $q/\epsilon(0)$. On the other hand, the dielectric constant clearly cannot be used for low velocities in which the projectile effectively sees the cloud of electrons of a single atom at a time. Since this cloud has many of the properties of a Fermi gas, the screening of the point charge q should be characterized by the potential $(qe^{-\lambda r})/r$. Lindhard[11] has shown how the correct screening can be accounted for in a natural way by including spatial as well as temporal variations of material response in the macroscopic Maxwell equations. This is achieved by allowing $\epsilon=\epsilon(k,\omega)$ to depend on the wavelength as well as the frequency of the excitation. Ziman[13] has emphasized the importance of such a description for a wide variety of physical phenomena. For the case of static screening one used the result that $\epsilon(k,0)\simeq 1+\lambda^2/k^2$ for small k. Thus, since the Fourier transform of the potential q/r is $q/2\pi^2k^2$, one finds that the electric potential in k space in a degenerate Fermi gas is $q/2\pi^2(k^2+\lambda^2)$ which is the transform of the spatial field $qe^{-\lambda r}/r$, as required.

In this paper we will be applying Lindhard's technique to the slowing of monopoles. We regard the success of the technique in describing measured proton stopping power as evidence supporting our calculations for monopoles. However, due to the somewhat unfamiliar nature of Lindhard's calculations we will derive our stopping-power result in several different ways, each of which illuminates a separate feature of the problem. We should emphasize at this point that our results are not directly applicable to problems relating to the electronic excitation and ionization of particle detectors, although the techniques employed here should be useful as a guide in thinking about such problems, We begin by considering a heuristic calculation of monopole stopping power in conductors.

SIMPLE TREATMENT OF MONOPOLE STOPPING POWER IN CONDUCTORS

Consider a cylindrical coordinate system (ρ,ϕ,z) in which the monopole, assumed to be infinitely

massive, travels on the z axis with velocity V. We divide the (infinite) conducting medium into volume elements, each consisting of a loop at constant ρ and z with cross section $dz\,d\rho$. At the moment the monopole is at $z=0$, the electric field it produces in each "wire" is determined by the Biot-Savart law for magnetic charges (we neglect screening; this would be invalid in superconductors but it is a good approximation for ordinary conductors):

$$E=gV\rho/cR^3\,, \tag{1}$$

where g is the monopole charge and $R^2=z^2+\rho^2$. The field is parallel to the loop and thus has a nonvanishing line integral around it so that a current will flow. From Ohm's Law, the energy dissipated by this current flow is given by

$$d\epsilon=\sigma E^2 2\pi\rho\,d\rho\,dz\,dt\,, \tag{2}$$

where $2\pi\rho E$ is the voltage around the wire and $\sigma E\,d\rho\,dz$ is the current in the wire (σ is the conductivity). The quantity $d\epsilon$ is the energy dissipated in the time dt. If we define monopole stopping power S as the total energy dissipated per unit distance traveled by the monopole, then

$$S=\frac{1}{V}\int\frac{d\epsilon}{dt\,d\rho\,dz}d\rho\,dz$$
$$=\frac{1}{V}\int\sigma E^2 2\pi\rho\,d\rho\,dz\,. \tag{3}$$

The integral over z can be carried out to yield

$$S=\frac{3\pi^2 g^2 V}{4c^2}\int_0^\infty\sigma\,d\rho/\rho^2\,. \tag{4}$$

One might be tempted to assume that $\sigma=\sigma(\omega=0)$ is the standard zero-frequency conductivity and that a lower limit on the impact parameter of $\sim a_0$ should be imposed. This choice might be thought to be justified on the basis of the discrete nature of matter at this scale. The resulting stopping power would be

$$S=\frac{3\pi^2 g^2 V\sigma(0)}{4c^2 a_0}\,. \tag{5}$$

This is somewhat unsatisfactory in that we have used a cutoff impact parameter in such a way that the stopping power is quite sensitive to the choice of its value. One is generally better off and more nearly correct if the cutoff appears in a logarithm. *Equation (5)* is in fact *incorrect* and we can understand why if we consider the properties of electrical conductivities in real metals.

In the Drude theory of conductivity[13] σ is expressed as a function of ω in the form

$$\sigma(\omega)=\sigma(0)/(1+\omega^2\tau^2)\,, \tag{6}$$

where τ is the mean time for the collisions of conduction electrons with positive ions in the conductor. Since the properties of conductivity are determined by those electrons at the surface of the Fermi sphere, the collision mean free path Λ is related to τ via

$$\Lambda=\tau v_F\,, \tag{7}$$

where v_F is the Fermi velocity of the conduction electrons. It is shown elsewhere[13] that $\sigma(0)=\omega_p^2\tau/4\pi$ where $\omega_p^2=4\pi Ne^2/m$ is the square of the plasma frequency of the conduction electrons. Ziman also shows[13] that an approximate expression for Λ is

$$\Lambda\approx 50aT_m/T\,, \tag{8}$$

where a is the lattice constant, T_m is the melting temperature of the metal, and T is the actual temperature. This expression is claimed to be valid within an order of magnitude.

To apply these results to our theory we would need to relate impact parameter to frequency. It is natural to set ω equal to the inverse of the "collision time" ρ/V so that $\sigma(\rho)=\sigma(0)/(1+V^2\tau^2/\rho^2)$. Equation (5) would then be valid in the limit $V^2\tau^2\ll a_0^2$ or, equivalently, when V/v_F is small compared to the ratio a_0/Λ which is in turn less than 1. However, by using Eq. (6) we are making a serious error. This is due to the fact that Eq. (6) is derived under the assumption[13] that the Fermi velocity is much less than the ratio ω/k where k is the wave vector of the excitation. For the present case k is the inverse of the characteristic length over which fields change appreciably so that $k\sim 1/\rho$. Thus, $\omega/k\sim V$, which we have just seen must be smaller than the Fermi velocity in order for Eq. (5) to apply. Since the opposite limit must apply for Eq. (6) to be valid, we see that greater care must be applied to obtain the correct stopping power. This is where we can take advantage of the work done by Lindhard.[11] He analyzed the properties of electron gases and found that they could be described by Maxwell's equations:

$$\vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{H}=\frac{1}{c}\frac{\partial\vec{D}}{\partial t}+\frac{4\pi}{c}\vec{J}_0\,,$$
$$\vec{\nabla}\cdot\vec{D}=4\pi\rho_0,\quad \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial B}{\partial t}\,, \tag{9}$$

where ρ_0 and $\vec{J}_0$ are *extraneous* charge and current densities and where the Fourier transforms of the fields are related by

$$\vec{B}(\vec{k},\omega)=\mu(k,\omega)\vec{H}(\vec{k},\omega)\ ,$$
$$\vec{D}(\vec{k},\omega)=\epsilon(k,\omega)\vec{E}(\vec{k},\omega)\ . \tag{10}$$

The conductivity is included in the definition of the dielectric constant[11]:

$$\sigma^{\mathrm{tr},l}=\frac{\omega}{4\pi}\,\mathrm{Im}[\epsilon^{\mathrm{tr},l}(k,\omega)-1]\ , \tag{11}$$

where tr (l) refers to the component of the field transverse (longitudinal) to the $\vec{k}$ vector. In the formulation where one uses the magnetic permeability, ϵ,μ and ϵ^l, ϵ^{tr} are related by[11]

$$\epsilon(k,\omega)=\epsilon^l(k,\omega)\ ,$$
$$k^2\left[1-\frac{1}{\mu(k,\omega)}\right]=\frac{\omega^2}{c^2}[\epsilon^{\mathrm{tr}}(k,\omega)-\epsilon^l(k,\omega)]\ . \tag{12}$$

Lindhard gives several results for ϵ and μ depending on the physical situation. For the semiclassical approximation (which is essentially the same used in the Thomas-Fermi description of atoms) one assumes the de Broglie wavelength of electrons in a degenerate Fermi gas to be small compared to other relevant scales. For this case[11]

$$\epsilon^{\mathrm{tr}}(k,\omega)=1+\frac{3}{4}\frac{\omega_p^2}{\omega}\left[\frac{-2\omega'}{(v_F^2k^2)}+\left[\frac{1-\omega'^2/v_F^2k^2}{v_Fk}\right]\ln\left[\frac{\omega'-v_Fk}{\omega'+v_Fk}\right]\right]\ , \tag{13a}$$

where $\omega'=\omega+i/\tau$ and

$$\epsilon^l(k,\omega)=1+\frac{3\omega_p^2}{v_F^2k^2}\left[1+\frac{\omega'}{2kv_F}\ln\left[\frac{\omega'-v_Fk}{\omega'+v_Fk}\right]\right]\ . \tag{13b}$$

In the above, the magnetic moment of the electrons is assumed to be zero even though the spin is taken into account insofar as it relates to the Fermi-Dirac statistics. If one takes the dynamical effects of spin into account, an additional component of the transverse permeability is obtained[11]:

$$\delta\epsilon^{\mathrm{tr}}=-\frac{k^2p_F^2c^2\mu_B^2}{\pi^2\omega^2\hbar^3v_F}\left[1+\frac{\omega'}{2kv_F}\ln\left[\frac{\omega'-kv_F}{\omega'+kv_F}\right]\right]\ , \tag{13c}$$

where $p_F=mv_F$ and $\mu_B=e\hbar/2mc$ is the Bohr magneton.

When the quantum-mechanical effects of the finite wavelength of the electrons are taken into account, the permeabilities for a degenerate Fermi gas become (excluding dynamical spin effects)[11]

$$\epsilon^l=1+\frac{3\omega_p^2}{k^2v_F^2}f\ , \tag{14a}$$

where

$$f=\tfrac{1}{2}+\frac{1}{8z}[1-(z-u')^2]\ln\left[\frac{z-u'+1}{z-u'-1}\right]+\frac{1}{8z}[1-(z+u')^2]\ln\left[\frac{z+u'+1}{z+u'-1}\right] \tag{14b}$$

with the logarithms being principal values and where $u'=\omega'/kv_F$ and $z=k/2k_F$. Lindhard also gives[11]

$$\epsilon^{\mathrm{tr}}=1-\frac{\omega_p^2}{\omega^2}f^{\mathrm{tr}} \tag{14c}$$

with[11]

$$f^{\mathrm{tr}}=\tfrac{3}{8}(z^2+3u'^2+1)-\frac{3}{32z}\left[[1-(z-u')^2]^2\ln\left[\frac{z-u'+1}{z-u'-1}\right]+[1-(z+u')^2]^2\ln\left[\frac{z+u'+1}{z+u'-1}\right]\right]\ . \tag{14d}$$

For the present simplified calculation in which monopole stopping power is equated to conductive losses, we will neglect dynamical spin effects. Furthermore, we will be interested only in the transverse conductivity since $\vec{\nabla}\cdot\vec{D}=0\Rightarrow\vec{k}\cdot\vec{E}(\vec{k},\omega)=0$ (this applies both to monopole stopping and to optical response of metals for which the Drude theory is intended). Thus,

$$\sigma=\frac{\omega}{4\pi}\,\mathrm{Im}(\epsilon^{\mathrm{tr}})\ . \tag{15}$$

We first show that the Drude result [Eq. (6)], is a consequence of Eqs. (15) and (13a) in the limit $k \rightarrow 0$. Holding ω constant, we find for small k that the semiclassical approximation (which should definitely be valid for the optical case since the electromagnetic wavelengths are much larger than the Fermi wavelength) is

$$\epsilon^{\mathrm{tr}} \approx 1+\frac{3}{4}\frac{\omega_p^{\,2}}{\omega}\left[-\frac{2\omega'}{v_F^{\,2}k^2}+\frac{1}{v_F k}\left[1-\frac{\omega'^2}{v_F^{\,2}k^2}\right]\left[-2\frac{v_F k}{\omega'}-\frac{2}{3}\frac{v_F^{\,3}k^3}{\omega'^3}\right]\right] \tag{16}$$

or

$$\epsilon^{\mathrm{tr}} \approx 1-\frac{\omega_p^{\,2}}{\omega\omega'}=1-\frac{\omega_p^{\,2}\tau^2}{1+\omega^2\tau^2}+i\frac{\omega_p^{\,2}\tau}{\omega(1+\omega^2\tau^2)}$$

and

$$\sigma=\frac{\omega_p^{\,2}\tau}{4\pi}/1+\omega^2\tau^2$$

which is identical to Eq. (6).

We now take the opposite limit, i.e., $kv_F \gg \omega$ which corresponds to the stopping of low-velocity monopoles. For the moment we neglect damping which implies that $kv_F \gg 1/\tau$ or, in terms of our present calculation, $\rho \ll v_F\tau$. In this case, it is easy to show that

$$\sigma(k,\omega) \approx \frac{3\omega_p^{\,2}}{16 v_F k} . \tag{17}$$

By replacing k with $1/\rho$ and inserting the result in Eq. (4) we find that

$$S=\frac{9\pi^2}{64}\frac{\omega_p^{\,2}g^2V}{c^2v_F}\ln(\rho_{\max}/\rho_{\min}) \tag{18}$$

and $\rho_{\max}$ and $\rho_{\min}$ are logarithmically divergent cutoffs which prevent S from being infinite. It is easy to see where $\rho_{\max}$ originates. When $\rho \gtrsim v_F\tau$ our neglect of damping (i.e., of the i/τ term) is incorrect. If one assumes that $\omega=0$ and that $kv_F \ll 1/\tau$, then it is found that $\sigma=\omega_p^{\,2}\tau/(4\pi)$ $=\sigma(0)$. Thus, for $\rho \gg v_F\tau$, σ is constant and the integral for S decreases as $1/\rho^2$, so that the integral converges. It is clear then that

$$\rho_{\max}=v_F\tau=\Lambda . \tag{19}$$

Since the semiclassical approximation does not adequately consider the wave nature of the conduction electrons, it cannot yield the correct minimum value for ρ. However, it is easy to obtain $\rho_{\min}$ in an *ad hoc* argument reminiscent of semiclassical treatments of relativistic stopping power. For conduction electrons confined in a wire of radius ρ, the angular momentum relative to the monopole is $\sim\rho m v_F$ where we use the fact that $v_F \gg V$. Since angular momentum is quantized in units of $\hbar$, the smallest wire with a nonzero area element corresponds to a radius

$$\rho_{\min}=\hbar/mv_F=1/k_F . \tag{20}$$

This is the same order of magnitude as our previous choice. However, the present interpretation is to be preferred in that we are not saying that the technique fails at some scale size. In fact, Lindhard's fully quantal permeabilities should be valid at all scales and we will see that $\rho_{\min}$ is proven to be $\sim 1/k_F$ in this theory. To conclude this section, we write our result for monopole stopping power in conductors obtained in our model:

$$S=\lambda\frac{\omega_p^{\,2}g^2V}{c^2v_F}\ln(\Lambda k_F) , \tag{21}$$

where λ is a number of order unity.

STOPPING-POWER CALCULATION USING MAXWELL'S EQUATIONS

We now consider monopole stopping power from a more rigorous point of view. Namely, we calculate the energy loss by evaluating the force acting on the monopole due to the fields induced in the material. In this way we do not have to introduce the dependence on wavelength and frequency in the artificial manner above. Furthermore, screening effects are taken into account within a rigorous framework. Although this will turn out not to be important for the case of monopole stopping power it is apparent that it would be absolutely necessary for the energy loss of electric particles.

To begin, we will not specify the nature of the material through which the monopole passes. We assume, however, that the velocity is so small that we can neglect all time derivatives in Maxwell's equations. Nevertheless, it is important to first write down the symmetric Maxwell equations[14] to clarify the roles of $\vec{B}$ and $\vec{H}$ in monopole stopping power:

$$\vec{\nabla}\cdot\vec{B}=4\pi\rho_m,\quad \vec{\nabla}\times\vec{H}=\frac{1}{c}\frac{\partial\vec{D}}{\partial t}+\frac{4\pi}{c}\vec{J}_e\ ,$$
$$\vec{\nabla}\cdot\vec{D}=4\pi\rho_e,\quad \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}-\frac{4\pi}{c}\vec{J}_m\ , \tag{22}$$

where ρ_m and $\vec{J}_m$ are extraneous magnetic monopole charge and current densities and ρ_e and $\vec{J}_e$ are the analogous electric monopole densities. Note that the relation $\vec{J}_e(\text{induced})=\sigma\vec{E}$ is contained within the definition of the dielectric permeability which relates $\vec{D}$ and $\vec{E}$ via Eq. (10).

It is straightforward to show from Eq. (22) that

$$\vec{\nabla}\cdot\vec{S}+\frac{1}{4\pi}\left[\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right]=-\vec{J}_m\cdot\vec{H}-\vec{J}_e\cdot E\ , \tag{23}$$

where $\vec{S}=(c/4\pi)(\vec{E}\times\vec{H})$ is the Poynting vector. The obvious interpretation of this is that it is the $\vec{H}$ and not the $\vec{B}$ field that does work on the monopole. We will see that if it were the other way around, the monopole stopping power would be strictly zero at low velocities, in contradiction to the results of the previous section.

To proceed, let $\rho_m=g\delta(\vec{r}-\vec{V}t)$. Then

$$\vec{\nabla}\cdot\vec{B}=4\pi g\delta(\vec{r}-\vec{V}t)\ . \tag{24}$$

In terms of Fourier components this equation reads

$$i\vec{k}\cdot\vec{B}(\vec{k},\omega)=\frac{g}{2\pi^2}\delta(\omega-\vec{k}\cdot\vec{V})\ . \tag{25}$$

In the limit of very small velocities we neglect time derivatives in Eq. (22). [This can be shown to be legitimate by using the full Eq. (22) to derive an expression for monopole stopping power; if this is done it is found that as long as $V<v_F$ it is legitimate to neglect the time derivatives.] Since $\vec{\nabla}\times\vec{H}=0$ for the low-velocity approximation ($\vec{J}_e=0$ in either dielectrics or conductors), we can write $\vec{H}=-\vec{\nabla}\phi$ so that in $\vec{k},\omega$ space

$$\vec{H}(\vec{k},\omega)=-i\vec{k}\phi(\vec{k},\omega)\ . \tag{26}$$

Since $\vec{B}(\vec{k},\omega)=\mu(k,\omega)\vec{H}(\vec{k},\omega)$ we have

$$\phi(\vec{k},\omega)=\frac{g}{2\pi^2k^2\mu(k,\omega)}\delta(\omega-\vec{k}\cdot\vec{V})\ . \tag{27}$$

The stopping power of a monopole is merely the force exerted on it by the medium, which is given by $g\vec{H}\cdot\vec{V}/V$ (by symmetry the electric force on the monopole must vanish). Therefore, we have

$$S=-i\frac{g^2}{2\pi^2}\int d^3k\,d\omega\frac{\delta(\omega-\vec{k}\cdot\vec{V})}{k^2\mu(k,\omega)}\frac{\vec{k}\cdot\vec{V}}{V}\ . \tag{28}$$

Since the stopping power is real we obtain

$$S=-\frac{g^2}{\pi V^2}\int_0^\infty\frac{dk}{k}\int_{-kV}^{kV}\mathrm{Im}\left[\frac{1}{\mu(k,\omega)}\right]\omega\,d\omega\ . \tag{29}$$

(We use the minus sign here since we will interpret stopping power as positive.)

We now restrict ourselves to the case of a degenerate Fermi gas of electrons. For the moment we neglect dissipation (i.e., $\tau=\infty$) and consider Eqs. (14a) and (14b), which describe such a situation in a fully quantal framework. Separating f into real and imaginary components $f=f_1+if_2$, Lindhard[11] shows that

$$f_1(u,z)=\frac{1}{2}+\frac{1}{8z}[1-(z-u)^2]\ln\left|\frac{z-u+1}{z-u-1}\right|+\frac{1}{8z}[1-(z+u)^2]\ln\left|\frac{z+u+1}{z+u-1}\right| \tag{30a}$$

and

$$f_2(u,z)=\begin{cases}\dfrac{\pi}{2}u\ , & z+u<1\ ,\\ \dfrac{\pi}{8z}[1-(z-u)^2]\ , & |z-u|<1<z+u\ ,\\ 0\ , & |z-u|>1\ ,\end{cases} \tag{30b}$$

where $z=k/2k_F$ and $u=\omega/kv_F$. Similarly for ϵ^{tr} we have Eqs. (14c) and (14d) with $f^{\mathrm{tr}}=f_1^{\mathrm{tr}}-if_2^{\mathrm{tr}}$:

$$f_1^{\mathrm{tr}}(u,z)=\tfrac{3}{8}(z^2+u^2+1)-\frac{3}{32z}\left\{[1-(z-u)^2]^2\ln\left|\frac{z-u+1}{z-u-1}\right|+[1-(z+u)^2]^2\ln\left|\frac{z+u+1}{z+u-1}\right|\right\} \tag{31a}$$

and

$$f_2^{\rm tr}(u,z)=\begin{cases}\frac{3}{4}\pi u(1-u^2-z^2)\,, & u+z<1\,,\\ \frac{3}{32}\frac{\pi}{z}[1-(u-z)^2]^2\,, & |u-z|<1<u+z\,,\\ 0\,, & |u-z|>1\,.\end{cases} \tag{31b}$$

By using Eqs. (12) and (29) we easily obtain

$$S=\frac{g^2{v_F}^2{\omega_p}^2}{\pi V^2c^2}\int_0^\infty\frac{dz}{z}\int_{-V/v_F}^{V/v_F}(f_2^{\rm tr}-3u^2f_2)u\,du\,. \tag{32}$$

In the limit $V/v_f\ll 1$ we have $f_2^{\rm tr}=\frac{3}{4}\pi u(1-z^2)$ for $z<1$ and $f_2^{\rm tr}=0$ for $z>1$; we also find that $f_2=(\pi/2)u$ for $z<1$ and $f_2=0$ for $z>1$. Keeping only those terms to lowest order in u we find

$$S=\frac{g^2{\omega_p}^2V}{2c^2v_F}\left[\ln\frac{1}{z_{\rm min}}-\frac{1}{2}\right]. \tag{33}$$

Note the similarity to Eq. (18). The cutoff $z_{\rm min}$ is clearly related to our neglect of damping as we saw in our previous discussion of $\rho_{\rm max}$. In that context we found that $k_{\rm min}=1/\Lambda$ for the case of conductors. Another way of seeing this is to interpret k/k_F as the scattering angle of an electron off the monopole. According to the uncertainty principle, one cannot simultaneously localize position and momentum so that $k_{\rm min}$ is related to the delocalization of the electron δb via $k_{\rm min}\sim 1/\delta b$. The quantity δb is essentially the lateral breadth of the electron wave packet which is scattering off of the monopole. If $k<k_{\rm min}$, then the uncertainty principle will be violated. For the case of conduction electrons, $\delta b\sim\Lambda$ (any broad plane wave will be broken into small wavelets by scattering off the positive ions of the lattice). For ordinary atoms, $\delta b\sim a_0$ since electrons are localized in the atoms. Thus, for conductors,

$$1/z_{\rm min}=2k_F\Lambda \tag{34}$$

and for nonconductors

$$1/z_{\rm min}=2k_Fa_0\,. \tag{35}$$

To calculate stopping power for atoms in nonconducting liquids and solids we will use the *total* number of electrons in the medium to determine Fermi energies and plasma frequencies:

$${\omega_p}^2=4\pi Ne^2/m,\quad \epsilon_F=\frac{\hbar^2}{2m}(3\pi^2N)^{2/3}\,, \tag{36}$$

where $N=Z_2\rho/A_2m_u$, Z_2 and A_2 denote the atomic number and weight, respectively, ρ denotes mass density, and m_u is an atomic mass unit. For simple conductors we will calculate, using Eq. (36), a "bulk" stopping power with $Z_2\rightarrow Z_2-1$ and a "conduction" stopping power with $Z_2\rightarrow 1$.

It is important to note that in the above derivation, z was naturally limited from above by 1 in the limit $u\ll 1$. This justifies our choice of $\rho_{\rm min}$ in the previous section.

It is instructive at this point to consider the stopping power of electrically charged particles. By using the same approximations as we have (i.e., neglect of damping, of time derivatives and $u\ll 1$), Lindhard[11] has shown that for particles with electric charge Z_1e,

$$S_e=\frac{{Z_1}^2e^2{\omega_p}^2V}{{v_F}^3}C_1(\chi)\,, \tag{37}$$

where $\chi^2=e^2/(\pi\hbar v_F)$ and

$$C_1(\chi)=\int_0^1\frac{z^3dz}{[z^2+\chi^2f_1(0,z)]^2}\,. \tag{38}$$

If one approximates $f_1(0,z)\approx 1$ it is found that

$$S_e=\frac{2m^2{Z_1}^2e^4V}{3\pi\hbar^3}\left[\ln\frac{137v_F}{c}+\ln\pi-1+\frac{2c}{137\pi v_F}\right] \tag{39}$$

if χ^2 is assumed to be small compared to unity. In the above, it is also assumed that the projectile does not capture orbital electrons. In actual fact for large values of Z_1 this is not valid and the projectile charge dependence is reduced from its quadratic nature. Lindhard and Scharff[15] have found that the expression

$$S_e=\frac{8\pi e^2a_0V}{\alpha c}\frac{{Z_1}^{7/6}N}{({Z_1}^{2/3}+{Z_2}^{2/3})^{3/2}} \tag{40}$$

is a good approximation to data. For protons, however, Eq. (39) should be reasonably accurate.

Note that Eq. (39) is not divergent even though it was obtained in the limit of zero damping. This illustrates the key difference between electric- and

magnetic-particle stopping power. If a monopole is slowing in a conductor or a degenerate Fermi gas its maximum impact parameter is limited by dissipation of eddy currents (if the monopole has a large velocity in a dielectric the adiabatic limit determines its cutoff). For a number of years it was believed[16] that similar limitations applied for electric particles. However, the failure to observe temperature-dependent stopping power of α particles in conductors prompted Kramers[17] to reexamine the problem, and he found that the plasma frequency provided the natural impact parameter cutoff via the adiabatic relation v/ω_p; the electric particle is capable of inducing coherent oscillations of the conduction electrons, producing an effective energy level of the medium with an energy $\hbar\omega_p$. Such modes are inacessible to monopoles due to the lack of a longitudinal electric field. For very low projectile velocities there is a slightly different, although related, cutoff. This has to do with the notion of screening. In the limit $k \to 0$ for $\omega = 0$ Eq. (13b) reduces to

$$\epsilon^l(k,\omega) \approx 1 + \frac{3\omega_p{}^2}{v_F{}^2 k^2} . \tag{41}$$

This implies that the potential in the vicinity of a particle of charge $Z_1 e$ is of the form

$$\phi = Z_1 e \exp(-\lambda r)/r , \tag{42}$$

where $\lambda^2 = 3\omega_p{}^2/v_F{}^2$. If electrons are more distant than $1/\lambda$ from the ion they do not scatter off it since they do not see it due to the screening of the longitudinal field. Again, the same effect will not apply for magnetic monopoles due to the absence of a longitudinal interaction.

The importance of these remarks lies in the observation that since the cutoff impact parameter for monopoles is larger than for electric charges (were it not for screening, electric-charge stopping would be limited by dissipation, as is the case for monopoles), the term $\ln(1/z_{\min})$ is at least as large as the corresponding one for electric charges. Thus, if one is not convinced of the numerical accuracy of these calculations of monopole stopping power, one can at least obtain a firm lower limit from available data.

It is of interest at this point to consider what effects the dynamical aspects of electron spin, i.e., the magnetic moment, have on the above results. One might be tempted to incorporate Eq. (13c) into the calculations in order to do this. However, when this is tried, it is found that the magnetic moment's contribution to stopping power diverges as $k_{\max}{}^2$. Physically, this is due to the form of the spin-monopole interaction potential, $-\vec{\mu}\cdot\vec{B}$. The force on the magnetic moment $\vec{\mu}$ is $(\vec{\mu}\cdot\vec{\nabla})\vec{B}$ so that it varies as $1/r^3$ where r is the electron-monopole separation. Thus, in a collision of a monopole with a stationary magnetic moment at impact parameter b, the momentum transfer goes as $(1/b^3)(b/V)$ and the energy transfer goes as $1/(b^2V)^2)$ so that when one integrates over $2\pi b\, db$ a result $1/b_{\min}{}^2$ is obtained. With the interpretation $b_{\min} = 1/k_{\max}$, we get the above divergence for $k_{\max}$. With a divergence this severe it would be foolhardy to attempt to estimate the size of the contribution to stopping power. The reason for this problem clearly has to do with the neglect of the wave nature of the electron in Eqs. (13a)–(13c). Unfortunately, we do not know of an expression for $\delta\epsilon^{\rm tr}$ which takes into account the electron spin in a fully quantal manner. However, we can put a limit on the magnitude of the effect by considering the Fermi-Teller approach to low-velocity stopping power which we take up in the next section.

THE FERMI-TELLER APPROACH TO LOW-VELOCITY STOPPING POWER

Fermi and Teller[18] have used a very simple but physically illuminating approach to estimate the stopping power of slow negative muons in matter. As with Lindhard, the projectile was assumed to be passing through a degenerate Fermi gas. By restricting collisions between the electrons and the projectile to those for which the energy transfer to the electrons is sufficient to raise the total energy greater than ϵ_F, they obtained the result

$$S_e = \frac{4}{3\pi}\frac{m^2 Z_1{}^2 e^4 V}{\hbar^3}\ln(1/\psi_{\min}) , \tag{43}$$

where $\psi_{\min}$ is the minimum allowed scattering angle between the directions of an incident and scattered electron momenta. They assumed this minimum angle is determined by the maximum permissible impact parameter due to screening and they obtained $\psi_{\min} \sim (\alpha c/v_F)^{1/2}$. Thus, aside from the additional term in the bracket of Eq. (39), the Fermi-Teller result is the same as the Lindhard result. This lends us confidence that the Fermi-Teller approach can also be used to estimate monopole stopping power. The advantage of this technique is that it enables us to include the dynamical effects of the electron spin, as we shall presently see.

Consider a slow, heavy projectile passing through a degenerate electron gas. Assume that the projectile velocity $V \ll v_F \ll c$. In this limit there is no need to distinguish between angles measured in the laboratory frame and those measured in the rest frame of the projectile. This approximation would not be permissible if we considered energy loss in a classical gas in which the projectile would be equally likely to lose as to gain energy unless the relative number of overtaking to approaching collisions were properly taken into account; this is in fact the basis of Fermi's mechanism for the acceleration of cosmic rays by massive magnetic irregularities in the interstellar medium. The energy transfer to an electron is given by

$$\Delta T = pV(\cos\theta' - \cos\theta) , \tag{44}$$

where θ is the angle between $\vec{p}$ and $\vec{V}$, $\vec{p}$ is the incident electron momentum, and θ' is the angle between the scattered electron's velocity and $\vec{V}$. The time rate of energy transfer to the Fermi gas is given by

$$\frac{dE}{dt} = \int_{\epsilon,\Omega,\Omega'} n(\epsilon) d\epsilon \frac{d\Omega}{4\pi} v \frac{d\sigma}{d\Omega'} d\Omega' \times \Delta T(\theta',\theta,\epsilon) F(\theta',\theta,\epsilon) , \tag{45}$$

where

$$v = p/m , \quad \epsilon = \tfrac{1}{2} m v^2 ,$$

$$n(\epsilon) = \begin{cases} \dfrac{m}{\pi^2 \hbar^3} \sqrt{2m\epsilon} , & \epsilon < \epsilon_F , \\ 0 , & \epsilon > \epsilon_F \end{cases}$$

and $d\Omega = \sin\theta \, d\theta \, d\phi$, $d\Omega' = \sin\theta' d\theta' d\phi'$ and where $d\sigma/d\Omega'$ is the differential scattering cross section to be evaluated at the relative velocity v (not V). The function $F(\theta',\theta,\epsilon)$ is defined by

$$F(\theta',\theta,\epsilon) = \begin{cases} 1 , & \text{if } \epsilon + \Delta T > \epsilon_F , \\ 0 & \text{otherwise} . \end{cases} \tag{46}$$

Fermi and Teller evaluated dE/dt for the case of a Rutherford cross section,

$$\frac{d\sigma}{d\Omega'} = \frac{Z_1^2 e^4}{4p^2 v^2 \sin^4(\psi/2)} , \tag{47}$$

where $\cos\psi = \sin\theta \sin\theta' \cos(\phi' - \phi) + \cos\theta \cos\theta'$. By performing the indicated integrations in Eq. (45) they obtained Eq. (43). Note that since only the screening angle had to be inserted after the fact, the Fermi-Teller approach is apparently handling the wave nature of the electron properly. This is due to the fact that by utilizing a differential-cross-section procedure one is in fact dealing with momentum transfer rather than impact parameters. In the Fermi-Teller approach then, one is in principle assuming the electron wave packets to be infinitely broad. This leads us to believe that by using the electron-monopole cross section evaluated by Kazama, Yang, and Goldhaber[19] we will be considering dynamical effects of electron spin. This is due to the fact that in Ref. 19 the Dirac equation was solved to get the cross section. It is convenient to express this cross section as a ratio to the classical "Rutherford" cross section for monopoles;

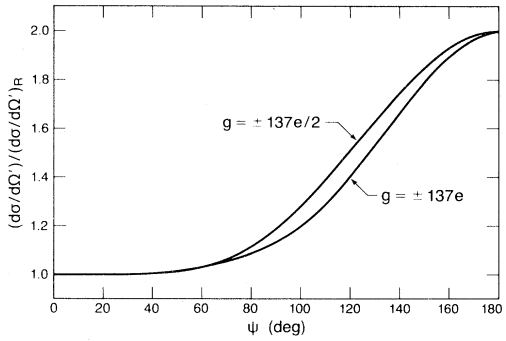

FIG. 1. Ratio of the Kazama-Yang-Goldhaber (Ref. 19) electron-monopole cross section to the classical cross section [Eq. (48)] as a function of center-of-mass scattering angle.

$$\left[\frac{d\sigma}{d\Omega'}\right]_R = \frac{g^2 e^2}{4p^2 c^2 \sin^4(\psi/2)} . \tag{48}$$

In Fig. 1 $(d\sigma/d\Omega')/(d\sigma/d\Omega')_R$ is plotted for the two cases $g = \pm 137e/2$ and $g = \pm 137e$. Note that except for large scattering angles ($\gtrsim 40°$) the exact cross section is very nearly equal to the Rutherford cross section. This leads us to write

$$\frac{d\sigma}{d\Omega'} = \left[\frac{d\sigma}{d\Omega'}\right]_R + \left[\frac{d\sigma}{d\Omega'}\right]_\Delta . \tag{49}$$

In Fig. 2 these two contributions are compared for the cases $g = \pm 137e/2$ and $g = \pm 137e$. The values for $(d\sigma/d\Omega')_\Delta$ were obtained from Ref. 19 where they were presented as the data in Fig. 1 to $\sim 1\%$ accuracy. Thus, for $\psi < 40°$ we can only claim an upper limit for $(d\sigma/d\Omega')_\Delta$ which is denoted by curve *a*. For $\psi > 40°$ the upper limit is given by curve *b*. The stopping power due to the Rutherford component can be obtained from the Fermi-Teller calculation by replacing $Z_1 e$ with $g v_F/c$. We have set $v = v_F$ since for $V \ll v_F$ only those

electrons with v near the surface of the Fermi sphere are capable of being ejected from the sphere. So

$$S_R=\frac{4}{3\pi}\frac{m^2g^2e^2Vv_F^2}{\hbar^3c^2}\ln(1/\psi^m_{\min}) . \tag{50}$$

In this expresson $\psi^m_{\min}$ is a minimum allowed electron scattering angle. Since screening is not important for monopoles we know that

$$\psi^m_{\min}(\text{monopoles})<\psi_{\min}(\text{electric}) . \tag{51}$$

This enables us to put a lower limit on monopole stopping power. It is easy to see that

$$S_R/S_p>\left[\frac{g}{e}\right]^2\left[\frac{v_F}{c}\right]^2 , \tag{52}$$

where S_P is proton stopping power. Since $S=S_R+S_\Delta$ the total monopole stopping power must exceed the limit above. For $g=137e$, we have the useful result that monopole stopping power must exceed proton stopping power (for $v_F\sim\alpha c$). The determination of a limit for S_Δ is easily done. We can write $S_\Delta<S_a+S_b$ (a and b correspond to curves a and b in Fig. 2). By definition, $S_a=0.01S_R$. The general expression for S can be shown to reduce to

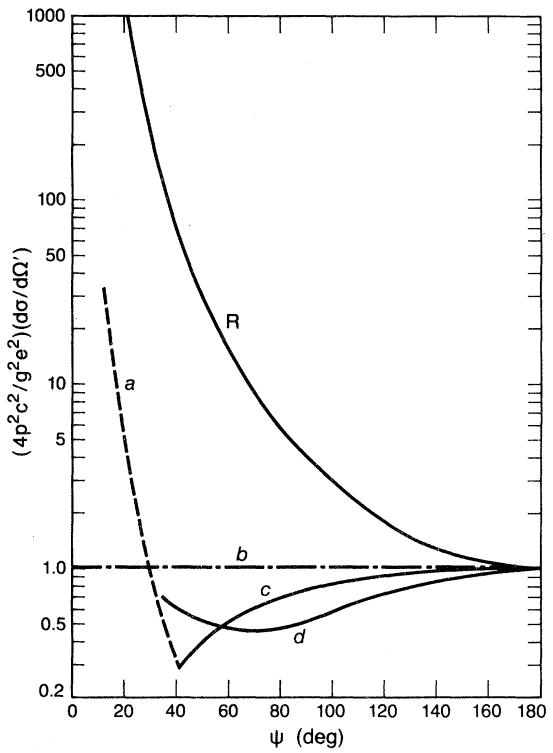

FIG. 2. Comparison of $(d\sigma/d\Omega')_R$ to $(d\sigma/d\Omega')_\Delta$ [see Eq. (49)]. Units are indicated on the vertical scale. Curve c corresponds to $(d\sigma/d\Omega')_\Delta$ for $g=\pm137e/2$ and curve d corresponds to the case for $g=\pm137e$. Curves a ($\psi\lesssim30°$) and b ($\psi\gtrsim30°$) are upper limits for the $(d\sigma/d\Omega')_\Delta$ contribution to the total cross section.

$$S=\frac{2m^2\epsilon_F^2}{\pi^2\hbar^3}V\int_0^\pi d\theta'\int_{\theta'}^\pi d\theta\int_0^{2\pi}d\phi'\sin\theta\sin\theta'(\cos\theta'-\cos\theta)^2\frac{d\sigma}{d\Omega'} , \tag{53}$$

where $d\sigma/d\Omega'$ is evaluated at the Fermi velocity of the electron gas (not at the projectile velocity since $V\ll v_F$). Note that there was a typographical error in the Fermi and Teller paper and that the term $(\cos\theta'-\cos\theta)$ appeared rather than its square. The contribution due to curve b in Fig. 2 corresponds to $d\sigma/d\Omega'=g^2c^2/(4p^2c^2)$ so that

$$S_b=\frac{2m^2\epsilon_F^2}{\pi^2\hbar^3}V\frac{g^2e^2}{8m\epsilon_Fc^2}\times\int_{-1}^1dx'\int_{-1}^{x'}dx(x'-x)^2\pi \tag{54}$$

which can be expressed in terms of S_R as

$$S_b=S_R\Big/\left[4\ln\frac{1}{\psi^m_{\min}}\right] . \tag{55}$$

Since

$$\ln\left[\frac{1}{\psi^m_{\min}}\right]>S_p\Big/[(4/3\pi)m^2e^4V/\hbar^3]$$

the ratio S_b/S_R can be quantitatively estimated by using actual proton stopping-power measurements. Taking silicon as the target material, we find that[20]

$$S_p=(210\text{ MeV/cm})\sqrt{E}\quad(\text{silicon}) , \tag{56}$$

where E is the proton kinetic energy in keV. Thus, $\ln(1/\psi^m_{\min})>0.48$ so that $S_b<0.5S_R$. So the Fermi-Teller approach yields the result

$$S_{FT}=\frac{4}{3\pi}\frac{m^2g^2e^2v_F^2}{\hbar^3c^2}V\ln\left[\frac{1}{\psi^m_{\min}}\right](1+b) , \tag{57}$$

where $0<b<0.5$. By noting that $z_{\min}=\psi^m_{\min}/2$ we can take the ratio of $S_{\rm FT}$ to S_L [the Lindhard result, Eq. (33)] to find

$$S_{\rm FT}/S_L=2(1+b)\Big/\left[1+0.19/\ln\frac{1}{\psi^m_{\min}}\right]. \tag{58}$$

For $\ln(1/\psi^m_{\min})=0.5$ this ratio varies from 1.4 to 2.1 depending on the value of b. This suggests that spin effects do not dominate monopole stopping power at small velocities although a significant contribution due to interaction of the electron magnetic moment cannot be ruled out. In fact, Ullman has argued[21] that since the energy of interaction with an electron at a distance a_0 is similar for a proton and a monopole, their low-velocity stopping powers should be similar, a result we have arrived at by different arguments. The magnetic interaction is given by $ge\hbar/2mca_0{}^2$ and the electric interaction is e^2/a_0. For $g=137e$ these energies are the same within a factor of 2. One might expect, therefore, that by coupling to the spin of an electron a slowly moving monopole could excite energy levels with efficiency comparable to excitation by a proton. Classically, this corresponds to the situation where a force is exerted on the electron by means of the gradient of the monopole's magnetic field. More detailed calculations will be required to achieve a more quantitative evaluation of the role of the electron's spin in the slowing of slow monopoles.

In calculating the parameters for the Fermi gas we assume properties of the bulk medium. It would be more accurate (and absolutely essential for gas absorbers) but much more laborious to take suitable averages throughout the medium by varying impact parameters and taking into account the rarefied spaces between atoms. In so doing one would require realistic radial profiles of electron densities in atoms as provided by Thomas-Fermi or Hartree-Fock descriptions. However, we will see that for silicon the Lindhard description accounts well for actual proton data even with a naive use of bulk properties. In this case $v_F=0.106c$. The Lindhard expression [Eq. (39)] for protons in the limit $\chi^2\ll 1$ (for the present case, $\chi^2=0.22$) gives the result

$$S_p(\mathrm{Si})=(143\ \mathrm{GeV/cm})\beta$$
$$=(209\ \mathrm{MeV/cm})\sqrt{E}, \tag{59}$$

where E is in keV. This is remarkably close to the experimental value quoted previously. It is probably only an accident that the agreement is so good since it is difficult to measure stopping power to within 30% for these small velocities. Emboldened by this success we now use Lindhard's expression for monopole stopping power with $\psi^m_{\min}=\alpha c/v_F$ [see Eq. (35)] to find

$$S(\mathrm{Si})=\left[180\ \frac{\mathrm{GeV}}{\mathrm{cm}}\right]\left[\frac{g}{137e}\right]^2\beta\ . \tag{60}$$

Since the Lindhard technique does not take into account electron spin in a dynamical way, this must be interpreted as a lower limit to monopole stopping power. It is conceivable that spin interactions could increase it by as much as a factor of 2. As we have seen earlier, a more firm lower limit on monopole stopping power for $g=\pm 137e$ is provided by measurement of proton stopping power.

The above expression for monopole stopping power is valid only for $V<0.01c$. At large velocities ($V>0.1c$) we can use an earlier result[9] for silicon:

$$S(\mathrm{Si})=\begin{cases}0.72\ \dfrac{\mathrm{GeV}}{(\mathrm{g/cm^2})}(8.18+\ln\beta^2\gamma^2), & g=\pm 137e/2,\ \beta>0.1,\\ 2.89\dfrac{\mathrm{GeV}}{(\mathrm{g/cm^2})}(7.73+\ln\beta^2\gamma^2), & g=\pm 137e,\ \beta>0.1,\end{cases} \tag{61}$$

which we compare to the low-velocity results

$$S(\mathrm{Si})=\begin{cases}20\ \dfrac{\mathrm{GeV}}{(\mathrm{g/cm^2})}\beta, & g=\pm 137e/2,\ \beta<0.01,\\ 78\dfrac{\mathrm{GeV}}{(\mathrm{g/cm^2})}\beta, & g=\pm 137e,\ \beta<0.01\end{cases} \tag{62}$$

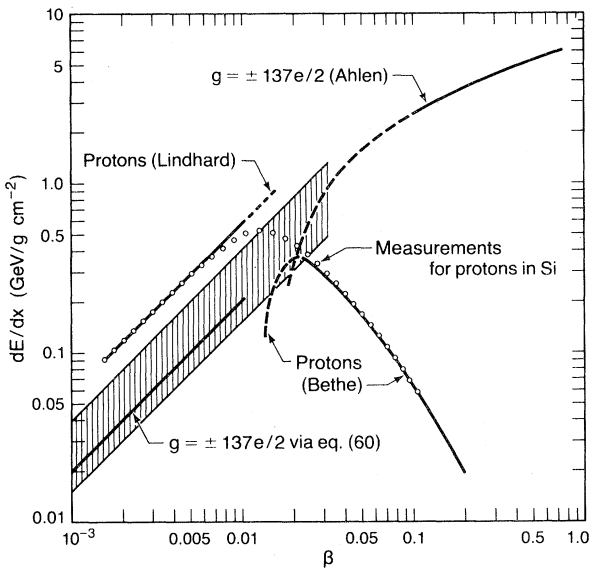

FIG. 3. Stopping powers in silicon for protons and for monopoles with $g=\pm 137e/2$. Solid lines are calculations and are taken from Ref. 6 (high-velocity-proton curve labeled Bethe), Ref. 9 (high-velocity-monopole curve labeled Ahlen), Ref. 11 (low-velocity-monpole curve labeled Lindhard) and Eq. (60) of this work (low-velocity-monopole curve). Dashed lines are extrapolations of the various theories into regions of questionable validity. Note that the Bethe theory as shown does not include shell corrections. The shaded region indicates estimated range of errors for the slow-monopole stopping power. The open circles are the averaged values of high-quality measurements of proton stopping power in silicon and are taken from Ref. 20.

in Fig. 3. Open circles corresponding to measured proton stopping are also shown, as are calculations of low-velocity proton stopping power from the Lindhard expression and high-velocity calculations from the Bethe equation (for which shell corrections were not used). The shaded region corresponds to a very conservative range of possible monopole stopping powers and it overlaps the extrapolation of high-velocity stopping power when extended into the region $0.01<\beta<0.1$.

Let us now consider the monopole stopping power in a conductor. For such a system there are two separate contributions to monopole stopping power: from the bound electrons for which $\psi^m_{\min}=\alpha c/v_F$, and the conduction electrons for which $\psi^m_{\min}=(\alpha c/v_F^c)/(50aT_m/a_0T)$ where v_F^c is the Fermi velocity of the conduction electrons. Similarly we distinguish $\omega_p^{c\,2}$ from $\omega_p{}^2$. We have for alkali metals like sodium the following relations: $\omega_p{}^2=(Z_2-1)\omega_p^{c\,2}$ and $v_F=(Z_2-1)^{1/3}v_F^c$.

Taking the particular example of Na at 20° C, for which $T_m=98°$ C and $a=6.4a_0$, we have for the bulk electrons $\ln(1/z_{\min})-\frac{1}{2}=0.23$ and for the conduction electrons $\ln(1/z_{\min})-\frac{1}{2}=6.16$. The bound-electron stopping power is

$$\frac{g^2\omega_p{}^2}{2c^2v_F}V(0.23)$$

while the conduction-electron stopping power is

$$\frac{g^2\omega_p{}^2}{2c^2v_F}V\left[\frac{1}{Z_2-1}\right](Z_2-1)^{1/3}(6.16)$$

so the total stopping power is

$$\frac{g^2\omega_p{}^2}{2c^2v_F}V(0.23+1.33)\,,$$

where the second term in parentheses corresponds to the contribution from conduction electrons while the first term corresponds to bound electrons. The conduction electron's contribution apparently dominates. We can compare this result to that obtained by Martem'yanov and Khakimov[22] who used a variant of Landau's[12] method for stopping-power calculation to obtain the rate of energy loss by slow monopoles in conductors. They found that

$$S=\frac{4\pi^2Ne^2g^2}{mc^2v_F^cZ_2}V,$$

where N is the bulk electron density. Expressed in our form their result is

$$S=\frac{\pi g^2\omega_p{}^2}{c^2v_F(Z_2-1)^{2/3}}V\,. \tag{63}$$

For the case of Na, the ratio of our result to theirs is 1.15, which is quite good agreement. However, this is probably an accident and should not be taken too seriously since the authors of Ref. 22 assumed the Drude conductivity [Eq. (6)] which we have shown to be inappropriate for calculations of stopping power.

LIMITATIONS OF THE CALCULATIONS

For each of the three approaches utilized above to calculate the stopping power of slow monopoles, we have assumed that the stopping material is composed of a gas of noninteracting electrons at zero temperature. The existence of a positively charged background has been incorporated only with regard to the damping of the electron gas and

to provide a physical rationale for preventing the Coulomb disruption of the electron gas. An additional requirement for the validity of our calcultaions has been that typical wavelengths of the electrons in the Fermi gas are smaller than those of the excitations induced in the gas by the atomic nuclei and by the monopole. This so-called semiclassical approximation is reflected in our use of a quasicontinuous electron energy density function in Eq. (45) for our calculation of stopping power via the Fermi-Teller technique. For the Lindhard technique we have not explicitly required the semiclassical approximation insofar as we used the fully quantal expression for the magnetic permeability. Thus, in arriving at Eq. (33) we made use of the limit $V/v_F \ll 1$ to eliminate the contribution to stopping power from those Fourier components for which $k > 2k_F$. However, since $z_{\min}$ must be less than 0.61 in order for there to be positive stopping power, it is necessary that $k_F a_0 \gtrsim 0.8$ or that $\lambda_F \lesssim 7.9 a_0$ for the case of nonconductors. The latter inequality is not too much less restrictive than the criterion for the validity of the semiclassical approximation. Since this is the same criterion required for the Thomas-Fermi model, which is valid for $Z_2 \gtrsim 10$, we expect our calculations to be valid for the same range of atomic numbers of nonconducting stopping materials. For conductors, of course, the description of the conduction electrons as comprising a degenerate Fermi gas is quite valid and our confidence in the stopping-power calculation is strong.

It is essential in determining the validity of our calculations to consider the possible effects of very strong magnetic fields on the validity of Lindhard's expression for ϵ and μ. The expression becomes invalid if the field strength is such that currents induced in directions other than that of the magnetic field are strongly suppressed. This happens if the electron's period of gyration in the magnetic field is small compared to the damping time. We now calculate this classically, to determine the gyration frequency as a function of impact parameter in the field of the monopole.

Consider an electron traveling with velocity v_F on a trajectory with impact parameter b in the field of a monopole with charge g. Its distance from the monopole as a function of time is given as

$$r = (b^2 + v_F^2 t^2)^{1/2} .$$

The instantaneous gyration frequency is then

$$\omega = \frac{eB}{mc} = \frac{e}{mc} \frac{g}{(b^2 + v_F^2 t^2)} .$$

The angle of rotation in traveling from $t = 0$ to $t = \tau$, where τ is the damping time, is

$$\phi = \int_0^\tau \omega \, dt = \frac{eg}{mc} \int_0^\tau \frac{dt}{b^2 + v_F^2 t^2}$$

$$= \frac{eg}{mcbv_F} \tan^{-1} \frac{v_T \tau}{b} .$$

For a Dirac monopole and a rotation of 2π,

$$\tan^{-1} \frac{v_F \tau}{b} = \frac{4\pi b \beta_F}{\alpha a_0} .$$

If we take $\beta_F \approx 0.01$, then

$$\frac{v_F \tau}{b} = \tan \left[\frac{17b}{a_0} \right] .$$

For conductors we assume $v_F \tau \gg b$ and

$$17 \frac{b}{a_0} \approx \pi/2 ;$$

so

$$b \approx 0.09 a_0 ,$$

and we see that the assumption is justified. For nonconductors $v_F \tau \sim a_0$, and we obtain numerically

$$b \approx 0.08 a_0 .$$

In our first two calculations of monopole stopping power in this paper we used a minimum impact parameter corresponding to $b \sim 1/k_F \sim 0.73 a_0$, for $\beta_F = 0.01$. Clearly, the Lindhard equations for ϵ and μ would be valid in this region.

Another conceivable limitation of our calculations is that due to the kinematics of transferring energy from slow heavy projectiles to atomic systems. From Eq. (44), we can estimate the maximum possible energy transfer to be $2mv_F V$. For conductors, the energy levels of the conduction electrons are continuous and energy transfer is possible for all values of the monopole velocity V. In insulators, gases, semiconductors, etc., there are threshold excitation and ionization levels which must be exceeded by the kinematic limit in order for the system to be excited or ionized. For systems in which the minimum excitation energy is several electron volts, one would expect the kinematic limitation to result in cessation of electronic energy loss at a monopole velocity somewhere between $10^{-4}c$ and $10^{-3}c$. In addition, one might expect a more complicated velocity dependence than the linear one derived in the preceding sections for monopole velocities greater than the kinematic limit. However, we suspect that these

limitations are not too severe with regard to monopole stopping power (although they may be for considerations of various types of detector response). The reason for this is based on an analogy with the stopping power of electrically charged particles. One would naively expect our arguments regarding kinematics to apply to the stopping power of electrically charged particles as well as of magnetic particles. However, experimental observations[20,23] indicate that in silicon, the electronic stopping power for proton velocities ranging from $\sim 10^{-3}c$ to $10^{-2}c$ is very well described as being proportional to velocity, with no evidence of threshold effects. A similar result has been observed[24] for the electronic stopping power of hydrogen ions in carbon foils for projectile velocities ranging from 6×10^{-4} to 3×10^{-3}. Lindhard[25] has noted that the electronic stopping power of ions heavier than hydrogen is similarly characterized by a linear dependence of stopping power on velocity and that the distinction between conduction electrons and tightly bound electrons is not as strong as one might suspect. He argues that the total kinetic energy of the projectile is more than adequate for the production of electron excitations by means of the nature of the quasielastic collision process in which a projectile with its electron cloud forces its way through the electron cloud of the target atom. With this point of view, the threshold for the excitation of an atomic energy level is determined by the requirement that the total center-of-mass kinetic energy exceeds the excitation energy. For a GUT monopole and an excitation energy of 3 eV, the threshold velocity would correspond to $\beta\sim 2\times 10^{-5}$ for a collision with a carbon atom. We will see in the next section that the energy loss of monopoles at such low velocities is dominated by the kinetic energy gained by the recoiling atoms in such collisions. Nevertheless, it is important to realize that electronic excitation is also possible at very low velocities.

Another example which helps to illuminate how the kinematic constraints on electron energy loss can be overcome is that of Cherenkov radiation. This phenomenon involves the loss of energy by a projectile in parcels which may be much smaller than any electronic excitation levels of the stopping material. The energy parcels are in fact lost not to the stopping material itself, but to the combined system which includes both the electronic states of the medium and the electromagnetic field. In dense materials where the index of refraction is significantly different from unity it is in fact impossible to discuss separately the electronic and the electromagnetic fields. One might expect that in passing through a degenerate Fermi gas, analogous states with low-lying energy are made accessible to slow-moving projectiles. We are encouraged regarding the validity of this analogy by the fact that the usual derivation of the Cherenkov intensity relation follows techniques similar to those used by Lindhard, in that stopping effects are analyzed by means of the properties of the electric and magnetic permeabilities. For the case of monopoles this argument can be made more specific by noting that the energy levels of an electron in a magnetic field perpendicular to the plane of motion of the electron are quantized with the values[26]

$$E_l=\frac{e\hbar B}{mc}(l+\tfrac{1}{2})\ .$$

For an electron in the vicinity of a monopole this reduces to

$$E_l=7.6\text{ eV }(g/137e)(l+\tfrac{1}{2})/[r\ (\text{Å})]^2\ ,$$

where r is the monopole-electron separation. To lose energy in small parcels, the monopole merely has to provide for the existence of these energy levels and to excite the electron into low-lying levels (for $r=$ several Å) and then to have the electromagnetic field induce the decay of the levels. In fact this must be what is actually occurring in the Lindhard technique for calculating monopole stopping power. The reaction force on the monopole is due to the diamagnetism of the stopping material. This is a consequence of Lenz's law in which an induced electric current opposes the flux which induces the electric field in the first place. The parcels of energy loss correspond to the quanta of the magnetic field which is produced to oppose the field of the monopole. If the quanta actually correspond to energies sufficient to raise the atoms of the stopping material to excited states, then the monopole leaves the medium with net atomic excitation energy which will ultimately yield fluoresence, Auger emission, heat generation via internal conversion, etc. In a sense, one can compare the relationship of the excitation of atoms by passing monopoles to the emission of low-energy quanta to the relation of resonance fluorescence to Rayleigh scattering for real photons. A similar relation applies to scintillation and Cherenkov radiation. There are, of course, major distinctions between Cherenkov radiation and the "emission" of the quanta associated with slow-monopole energy loss. Our intention here has been merely to point out

the possibility of exciting electronic-electromagnetic modes in which atoms themselves are not individually excited. This distinction is quite important when one considers the response of particle detectors to slow monopoles. It is possible for the electronic stopping power to be quite linear in monopole velocity while detector responses display different velocity dependences.

COMPARISONS WITH OTHER WORK

We have already briefly touched upon the estimates of the stopping power of slow monopoles obtained by Ullman[21] and by Martem'yanov and Kh.Khakimov.[22] In each case, the results obtained agree roughly with those obtained here, but it is not clear that this is not an accident. For example, we have seen that the authors in Ref. 22 used the Drude conductivity which we have shown to be invalid. In fact, Akerlof[27] has used the same assumptions as in Ref. 22, but he has obtained stopping power in conductors several orders of magnitude larger than found in Ref. 22. It is likely that Martem'yanov and Kh.Khakimov made canceling errors in their calculations. Similarly, Ullman's[21] educated guess was most likely just a lucky one in predicting the similarity between monopole and proton stopping power. His conclusions were based on the monopole-electron magnetic moment interaction. However, we have seen here that it is the interaction between the monopole's electric field and the electronic charge that determines the similarity. It was shown above that the role of the monopole-electron moment interaction is not clear at the present time, but that it probably does not dominate the interaction.

In the past year a number of other papers have been written which deal with the interactions of slow monopoles with matter. These have achieved rather wide distribution in preprint form and we, therefore, feel an obligation to comment here on their content. We feel that substantial misconceptions could arise if some of these papers are viewed uncritically. Trefil[28] has claimed to calculate slow-monopole stopping power by incorporating second-order electron motion into the formalism. He does so by eschewing the low-velocity approximation used in Sec. 13.2 of Ref. 14 in which Bessel functions are expressed as logarithms. However, by adapting the other features of Ref. 14 to apply to monopole stopping power he does not avoid the fundamental assumption of the impulse approximation in which atomic electrons are exposed to purely time-dependent perturbations. We have shown here that such an approximation will lead to erroneous results and that the spatial dependence of the perturbation must be included for slow monopoles. Hayashi[29] has attempted to evaluate the stopping power of slow monopoles by modifying the Fermi-Teller calculations much as we have done here. However, by incorrectly using the prescription $Z_1 e \rightarrow gV/c$, a V^3 dependence of monopole stopping power was inferred. The error here was in assuming the monopole-electron interaction to be determined by the monopole velocity rather than the electron velocity. McIntyre and Webb[30] have followed the same general scheme of Hayashi in replacing $Z_1 e$ by gV/c. However, they chose an expression from Lindhard and Scharff[15] for electric-charge stopping power which was proportional to Z_1 rather than Z_1^2 due to the effects of electron capture and loss. The resulting monopole stopping power was proportional to V^2. This is incorrect both for the reason stated above with regard to Hayashi's result and due to the fact that the monopole charge would not be screened by captured electrons even in the event that such would happen. Thus, comparison to the stopping power of screened electric charges is clearly invalid. It is unfortunate that results such as those referred to above have been quoted with the purpose of demonstrating either the feasibility or impracticality of using conventional excitation or ionization particle detectors to search for GUT monopoles. It never seems to be realized that there may be significant distinctions between energy loss by a projectile and detector response, which we emphasized in the preceding section.

NUCLEAR STOPPING POWER

In the above sections we have considered the energy loss by monopoles in electronic collisions. We now want to consider briefly the loss in collisions with atoms, which might be expected to dominate at very low velocities in analogy with the stopping power of electric charges. One might be tempted to obtain this so-called nuclear stopping by monopoles by making the substitution $Z_1 \rightarrow g\beta/e$ in an expression such as that obtained in Ref. 31. However, this cannot be correct since the nuclear stopping of electric particles strongly reflects the fact that at low velocities the projectiles are almost completely neutralized by the capture of orbital electrons. Owing to the absence of magnetic charge in the stopping material such shielding is

not possible for monopoles.

In what follows we will apply classical techniques to the problem. This is probably adequate in view of the fact that the Coulomb parameter $Z_1Z_2\alpha/\beta \rightarrow \alpha(g/e)Z_2$ for monopoles. If $Z_2 \gg 1$, then this quantity is large compared to unity and it follows that the classical approximation is valid. We will assume that the atoms of the absorbing material have no magnetic moment. Then the interaction is dominated by the transverse electric field [Eq. (1)]. When the monopole is well within the atom the coherent response of the nucleus will dominate that of the electrons and one can neglect the latter. For large impact parameters the atom can respond to an applied electric field only through its induced electric dipole moment. This will lead us to utilize the atomic radius as a cutoff impact parameter.

By using Eq. (1) it is easy to show that the momentum transfer to a nucleus of charge Z_2e, and mass $M_2=A_2m_u$ for an impact parameter ρ is

$$\Delta p=\frac{2gZ_2e}{c\rho} , \tag{64}$$

where the nucleus is assumed to be at rest compared to the monopole, a good approximation as long as the monopole velocity $V \gtrsim 10^{-5}c$. The energy transfer is

$$\Delta E=\frac{2g^2Z_2^2e^2}{M_2c^2\rho^2} . \tag{65}$$

By integrating over impact parameters,

$$S_{\text{nuc}}=\frac{4\pi N_a g^2Z_2^2e^2}{M_2c^2}\ln\frac{\rho_{\max}}{\rho_{\min}} , \tag{66}$$

where N_a is the number density of atoms and S_{nuc} denotes nuclear energy loss per unit length. This is familiar as a crude but effective technique for evaluating high-velocity stopping power (see Ref. 14). Here, as there, $\rho_{\min}$ is estimated by setting $\Delta E(\rho_{\min})=2M_2V^2$, the kinematic limit of energy transfer. This gives

$$\rho_{\min}=\frac{gZ_2e}{M_2Vc} . \tag{67}$$

For electronic stopping power one uses the adiabatic limit for $\rho_{\max}$. This is clearly not appropriate here since nuclei can absorb energy in any amount. However, since we are here concerned with energy loss to the recoil of *atoms* instead of *individual electrons* we use the fact that for $\rho \gtrsim a_0$ the momentum transfer is not nearly as large as indicated by Eq. (64) due to the opposing effect of the atomic electrons. Hence, we set $\rho_{\max}=a_0$ to obtain

$$S_{\text{nuc}}=\frac{4\pi N_a g^2Z_2^2e^2}{M_2c^2}\ln\frac{M_2Vca_0}{gZ_2e} . \tag{68}$$

In this equation we neglect subtleties which allow for slight dependences of atomic radius on atomic number. For silicon, Eq. (68) yields

$$S_{\text{nuc}}(Si)=\left[0.79\frac{\text{MeV}}{\text{g/cm}^2}\right]\left[\frac{g}{137e}\right]^2 \times\left[13.12+\ln\beta-\ln\left[\frac{g}{137e}\right]\right] . \tag{69}$$

For $\beta=10^{-3}$, $g=\pm 137e/2$, $S_{\text{nuc}}=1.4$ MeV/(g/cm^2) which is about 7% of the electronic stopping power. For $\beta=10^{-2}$ S_{nuc} is only 1% of the electronic stopping power.

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