Relativistic corrections to the electric dipole one-photon transition rates of charmonium

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We calculate the relativistic corrections of leading order to the electric dipole one-photon transition rates of charmonium using an approximately relativistic theory of the composite system and the simple harmonic-oscillator-type confining potential. We find that the corrections are large, of the order of -30% to -70%, for the decay rates.

I. INTRODUCTION

Recently,[1] we have presented a new treatment of the interaction of a composite system with the quantized radiation field, within the framework of an approximately relativistic theory,[2-10] which is relativistic to order $1/c^2$ in the sense that it has a representation[2] of the ten generators of the Poincaré group[3] to the same order. The generators are expressed in terms of the position, the momentum, and the spin variables of the fixed number of constituent particles which make up the composite system. The position and the momentum variables are canonically conjugate and the wave functions of the composite system involve only one time, the time of the observer, so that the formalism is quite similar to the ordinary nonrelativistic Schrödinger theory. Since the "no-interaction theorem"[11,12] does not hold in approximately relativistic theories, the position variables can satisfy the classical world-line condition of Currie, Jordan, and Sudarshan[11] to order $1/c^2$. In Ref. 1 we have shown that in this theory the interaction of a composite system of N spin-$\frac{1}{2}$ Dirac particles with the quantized radiation field should be represented, correct to relative order $1/c^2$, by the interaction Hamiltonian

$$H_I = i\sum_{\mu=1}^{N}\frac{e_\mu}{2c}\{[\vec{r}_\mu,H]\cdot\vec{A}_\mu+\vec{A}_\mu\cdot[\vec{r}_\mu,H]\}-\sum_{\mu=1}^{N}\frac{e_\mu}{m_\mu c}\vec{s}_\mu\cdot\vec{B}_\mu-\sum_{\mu=1}^{N}\frac{e_\mu}{2m_\mu{}^2c^2}\vec{s}_\mu\cdot(\vec{E}_\mu\times\vec{p}_\mu)$$

$$-i\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu{}^2c^2}\vec{s}_\mu\cdot(\vec{\nabla}_\mu\times\vec{E}_\mu)+\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu{}^3c^3}[p_\mu{}^2,\vec{s}_\mu\cdot\vec{B}_\mu]_+, \tag{1}$$

where H is the Hamiltonian of the isolated composite system. In Eq. (1) we have included only the terms which are linear in the quantized transverse vector potential $\vec{A}(r,t)$ and in the fields $\vec{E}$ and $\vec{B}$ since in this paper we are only interested in the one-photon transitions[13,14] of the composite system. The symbols e_μ and m_μ ($\mu=1,2,\ldots,N$) represent the electric charges and masses of the constituent particles, while $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ represent their position, momentum, and spin (spin-$\frac{1}{2}$) variables. Also,

$$\vec{O}_\mu=\vec{O}(\vec{r}_\mu,t), \tag{2}$$

where $\vec{O}_\mu$ stands for $\vec{A}_\mu$, $\vec{B}_\mu$, or $\vec{E}_\mu$ in Eq. (1). The advantages of using Eq. (1) for the interaction Hamiltonian have been spelled out in Ref. 1. In particular we have shown[1,15] that the one-photon and the two-photon transition amplitudes of the composite system obtained on the basis of this interaction Hamiltonian satisfy the requirement of relativistic invariance to relative order $1/c^2$, provided the position operators satisfy the classical world-line condition.[11]

To order $1/c^2$, the Hamiltonian H (of the isolated composite system) occurring in Eq. (1), is given by

$$H=\sum_{\mu=1}^{N}\frac{p_\mu{}^2}{2m_\mu}-\sum_{\mu=1}^{N}\frac{p_\mu{}^4}{8m_\mu{}^3c^2}+U^{(0)}+U^{(1)}, \tag{3}$$

where $U^{(0)}$ and $U^{(1)}$ are the potential terms of the zeroth and first order in $1/c^2$, respectively. The relativistic center-of-mass (c.m.) variables are defined in such a way that when expressed in terms of them, the Hamiltonian and the other generators of the Poincaré group assume the single-particle form. Krajcik and Foldy[8] have shown that to order $1/c^2$ the relativistic relations between the constituent and the c.m. variables are given by

$$\vec{r}_\mu=\vec{\rho}_\mu+\vec{R}-\frac{1}{2c^2}\left[\frac{\vec{\rho}_\mu\cdot\vec{P}}{M}\left(\frac{\vec{\pi}_\mu}{m_\mu}+\frac{\vec{P}}{2M}\right)+\text{H.c.}\right]-\frac{1}{2c^2}\sum_{\nu=1}^{N}\left[\frac{\pi_\nu^2\vec{\rho}_\mu}{2m_\nu M}+\text{H.c.}\right]+\sum_{\nu=1}^{N}\frac{(\vec{\rho}_\nu\times\vec{\pi}_\nu)\times\vec{P}}{2M^2c^2}$$
$$-\frac{\vec{\sigma}_\mu\times\vec{P}}{2m_\mu Mc^2}+\sum_{\nu=1}^{N}\frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2}+\sum_{\nu=1}^{N}\frac{\vec{\sigma}_\nu\times\vec{P}}{2M^2c^2}-\frac{1}{M}\vec{W}^{(1)}-\frac{i}{M}\left[\int_0^{\vec{P}}d\vec{P}\cdot\vec{W}^{(1)},\vec{\rho}_\mu\right], \tag{4}$$

$$\vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}+\left[\frac{\pi_\mu^2}{2m_\mu}-\frac{m_\mu}{M}\sum_\nu\frac{\pi_\nu^2}{2m_\nu}+\frac{\vec{\pi}_\mu\cdot\vec{P}}{2M}\right]\frac{\vec{P}}{Mc^2}-\frac{i}{M}\left[\int_0^{\vec{P}}d\vec{p}\cdot\vec{W}^{(1)},\vec{\pi}_\mu\right], \tag{5}$$

$$\vec{s}_\mu=\vec{\sigma}_\mu-\frac{\vec{\sigma}_\mu\times(\vec{\pi}_\mu\times\vec{P})}{2m_\mu Mc^2}-\frac{i}{M}\left[\int_0^{\vec{P}}d\vec{p}\cdot\vec{W}^{(1)},\vec{\sigma}_\mu\right], \tag{6}$$

where $\vec{P}$ is the total or the c.m. momentum of the composite system. In Eqs. (4)–(6), $\vec{W}^{(1)}$ is an operator[8] of order $1/c^2$ which forms a part of the Lorentz-boost operator. It depends explicitly on the internal interaction among the constituent particles. The internal position and momentum variables $\vec{\rho}_\mu$ and $\vec{\pi}_\mu$ are not all independent. They satisfy the constraint equations

$$\sum_{\mu=1}^{N}\vec{\pi}_\mu=0,\quad\sum_{\mu=1}^{N}m_\mu\vec{\rho}_\mu=0. \tag{7}$$

In terms of the relativistic internal and c.m. variables, the Hamiltonian of the isolated composite system should have the form

$$H=(h^2+c^2P^2)^{1/2}, \tag{8}$$

where h is the internal Hamiltonian which depends only on the internal variables.[8] We expand the internal Hamiltonian h in powers of $1/c^2$ as

$$h=Mc^2+h^{(0)}+h^{(1)}+h^{(2)}+\cdots, \tag{9}$$

where $h^{(i)}$ is of the ith order in $1/c^2$ and M is the sum of the rest masses of the constituent particles. Using Eqs. (3)–(6), (8), and (9) we obtain

$$h^{(0)}=\sum_{\mu=1}^{N}\frac{\pi_\mu^2}{2m_\mu}+U^{(0)}(\vec{\rho}_\nu,\vec{\sigma}_\nu), \tag{10}$$

$$h^{(1)}=-\sum_{\mu=1}^{N}\frac{\pi_\mu^4}{8m_\mu^3c^2}+U^{(1)}\left[\vec{r}_\mu=\vec{\rho}_\mu+\vec{R},\vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P},\vec{s}_\mu=\vec{\sigma}_\mu\right]. \tag{11}$$

We will assume that $U^{(0)}$ is independent of the internal momenta. In terms of the c.m. variables, the total angular momentum of the composite system is

$$\vec{J}=\vec{R}\times\vec{P}+\vec{j}, \tag{12}$$

where the internal angular momentum or the spin of the composite system is

$$\vec{j}=\sum_{\mu=1}^{N}(\vec{\rho}_\mu\times\vec{\pi}_\mu)+\sum_{\mu=1}^{N}\vec{\sigma}_\mu. \tag{13}$$

We define $T(t_0)$ to be the probability amplitude of finding the composite system at time t_0 in state $|A\rangle$ with the simultaneous presence of a photon of energy ω, momentum $\vec{k}$, and polarization vector $\hat{\epsilon}_\alpha$ if the

composite system was originally (at time $t=0$) in state $|B\rangle$ with no photon being present. Using the interaction Hamiltonian of Eq. (1) in the first-order perturbation theory and Eqs. (3)–(11) we have shown elsewhere[1] that the one-photon electric dipole ($E1$) transition amplitude of the stationary composite system—including the relativistic corrections of leading order—can be written as

$$T_{E1}(t_0)=\frac{1}{\sqrt{V}}c\left[\frac{2\pi}{\omega}\right]^{1/2}k_{0I}\langle A|\hat{\epsilon}_\alpha\cdot(\vec{X}_0+\vec{X}_1)|B\rangle_I\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt', \tag{14a}$$

where $|A\rangle_I$ and $|B\rangle_I$ are eigenstates of the internal Hamiltonian $h=h^{(0)}+h^{(1)}$,

$$k_0=\frac{E_B^I-E_A^I}{c}=k\left[1+\frac{k}{2Mc}\right], \tag{14b}$$

$$\vec{X}_0=\sum_{\mu=1}^{N}e_\mu\vec{\rho}_\mu \tag{14c}$$

is the electric-dipole-moment operator, and

$$\vec{X}_1=-\frac{ik}{20}\sum_{\mu=1}^{N}\frac{e_\mu}{m_\mu c}\{2(\rho_\mu{}^2\vec{\pi}_\mu+\vec{\pi}_\mu\rho_\mu{}^2)-[\vec{\rho}_\mu(\vec{\rho}_\mu\cdot\vec{\pi}_\mu)+(\vec{\pi}_\mu\cdot\vec{\rho}_\mu)\vec{\rho}_\mu]\}$$
$$+\frac{iQ}{\omega_{BA}}\left[\vec{\nabla}_p U^{(1)}\Big|_{p=0}-\frac{1}{2Mc^2}\sum_{\mu=1}^{N}\frac{\pi_\mu{}^2}{m_\mu{}^2}\vec{\pi}_\mu\right]. \tag{14d}$$

It should be noted that the momentum k_0 is slightly different from the photon momentum k because of the recoil of the composite system as a result of the photon emission. When we go beyond the extreme nonrelativistic limit we can no longer neglect the recoil effect. In Eqs. (14), $\vec{X}_0$ gives the nonrelativistic limit of the $E1$ transition operator while $\vec{X}_1$ represents its relativistic correction of order $1/c^2$. In Eq. (14d) $Q=\sum_{\mu=1}^{N}e_\mu$ is the net electric charge of the composite system. Equation (14) is quite interesting because through it we have a compact expression for the $E1$ transition amplitude of any composite system of Dirac particles bound by any kind of internal interaction which can be represented by a potential. A particularly interesting aspect of Eq. (14) is the second term on the right-hand side of Eq. (14d) which depends explicitly on the interaction term $U^{(1)}$. It comes specifically from the use of the relativistic c.m. variables. Since it is proportional to the total charge Q, it vanishes when Q becomes zero. As a result, the relativistic correction to the $E1$ amplitudes are qualitatively different for the electrically charged and neutral composite systems. Also the relativistic corrections given by X_1 of Eq. (14d) are independent of spin except for a possible spin dependence coming from the term involving $U^{(1)}$. In any case, for electrically neutral composite systems the relativistic corrections to the $E1$ transition operator are all spin-independent.

In this paper we will apply Eqs. (14) to calculate the relativistic corrections of leading order to the one-photon $E1$ transitions of charmonium.[16–19] This application should be interesting for several reasons. First of all, the relativistic corrections, in general, are expected to be important for charmonium. The relativistic terms of order $1/c^2$ in the Hamiltonian give the fine and the hyperfine structure splittings in the energy spectrum of charmonium. From the magnitudes of these splittings we must conclude that such relativistic terms are by no means negligible. So we should also expect significant relativistic corrections of order $1/c^2$ in the one-photon transition probabilities of charmonium. Also, in the transitions we are considering, $\psi'\to\chi_j+\gamma$ and $\chi_j\to\psi+\gamma$ ($j=0,1,2$), the photon energies are not negligible compared to the rest masses of the particles involved. This situation makes it even more probable that the relativistic corrections to the amplitudes of these transitions are quite important. Second, the study of the relativistic corrections to the $E1$ decay rates may give valuable insights in the search for the correct form of the potential to be used in charmonium calculations. This will be especially true if the one-photon $E1$ transition probabilities of charmonium depend sensitively on the specific form of the potential chosen. In fact we already know that the nonrelativistic limit of the $E1$ decay rate is a rather sensitive function of the binding potential. For

example, the results of the nonrelativistic calculations[16] of the $E1$ decay rates of $\psi' \to \chi_j + \gamma$ ($j=0,1,2$) using the simple harmonic-oscillator potential agree fairly well with experiment, whereas the nonrelativistic results obtained from the use of a linear potential are too high by almost a factor of two.[18] Finding the correct form of the potential acting between the c and the $\bar{c}$ quark is one of the important problems in the phenomenological theory of charmonium. This potential should reproduce all the experimentally known properties of charmonium. So far the potential, including its relativistic corrections, was mostly considered only on the basis of the energy spectrum of charmonium.

Because of the uncertainty about the form of the confining potential in charmonium we will first derive a formula for the $E1$ decay rate valid for any form of the potential. But in order to make the numerical estimates we have to know the wave functions of charmonium and for this we have to specify the potential. In conformity with the usual practice we will here assume that there are two pieces to this potential, a confining potential which is responsible for the permanent binding of the quarks inside charmonium and a Coulomb-Breit potential[16] which is operative at short distances and which arises[16,20] due to the exchange of a massless gluon between the quark and the antiquark. Nobody has succeeded in deriving the confining potential from quantum chromodynamics. As a first attempt, in this paper, we choose the confining potential to be the potential of an isotropic simple harmonic oscillator. This model gives[17] the energy spectrum of charmonium—including the fine and the hyperfine structure splittings—to a fair degree of accuracy, although not as good as the results from a linear potential. Moreover the results of the nonrelativistic calculations[16] of the $E1$ decay rates of $\psi' \to \chi_j + \gamma$ using the simple-harmonic-oscillator (SHO) wave functions agree better with experiment than those obtained from other confining potentials. In order to show that this agreement is not just accidental and that such nonrelativistic results are reliable we should show that the leading relativistic corrections are small. As a result of our calculations using this model and Eqs. (14) we find that the corrections to the usual nonrelativistic results[16] of the $E1$ decay rates are quite large (-25% to -62% for $\psi' \to \chi_j + \gamma$ and -43% to -55% for $\chi_j \to \psi + \gamma$).

The format of the rest of the paper is as follows. In Sec. II we specialize Eq. (14) to the specific case of charmonium where $N=2$, the masses are equal, and the net charge Q of the composite system is zero. This simplifies the formula considerably. We then derive an interesting expression for the $E1$ decay rates—including the leading relativistic corrections—which is valid for any electrically neutral two-particle composite system with equal constituent masses and bound by an arbitrary potential. In Sec. III we specify the details of our model of charmonium and then using this model we calculate the matrix elements involved in the formulas for the $E1$ decay rates. In Sec. IV we make the numerical estimates and compare them with experiment. The results of our numerical calculations are summarized in Tables I and II. Finally, in Sec. V, we make some concluding remarks.

II. FORMULA FOR THE $E1$ DECAY RATE OF CHARMONIUM INCLUDING THE LEADING RELATIVISTIC CORRECTIONS

In this section we want to derive a general formula for the one-photon $E1$ decay rate of charmonium applicable for any interaction potential. For this purpose we first rewrite Eqs. (14) for the special case of charmonium, where

$$\begin{aligned} &N=2, \quad e_1=-e_2=+e_q=+\tfrac{2}{3}e, \\ &m_1=m_2=m, \quad \mu=\frac{m_1 m_2}{m_1+m_2}=\tfrac{1}{2}m, \\ &\vec{\pi}_1=-\vec{\pi}_2=\vec{\pi}, \quad \vec{\rho}_1-\vec{\rho}_2=\vec{\mathrm{q}}, \\ &\vec{\rho}_1=-\vec{\rho}_2=\tfrac{1}{2}\vec{\mathrm{q}}. \end{aligned} \tag{15}$$

It should be noted that $\vec{\pi}$ and $\vec{\mathrm{q}}$ as defined are canonically conjugate to each other. In Eqs. (15) e is the charge of the proton. Using Eqs. (15), the $E1$ transition operators $\vec{\mathrm{X}}_0$ and $\vec{\mathrm{X}}_1$ of Eqs. (14c) and (14d) become

$$\vec{\mathrm{X}}_0 = e_q \vec{\mathrm{q}}, \tag{16}$$

$$\vec{\mathrm{X}}_1 = -\frac{ik}{20}\frac{e_q}{mc}\{(q^2\vec{\pi}+\vec{\pi}q^2) - \tfrac{1}{2}[\vec{\mathrm{q}}(\vec{\mathrm{q}}\cdot\vec{\pi})+(\vec{\pi}\cdot\vec{\mathrm{q}})\vec{\mathrm{q}}]\}. \tag{17}$$

From Eq. (14a), the $E1$ decay rate or the transition

probability per unit time for the charmonium to make an $E1$ one-photon transition from state $|B\rangle_I$ to state $|A\rangle_I$ (irrespective of the direction of the emitted photon's momentum and polarization) is given by

$$W_{BA}=\frac{1}{\tau_{BA}}=\tfrac{4}{3}k_0{}^2k\,|\,{}_I\langle A\,|\,\vec{X}_0+\vec{X}_1\,|\,B\rangle_I\,|^2. \quad (18)$$

Since the operator $\vec{X}_1$ is of order $1/c^2$ compared to $\vec{X}_0$ we obtain, correct to relative order $1/c^2$,

$$|\,{}_I\langle A\,|\,\vec{X}_0+\vec{X}_1\,|\,B\rangle_I\,|^2=|\,{}_I\langle A\,|\,\vec{X}_0\,|\,B\rangle_I\,|^2+2\,\mathrm{Re}({}_I\langle A\,|\,\vec{X}_0\,|\,B\rangle_I\cdot{}_I\langle A\,|\,\vec{X}_1\,|\,B\rangle_I^*), \quad (19)$$

where the asterisk denotes the complex conjugate. In Eqs. (18) and (19) the state vectors $|A\rangle_I$ and $|B\rangle_I$ are eigenstates of the internal Hamiltonian $h=h^{(0)}+h^{(1)}$. Let us write

$$|A\rangle_I=|A\rangle_0+|A\rangle_1,\quad |B\rangle_I=|B\rangle_0+|B\rangle_1, \quad (20)$$

where $|A\rangle_0$ and $|B\rangle_0$ are eigenstates of the nonrelativistic internal Hamiltonian $h^{(0)}$, and $|A\rangle_1$ and $|B\rangle_1$ are their relativistic corrections of order $1/c^2$. Since the second term on the right-hand side of Eq. (19) is already of relative order $1/c^2$, and since we are interested in calculations correct only up to relative order $1/c^2$, we can use the eigenfunctions $|A\rangle_0$ and $|B\rangle_0$ in the evaluation of the matrix elements in this term. But in the calculation of the matrix element of the first term in Eq. (19) we should also include the relativistic corrections $|A\rangle_1$ and $|B\rangle_1$. Thus, correct to relative order $1/c^2$,

$$|\,{}_I\langle A\,|\,\vec{X}_0\,|\,B\rangle_I\,|^2=|\,{}_0\langle A\,|\,\vec{X}_0\,|\,B\rangle_0\,|^2+2\,\mathrm{Re}({}_0\langle A\,|\,\vec{X}_0\,|\,B\rangle_0\cdot{}_0\langle A\,|\,\vec{X}_0\,|\,B\rangle_1^* +{}_0\langle A\,|\,\vec{X}_0\,|\,B\rangle_0\cdot{}_1\langle A\,|\,\vec{X}_0\,|\,B\rangle_0^*). \quad (21)$$

Using Eqs. (16)–(21) and the result that

$$[\vec{q},h^{(0)}]=\frac{i}{\mu}\vec{\pi}, \quad (22)$$

we obtain, correct to relative order $1/c^2$,

$$\frac{1}{\tau_{BA}}=\tfrac{4}{3}e_q{}^2k_0{}^3\Big[\,|\,{}_0\langle A\,|\,\vec{q}\,|\,B\rangle_0\,|^2-\frac{k_0}{2Mc}\,|\,{}_0\langle A\,|\,\vec{q}\,|\,B\rangle_0\,|^2 +2\,\mathrm{Re}({}_0\langle A\,|\,\vec{q}\,|\,B\rangle_0\cdot\langle A\,|\,\vec{q}\,|\,B\rangle_1^*+{}_0\langle A\,|\,\vec{q}\,|\,B\rangle_0\cdot{}_1\langle A\,|\,\vec{q}\,|\,B\rangle_0^*) -\tfrac{1}{10}k_0{}^2\mathrm{Re}({}_0\langle A\,|\,\vec{q}\,|\,B\rangle_0\cdot{}_0\langle A\,|\,q^2\vec{q}\,|\,B\rangle_0^*) +\frac{3}{40}\frac{\omega_{BA}^I}{\mu c^2}\mathrm{Im}({}_0\langle A\,|\,\vec{q}\,|\,B\rangle_0\cdot{}_0\langle A\,|\,\vec{q}(\vec{q}\cdot\vec{\pi}+\vec{\pi}\cdot\vec{q})+\mathrm{H.c.}\,|\,B\rangle_0^*)\Big], \quad (23)$$

where k_0 is given by Eq. (14b) and

$$\omega_{BA}^I=E_B^I-E_A^I=ck_0\,. \quad (24)$$

The one-photon $E1$ decay rate given by Eq. (23) is applicable to any two-body composite system made up of spin-$\frac{1}{2}$ Dirac constituent particles of equal masses but opposite charges and bound by an arbitrary potential. In Eq. (23) the first term in the square brackets represents the nonrelativistic limit while the remaining terms represent the relativistic corrections of relative order $1/c^2$. The second term in the square brackets is specifically due to the recoil momentum of the composite system as a result of the emission of the photon. The third term is due to the relativistic modification of the wave function. Although Eq. (23) is valid for arbitrary internal interaction, in order to make numerical estimates of the matrix elements in this equation, we need specific expressions for the wave functions and for this we have to have a specific Hamiltonian, including the relativistic terms of order $1/c^2$. Once the Hamiltonian is specified, the remaining generators of the Poincaré group should be so chosen that they satisfy the Lie algebra of the group to order $1/c^2$. The important physical constraint in choosing the Hamiltonian is that it

should, at least to a reasonable degree, reproduce the known energy spectrum of charmonium. In the next section we describe a specific Hamiltonian, and using its eigenfunctions, we then calculate the matrix elements of Eq. (23) for the $E1$ decays $\psi' \to \chi_j + \gamma$ and $\chi_j \to \psi + \gamma$ $(j=0,1,2)$.

III. A SPECIFIC MODEL OF AN APPROXIMATELY RELATIVISTIC HAMILTONIAN AND THE CALCULATION OF THE RELEVANT MATRIX ELEMENTS

It is possible that quantum chromodynamics (QCD) (Refs. 16,17,20) is the fundamental theory of strong interactions. But, unfortunately, because of the complexities involved in a non-Abelian gauge theory, nobody has so far calculated from it the potential acting between a quark and an antiquark in a completely convincing manner. This situation makes the phenomenological theory of charmonium even more valuable. In accordance with the usual phenomenological approaches[16–18,20] we will here assume that the potential energy of a $q\bar{q}$ system should be

$$U = U_{\mathrm{AF}} + U_{\mathrm{C}}, \tag{25}$$

where U_{AF} is the so-called "asymptotically free" part of the potential which is applicable at short distances or at high values of the momentum transfer between the quark and the antiquark, while U_{C} is the confining potential which is applicable at large distances and is responsible for the permanent binding between the q and the $\bar{q}$. We assume U_{AF} to be due to the exchange of a massless vector gluon between the quark and the antiquark. So to order $1/c^2$, U_{AF} is exactly similar to the Fermi-Breit potential[16] which arises due to the exchange of a massless photon between the electron and the positron. Specific calculations[16,20] will give

$$\begin{aligned}
U_{\mathrm{AF}} = {} & \frac{K}{|\vec{r}_1-\vec{r}_2|} - \frac{K}{4m_1m_2c^2}\left[\vec{p}_1\frac{1}{|\vec{r}_1-\vec{r}_2|}\cdot\vec{p}_2 + \mathrm{H.c.}\right] \\
& - \frac{K}{4m_1m_2c^2}\left[\vec{p}_1\cdot(\vec{r}_1-\vec{r}_2)\frac{1}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)\cdot\vec{p}_2 + \mathrm{H.c.}\right] \\
& - \frac{K}{m_1m_2c^2}\frac{\vec{s}_2\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_1}{|\vec{r}_1-\vec{r}_2|^3} - \frac{K}{2m_1^2c^2}\frac{\vec{s}_1\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_1}{|\vec{r}_1-\vec{r}_2|^3} \\
& + \frac{K}{m_1m_2c^2}\frac{\vec{s}_1\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_2}{|\vec{r}_1-\vec{r}_2|^3} + \frac{K}{2m_2^2c^2}\frac{\vec{s}_2\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_2}{|\vec{r}_1-\vec{r}_2|^3} \\
& - \frac{K\pi}{m_1m_2c^2}\left(1+\tfrac{8}{3}\vec{s}_1\cdot\vec{s}_2\right)\delta^{(3)}(\vec{r}_1-\vec{r}_2) \\
& + \frac{K}{m_1m_2c^2}\frac{1}{|\vec{r}_1-\vec{r}_2|^3}\left[\vec{s}_1\cdot\vec{s}_2 - \frac{3\vec{s}_1\cdot(\vec{r}_1-\vec{r}_2)\vec{s}_2\cdot(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}\right],
\end{aligned} \tag{26a}$$

where

$$K = (e_1e_2 - \tfrac{4}{3}c\alpha_s). \tag{26b}$$

In Eq. (26b), α_s represents a dimensionless strong-interaction running coupling constant which depends on the c.m. energy of the $q\bar{q}$ system. Its value can be calculated, in principle, from asymptotically free QCD. For the problem at hand we will take its value to be[17]

$$\alpha_s \simeq 0.2, \tag{27}$$

where $U_{\mathrm{C}}^{(0)}$ is of zeroth order in $1/c^2$ while $U_{\mathrm{C}}^{(1)}$ is of first order in $1/c^2$. In this paper we choose $U_{\mathrm{C}}^{(0)}$ to be the isotropic simple harmonic (SH) potential. That is,

$$U_{\mathrm{C}}^{(0)} = \tfrac{1}{2}\mu\omega_0^2|\vec{r}_1-\vec{r}_2|^2, \tag{28}$$

where ω_0 is an unknown angular frequency which, in principle, can be obtained from the known energy spectrum of charmonium and also by other means[16,17] and μ stands for the reduced mass given by Eqs. (15). For the moment we will let $U_C^{(1)}$ to be arbitrary. It should be chosen on the basis of phenomenological considerations and in accordance with the constraint that the Hamiltonian together with the other generators of the Poincaré group satisfy the appropriate Lie algebra to order $1/c^2$. We write the Hamiltonian of charmonium, correct to order $1/c^2$, as

$$H=\sum_{\mu=1}^{2}\frac{p_\mu^{\,2}}{2m_\mu}-\sum_{\mu=1}^{2}\frac{p_\mu^{\,4}}{8m_\mu^{\,3}c^2}+\tfrac{1}{2}\mu\omega_0^{\,2}|\vec{r}_1-\vec{r}_2|^2+U_C^{(1)}+U_{AF}, \tag{29}$$

where U_{AF} is given by Eq. (26a).

In order to calculate the matrix elements in Eq. (23) we need the internal energy eigenfunctions of charmonium correct to order $1/c^2$. For this we need to know the internal Hamiltonians $h^{(0)}$ and $h^{(1)}$. In order to calculate them from Eq. (29) we first note, using Eq. (8), that

$$h=\lim_{P\to 0}H. \tag{30}$$

From Eqs. (4)–(6) we also obtain

$$\lim_{P\to 0}(\vec{r}_1-\vec{r}_2)=\vec{q},\quad \lim_{P\to 0}\vec{p}_\mu=\vec{\pi}_\mu,\quad \lim_{P\to 0}\vec{s}_\mu=\vec{\sigma}_\mu. \tag{31}$$

Then using Eqs. (30) and (31), we get from Eqs. (29) and (26a)

$$h^{(0)}=\frac{\pi^2}{2\mu}+\tfrac{1}{2}\mu\omega_0^{\,2}q^2+\frac{K}{q} \tag{32}$$

and

$$\begin{aligned}h^{(1)}=&-\frac{\pi^4}{4m^3c^2}+\frac{K}{2m^2c^2}\left[\vec{\pi}\frac{1}{q}\cdot\vec{\pi}\right]+\frac{K}{2m^2c^2}(\vec{\pi}\cdot\vec{q})\frac{1}{q^3}(\vec{q}\cdot\vec{\pi})\\&-\frac{K\pi}{m^2c^2}(1+\tfrac{8}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2)\delta^{(3)}(\vec{q})-\frac{3K}{2m^2c^2}\frac{1}{q^3}(\vec{\sigma}_1+\vec{\sigma}_2)\cdot(\vec{q}\times\vec{\pi})\\&+\frac{K}{m^2c^2q^3}[\vec{\sigma}_1\cdot\vec{\sigma}_2-3(\vec{\sigma}_1\cdot\hat{q})(\vec{\sigma}_2\cdot\hat{q})]+\lim_{P\to 0}U_C^{(1)}.\end{aligned} \tag{33}$$

From Eq. (33) we see that $h^{(1)}$ is arbitrary to the extent that $\lim_{P\to 0}U_C^{(1)}$ is arbitrary. We have to fix $\lim_{P\to 0}U_C^{(1)}$ on the basis of phenomenological considerations. In this connection we note that many calculations[17] of the energy spectrum (including the fine and the hyperfine structure splittings) of charmonium were done with the assumption that

$$\lim_{P\to 0}U_C^{(1)}=0. \tag{34}$$

Since the results of such calculations are in fair agreement[17] with experiment we will also make this assumption. In any case it is worthwhile to find the consequences of Eq. (34) for the one-photon $E1$ transition rates of charmonium. However it must also be pointed out that in some treatments of charmonium [for example, the work of Schnitzer (Ref. 25) and some others] Eq. (34) is not assumed. We can also show that corresponding to this Hamiltonian there is an appropriate Lorentz-boost operator which when commuted with H and other generators satisfies the Lie algebra of the Poincaré group to order $1/c^2$.

Next we notice that if we define

$$\begin{aligned}\vec{l}&=\vec{q}\times\vec{\pi},\\ \vec{s}&=\vec{\sigma}_1+\vec{\sigma}_2,\\ \vec{j}&=\vec{l}+\vec{s},\end{aligned} \tag{35}$$

then the operators l^2, s^2, j^2, and j_z commute with the unperturbed internal Hamiltonian $h^{(0)}$. So we can label the eigenstates of $h^{(0)}$ by the integers l, s, and j which are related to the eigenvalues of l^2, s^2, and j^2 in the usual way. It is the convention to take the unperturbed eigenfunction of $\psi(3.1\text{ GeV})$ to be the $1\,^3S_1$ state in the notation $n\,^{2s+1}l_j$ where n is a radial quantum number. In the same nota-

tion $\psi'(3.684\ \text{GeV})$ is the $2\,^3S_1$ state while the χ states are given the following designations:

$$\chi_0(3.414\ \text{GeV}) \rightarrow 1\,^3P_0,$$

$$\chi_1(3.508\ \text{GeV}) \rightarrow 1\,^3P_1,$$

$$\chi_2(3.552\ \text{GeV}) \rightarrow 1\,^3P_2.$$

In calculating the matrix elements in Eq. (31) we need to know the eigenfunctions of $h^{(0)}$ and their relativistic corrections of order $1/c^2$ due to $h^{(1)}$. For this purpose we can treat $h^{(0)}$ as the unperturbed Hamiltonian and $h^{(1)}$ as the perturbation. Here we come across a problem. We cannot solve the eigenvalue problem of even $h^{(0)}$ in closed form. So as an approximation we take

$$h^{(0)'}=\frac{\pi^2}{2\mu}+\tfrac{1}{2}\mu\omega_0{}^2q^2 \tag{36}$$

as the unperturbed Hamiltonian, and then as perturbation we get

$$h^{(1)'}=\frac{K}{q}+h^{(1)}, \tag{37}$$

where the operator $h^{(1)}$ is given by Eq. (33). This should be a quite reasonable approximation for the problem at hand for the following reason. We are interested in evaluating the matrix elements of $\vec{q}$, $q^2\vec{q}$, etc., between the eigenfunctions of $h^{(0)}$. These matrix elements are weighted towards the larger values of q. So the short-distance (small-q) behavior of the wave function, which is determined to a great extent by the K/q term in the potential, is not very important for their calculation. The large-distance behavior of the wave functions, which is very important in this case, is mostly determined by the SH potential. So we take the unperturbed wave functions of ψ, χ_j, and ψ' to be

$$\psi\rightarrow\phi_{1\,^3S_1}=\left[\frac{4\xi^3}{\sqrt{\pi}}\right]^{1/2}e^{-\xi^2q^2/2}Y_0^0\begin{bmatrix}\chi_{++}\\ \chi_{+-}\\ \chi_{--}\end{bmatrix}, \tag{38}$$

$$\chi_0\rightarrow\phi^{(0)}_{1\,^3P_0}=\left[\frac{8\xi^3}{3\sqrt{\pi}}\right]^{1/2}\xi qe^{-\xi^2q^2/2}\left[\frac{1}{\sqrt{3}}Y_1^{-1}\chi_{++}+\frac{1}{\sqrt{3}}Y_1^1\chi_{--}-\frac{1}{\sqrt{3}}Y_1^0\chi_{+-}\right], \tag{39}$$

$$\chi_1\rightarrow\phi^{(0)}_{1\,^3P_1}=\left[\frac{8\xi^3}{3\sqrt{\pi}}\right]^{1/2}\xi qe^{-\xi^2q^2/2}\begin{bmatrix}\frac{1}{\sqrt{2}}Y_1^1\chi_{+-}-\frac{1}{\sqrt{2}}Y_1^0\chi_{++}\\ \frac{1}{\sqrt{2}}Y_1^1\chi_{--}-\frac{1}{\sqrt{2}}Y_1^{-1}\chi_{++}\\ \frac{1}{\sqrt{2}}Y_1^0\chi_{--}-\frac{1}{\sqrt{2}}Y_1^{-1}\chi_{+-}\end{bmatrix}, \tag{40}$$

$$\chi_2\rightarrow\phi^{(0)}_{1\,^3P_2}=\left[\frac{8\xi^3}{3\sqrt{\pi}}\right]^{1/2}\xi qe^{-\xi^2q^2/2}\begin{bmatrix}Y_1^1\chi_{++}\\ \frac{1}{\sqrt{2}}Y_1^1\chi_{+0}+\frac{1}{\sqrt{2}}Y_1^0\chi_{++}\\ \frac{1}{\sqrt{6}}Y_1^1\chi_{--}+\left[\frac{2}{3}\right]^{1/2}Y_1^0\chi_{+-}+\frac{1}{\sqrt{6}}Y_1^{-1}\chi_{++}\\ \frac{1}{\sqrt{2}}Y_1^0\chi_{--}+\frac{1}{\sqrt{2}}Y_1^{-1}\chi_{+-}\\ Y_1^{-1}\chi_{--}\end{bmatrix}, \tag{41}$$

and

$$\psi'\rightarrow\phi^{(0)}_{2\,^3S_1}=\left[\frac{6\xi^3}{\sqrt{\pi}}\right]^{1/2}(1-\tfrac{2}{3}\xi^2q^2)e^{-\xi^2q^2/2}Y_0^0\begin{bmatrix}\chi_{++}\\ \chi_{+-}\\ \chi_{--}\end{bmatrix}, \tag{42}$$

where

$$\xi=(\mu\omega_0)^{1/2}, \tag{43}$$

Y_1^m's are the spherical harmonics, and χ_{++}, χ_{+-}, and χ_{--} are the three spin-triplet states with the z component of the total spin $\vec{s}$ having the values $+1$, 0, and -1, respectively. In Eqs. (38)–(42) the angular and spin functions written in a column correspond to the different values of the quantum number m_j and m_j decreasing from top to bottom. The eigenvalues $h^{(0)'}$ can be written as

$$E_n^{(0)}=[2(n-1)+1+\tfrac{3}{2}]\omega_0. \tag{44}$$

We also want to know the first-order corrections to the above wave functions due to $h^{(1)'}$. In finding this we should note that one term in $h^{(1)'}$, the so-called tensor force term

$$h_T^{(1)'}=\frac{K}{m^2c^2}\frac{1}{q^3}[\vec{\sigma}_1\cdot\vec{\sigma}_2-3(\vec{\sigma}_1\cdot\hat{q})(\vec{\sigma}_2\cdot\hat{q})]=-\frac{K}{4m^2c^2}\frac{1}{q^3}S_{12} \tag{45}$$

does not commute with the orbital-angular-momentum operator 1^2 although it still commutes with s^2, j^2, j_z, and the parity operator P. So this term can mix unperturbed eigenstates with different 1 values but with the same s, j, and m_j and with the same parity. Since $h^{(1)'}$ is diagonal in the subspaces of the degenerate eigenfunctions of $1\,^3S_1$ and $1\,^3P_j$ we can apply the formulas of nondegenerate perturbation theory to find the relativistic corrections of the wave functions of these states. With the state $2\,^3S_1$ we have to be more careful. Since the unperturbed states $2\,^3S_1$ and $1\,^3D_1$ are degenerate and since $h_T^{(1)'}$ of Eq. (45) can mix these two states we have to apply the full machinery of degenerate perturbation theory. If $\phi^{(1)}_{2\,^3S_1}$ denotes the first-order correction to the wave function, due to $h^{(1)'}$, we find that

$$\phi^{(1)}_{2\,^3S_1}\simeq c_2\phi^{(0)}_{1\,^3D_1}+\beta\phi^{(0)}_{1\,^3S_1}+\sum_{k\neq 2\,^3S_1,1\,^3S_1,1\,^3D_1}\frac{|k\rangle_{00}\langle k|h^{(1)'}|\phi^{(0)}_{2\,^3S_1}\rangle}{(E^{(0)}_{2\,^3S_1}-E^{(0)}_k)}, \tag{46}$$

where

$$c_2=\frac{1}{[(a_{11}-a_{22}/a_{12})^2+1]^{1/2}} \tag{47}$$

and

$$\beta=\frac{(\phi^{(0)}_{2\,^3S_1},h^{(1)'}\phi^{(0)}_{1\,^3S_1})}{(E^{(0)}_{2\,^3S_1}-E^{(0)}_{1\,^3S_1})}. \tag{48}$$

In Eq. (47), for $i,j=1,2$,

$$a_{ij}=(\phi_i^{(0)},h^{(1)'}\phi_j^{(0)}),$$

where

$$\phi_1^{(0)}=\phi^{(0)}_{2\,^3S_1},\quad \phi_2^{(0)}=\phi^{(0)}_{1\,^3D_1}. \tag{49}$$

For the first-order corrections $\phi^{(1)}_{1\,^3P_j}$ and $\phi^{(1)}_{1\,^3S_1}$ we find

$$\phi^{(1)}_{1\,^3P_j}=-\alpha_j\phi^{(0)}_{2\,^3P_j}-\sum_{k\neq 1P,2P}\frac{|k\rangle_{00}\langle k|h^{(1)'}|1\,^3P_j\rangle_0}{(E_k^{(0)}-E^{(0)}_{1\,^3P_j})}, \tag{50}$$

$$\phi^{(1)}_{1\,^3S_1}=-\beta\phi^{(0)}_{2\,^3S_1}-\delta\phi^{(0)}_{1\,^3D_1}+\sum_{k\neq 1S,2S,1D}\frac{|k\rangle_{00}\langle k|h^{(1)'}|1\,^3S_1\rangle}{(E^{(0)}_{1S}-E_k^{(0)})}, \tag{51}$$

where

$$\alpha_j=\frac{(\phi^{(0)}_{2\,^3P_j},h^{(1)'}\phi_{1\,^3P_j})}{2\omega_0}, \tag{52}$$

$$\delta=\frac{(\phi_{1\,^3S_1},h^{(1)'}\phi^{(0)}_{1\,^3D_1})}{2\omega_0}, \tag{53}$$

and

$$\phi^{(0)}_{2\,^3P_j}=\left[\frac{20\xi^3}{3\sqrt{\pi}}\right]^{1/2}\xi q(1-\tfrac{2}{5}\xi^2q^2)e^{-\xi^2q^2/2}\mathcal{Y}_{j,m_j}, \tag{54}$$

where $\mathcal{Y}_{jm_j}$ the angular and the spin-dependent part of the wave function, can be read off directly from Eqs. (39)–(41). Using Eqs. (38)–(42) for the wave functions and Eqs. (33) and (37) for the operator $h^{(1)'}$ we can evaluate the matrix elements of Eqs. (48), (49), (52), and (53). In doing this we have to be careful about the matrix element of the spin-spin interaction (the interaction energy of two stationary spin magnetic dipole moments) given by

$$V_{\sigma_1\sigma_2}=\frac{K}{m^2c^2}\frac{1}{q^3}[\vec{\sigma}_1\cdot\vec{\sigma}_2-3(\vec{\sigma}_1\cdot\hat{q})(\vec{\sigma}_2\cdot\hat{q})]. \tag{55}$$

First of all we should note that Eq. (55) is valid only for $q\neq0$. So in evaluating the matrix element of $V_{\sigma_1\sigma_2}$ between the S states we have to eliminate a small region around $q=0$ (in which region the integrand blows up) in the integral involved. Then we find, from the angular integrations, that the matrix element of $V_{\sigma_1\sigma_2}$ is zero between any S states. The matrix element of $V_{\sigma_1\sigma_2}$ between other states can be calculated by well-known techniques.[21] We now write down the results of our calculations for β, α_j, δ, a_{11}, a_{22}, and a_{12}:

$$\beta=\left[\frac{2}{3\pi}\right]^{1/2}\frac{K\xi}{2\omega_0}-\frac{5\sqrt{6}\xi^4}{16m^3c^2\omega_0}-\frac{1}{2\sqrt{6\pi}}\frac{K\xi^3}{m^2c^2\omega_0}, \tag{56}$$

$$\gamma_j=\left[\frac{8}{5\pi}\right]^{1/2}\frac{K\xi}{6\omega_0}-\frac{7\sqrt{10}}{16}\frac{\xi^4}{m^3c^2\omega_0}+\frac{\sqrt{2}}{10}\frac{K\xi^3}{\sqrt{5\pi}m^2c^2\omega_0}\{\tfrac{56}{3}+\tfrac{3}{2}[j(j+1)-4][j(j+1)-8]\}, \tag{57}$$

$$\delta=-\left[\frac{2}{15\pi}\right]^{1/2}\frac{K\xi^3}{m^2c^2\omega_0}, \tag{58}$$

$$a_{11}=\frac{5K\xi}{3\sqrt{\pi}}-\frac{75\xi^4}{16m^3c^2}+\frac{K\xi^3}{6\sqrt{\pi}m^2c^2}, \tag{59}$$

$$a_{21}=a_{12}=-\frac{2}{3\sqrt{5\pi}}\frac{K\xi^3}{m^2c^2}, \tag{60}$$

$$a_{22}=\frac{16K\xi}{15\sqrt{\pi}}-\frac{15\xi^4}{16m^3c^2}+\frac{56K\xi^3}{15\sqrt{\pi}m^2c^2}. \tag{61}$$

The first terms on the right-hand sides of Eqs. (56), (57), (59), and (61) represent the matrix elements of the Coulomb-type term K/q in $h^{(1)'}$ whereas the second and third terms are, respectively, the matrix elements of the relativistic correction to the kinetic energy and of the interaction terms of order $1/c^2$ in $h^{(1)}$. In Eqs. (58) and (60) only the term $V_{\sigma_1\sigma_2}$ of Eq. (55) contributes to the matrix elements.

We are now in a position to calculate all the matrix elements in Eq. (23). Below we give the results for the different decays.

A. $\psi'\to\chi_j+\gamma$

In calculating the experimentally observed decay rate, $1/\tau_{BA}$, from Eq. (23) we must sum the right-hand side of Eq. (23) over the internal-angular-momentum states of the final state $|A\rangle_I$ and average it over the

internal-angular-momentum states of the initial state $|B\rangle_I$. In the following we will represent such quantities by adding the symbol $\sum_{\text{spin}}$ to their left. We also notice that the SHO wave functions are such that the operator $\vec{q}$ cannot connect the energy eigenfunctions which differ in their energies by more than ω_0. So the infinite sums in Eqs. (46), (50), and (51) will not contribute to the matrix elements ${}_0\langle A|\vec{q}|B\rangle_1$ and ${}_1\langle A|\vec{q}|B\rangle_0$ in Eq. (23). This circumstance simplifies our calculations considerably. Using Eqs. (38)—(42) and (54) we then find

$$\mathrm{Im}\sum_{\text{spin}}{}_0\langle\chi_j|\vec{q}|\psi'\rangle_0{}_0\langle\chi_0|\vec{q}(\vec{q}\cdot\vec{\pi}+\vec{\pi}\cdot\vec{q})+\text{H.c.}|\psi'\rangle_0^*=\tfrac{4}{9}(2j+1)\frac{1}{\xi^2}, \tag{62}$$

$$\mathrm{Re}\sum_{\text{spin}}{}_0\langle\chi_j|\vec{q}|\psi'\rangle_0{}_0\langle\chi_j|q^2\vec{q}|\psi'\rangle_0^*=\tfrac{5}{9}(2j+1)\frac{1}{\xi^4}, \tag{63}$$

$$\sum_{\text{spin}}|{}_0\langle\chi_j|\vec{q}|\psi'\rangle_0|^2=(2j+1)\frac{1}{9\xi^2}, \tag{64}$$

and

$$2\,\mathrm{Re}\sum_{\text{spin}}\{{}_0\langle\chi_j|\vec{q}|\psi'\rangle_0{}_0\langle\chi_j|\vec{q}|\psi'\rangle_1^*+{}_0\langle\chi_j|\vec{q}|\psi'\rangle_0{}_1\langle\chi_j|\vec{q}|\psi'\rangle_0^*\}=\frac{(2j+1)}{9\xi^2}(\sqrt{10}\gamma_j-\sqrt{6}\beta+2\sqrt{5}c_2). \tag{65}$$

Combining the results of Eqs. (23) and (62)—(65) we obtain the sought-after result for the decay rate, namely,

$$\frac{1}{\tau_{\psi\chi_j}}=\tfrac{4}{3}e_q{}^2k_0{}^3\frac{1}{9\xi^2}(2j+1)\left[1+(\sqrt{10}\gamma_j-\sqrt{6}\beta+2\sqrt{5}c_2)-\frac{k_0}{2Mc}-\frac{1}{2}\frac{k_0{}^2}{\xi^2}+\frac{3}{10}\frac{\omega_{BA}^I}{\mu c^2}\right]. \tag{66}$$

The first term in the square brackets gives the nonrelativistic value while the remaining terms represent the corrections in our order of approximation. The second term in the square brackets also contains the first-order corrections due to the K/q term in the Hamiltonian. The remaining terms are purely relativistic.

B. $\chi_j\to\psi+\gamma$

First of all we note that due to the special nature of the SHO wave functions the infinite sum in Eq. (51) does not contribute to the matrix element ${}_1\langle\psi|\vec{q}|\chi_j\rangle_0$ which is the same as ${}_1\langle A|\vec{q}|B\rangle_0$ in Eq. (23). For the same reason, ${}_0\langle\psi|\vec{q}|\chi_j\rangle_1$ or ${}_0\langle A|\vec{q}|B\rangle_1$ is zero with the expressions given by Eq. (50) for $|\chi_j\rangle_1$. We can evaluate the integrals involved in the calculations of the matrix elements in Eq. (23) by making use of Eqs. (38)—(42) and (51). After some straightforward calculations, we find

$$\sum_{\text{spin}}|{}_0\langle\psi|\vec{q}|\chi_j\rangle_0|^2=\frac{1}{2\xi^2}, \tag{67}$$

$$2\,\mathrm{Re}\sum_{\text{spin}}({}_0\langle\psi|\vec{q}|\chi_j\rangle_0{}_1\langle\psi|\vec{q}|\chi_j\rangle_0^*+{}_0\langle\psi|\vec{q}|\chi_j\rangle_0{}_0\langle\psi|\vec{q}|\chi_j\rangle_1^*)=\frac{2}{\sqrt{6}\xi^2}(\beta+\sqrt{5}\delta), \tag{68}$$

$$\mathrm{Re}\left[\sum_{\text{spin}}{}_0\langle\psi|\vec{q}|\chi_j\rangle_0^*\right]=\frac{5}{4}\frac{1}{\xi^4}, \tag{69}$$

and

$$\mathrm{Im}\left[\sum_{\text{spin}}{}_0\langle\psi|\vec{q}|\chi_j\rangle_0{}_0\langle\psi|\vec{q}(\vec{q}\cdot\vec{\pi}+\vec{\pi}\cdot\vec{q})+\text{H.c.}|\chi_j\rangle_0^*\right]=-\frac{1}{\xi^2}. \tag{70}$$

We note that the results, in contrast to those of Eqs. (62)—(65) are independent of the quantum number j. Substituting Eqs. (67)—(70) in Eq. (23), we obtain

$$\frac{1}{\tau_{\chi_j\psi}}=\tfrac{4}{3}e_q{}^2k_0{}^3\frac{1}{2\xi^2}\left[1+\frac{4}{\sqrt{6}}(\beta+\sqrt{5}\delta)-\frac{k_0}{2Mc}-\frac{1}{4}\frac{k_0{}^2}{\xi^2}-\frac{3}{20}\frac{\omega_{BA}^I}{\mu c^L}\right]. \tag{71}$$

The first term in the square brackets gives the nonrelativistic value while the remaining terms represent the corrections. The second term in the square brackets also contains the first-order correction due to the K/q term in $h^{(1)'}$. The remaining terms are purely relativistic corrections of relative order v^2/c^2.

In the next section we attempt the numerical evaluations of Eqs. (66) and (71) for the decay rates and compare them with experiment whenever possible.

IV. NUMERICAL EVALUATION OF THE FORMULAS FOR THE DECAY RATES

In order to make the numerical evaluation of Eqs. (66) and (71) we should know the numerical values of the constants ξ, $\mu=\frac{1}{2}m$ (m = mass of the c quark), β, γ_j ($j=0,1,2$), and c_2. In order to calculate the constants β, γ_j, δ, and c_2 through Eqs. (56)–(61) and (47) we should also know the value of the constant K.

In the literature[16–18] the mass of the c quark is often taken to be

$$m\simeq 1.65 \text{ GeV}. \tag{72}$$

We will also adopt this value. There is, of course, some uncertainty about it. Several values, ranging from 1.1 to 2.1 GeV, have been used.[16,18] The order of magnitude of the relativistic corrections will not be affected by this variation.

The value of ξ can be estimated in at least two different ways.

(1) We recall Eq. (43) for ξ. For ω_0, the oscillator spacing, we take a rough mean between the values implied by the $1S$-$1P$ spacing and the $1S$-$2S$ spacing. Then we get

$$\omega_0\simeq 0.33 \text{ GeV}. \tag{73}$$

From Eq. (72) we also have

$$\mu=\tfrac{1}{2}m\simeq 0.83 \text{ GeV}. \tag{74}$$

Substituting Eqs. (73) and (74) in Eq. (43), we get

$$\xi\simeq 0.52 \text{ GeV}/c\,. \tag{75}$$

(2) There is also another way to calculate ξ. If we assume that the decay $\psi\to e^+ + e^-$ proceeds by the mechanism $\psi\to c+\bar{c}\to\gamma\to e^+ + e^-$, and that the c and the $\bar{c}$ quarks move nonrelativistically in ψ, the decay rate for $\psi\to e^+ + e^-$ is given by the formula[16,17,22]

$$\Gamma(\psi\to e^+e^-)=4\frac{e_q{}^2e^2}{M_0{}^2c^2}\,|R_{1S}(0)|^2, \tag{76}$$

where M_0 is the mass of the ψ particle and R_{1S} is its radial wave function, obtainable from Eq. (38). If we take[16]

$$\Gamma_{\text{exp}}(\psi\to e^+e^-)\simeq 6.5 \text{ keV} \tag{77}$$

and

$$e_q=+\tfrac{2}{3}e, \tag{78}$$

we obtain

$$|R_{1S}(0)|^2\simeq 0.48. \tag{79}$$

On the other hand, from Eq. (38),

$$|R_{1S}(0)|^2=\frac{4\xi^3}{\sqrt{\pi}}. \tag{80}$$

Comparing Eqs. (79) and (80),

$$\xi\simeq 0.6 \text{ GeV}/c, \tag{81}$$

in good agreement with the value given by Eq. (75). For our numerical estimates, we take the mean of the values given by Eqs. (75) and (81). That is, we assume

$$\xi\simeq 0.56 \text{ GeV}/c. \tag{82}$$

The constant K in Eqs. (56)–(61), according to Eq. (26b), is given by

$$K=-(e_q{}^2+\tfrac{4}{3}c\alpha_s). \tag{83}$$

Then, using Eqs. (26c) and (78), we obtain

$$\frac{K}{c}\simeq -0.27. \tag{84}$$

We have calculated the constants β, γ_j, c_2, and δ using Eqs. (56)–(61), (72), (73), (82), and (83). We obtain

$$\begin{aligned}&\beta\simeq -0.15,\quad \gamma_0\simeq -0.27,\\ &\gamma_1\simeq -0.21,\quad \gamma_2\simeq -0.17,\\ &\delta\simeq +0.011,\quad c_2\simeq -0.029.\end{aligned} \tag{85}$$

Using the numerical values given by Eqs. (72), (73), (82), (84), and (85), we have then made the numerical evaluations of Eqs. (66) and (71). For the energy differences ω_{BA}^{I} and the wave vectors k_0 in the formulas, we use their experimental values, obtained from the known mass differences. We summarize our results in Tables I and II.

Some comments are in order about Tables I and II. The first-order corrections due to the K/q term in $h^{(1)'}$ are given by the expressions $(\sqrt{10}\gamma_j^{\text{Coul}}-\sqrt{6}\beta^{\text{Coul}})$ for $\psi'\to\chi_j+\gamma$ decays and by $(4/\sqrt{6})\beta^{\text{Coul}}$ for $\chi_j\to\psi+\gamma$ decays where

$$\beta^{\text{Coul}}=(2/3\pi)^{1/2}\frac{K\xi}{2\omega_0}\simeq -0.11 \tag{86}$$

and

$$\gamma_j^{\text{Coul}}=\tfrac{1}{3}(2/5\pi)^{1/2}\frac{K\xi}{\omega_0}\simeq -0.06. \tag{87}$$

In Table I for $\psi'\to\chi_j+\gamma$ decays, the results of the nonrelativistic theory are somewhat larger than the experimental values. The corrections are large and negative, ranging from -62% for $\psi'\to\chi_0+\gamma$ to -25% for $\psi'\to\chi_2+\gamma$. The corrected values for the decay rates agree fairly well with the experimental values. The bulk of the correction comes from the relativistic modification of the wave function which tends to decrease the rates by as much as -65% for $\psi'\to\chi_0+\gamma$. The corrections in Table II for $\chi_j\to\psi+\gamma$ are also large and negative, ranging from -55% for $\chi_2\to\psi+\gamma$ decay rate to -43% for $\chi_0\to\psi+\gamma$. But in this case most of the corrections come from the relativistic modification of the $E1$ transition operator and due to the modification of the wave function by the K/q term in $h^{(1)'}$. The correction due to the relativistic modification of the wave function amounts to only -7%. Since the experimental values of the decay rates are not known for $\chi_j\to\psi+\gamma$ decays, we are not able to compare the predictions with experiment.

V. CONCLUDING REMARKS

In calculating the constant ξ from Eqs. (76) and (77) we had assumed that $R_{1S}(0)$ was given by the ground-state harmonic-oscillator wave function. This assumption may introduce some error in the calculation, especially because at small separations (or for small values of q) the wave function should be significantly influenced by the Coulomb-type potential K/q which is operative in that region. So it may seem that the value of ξ obtained from the energy spectrum using the formula $\xi=(\mu\omega_0)^{1/2}$ is more reliable. In Sec. IV we have seen that in this way we would get $\xi\simeq 0.52$ GeV/c. In fact if we calculate the corrected $R_{1S}(0)$ which includes the first-order (in perturbation theory) corrections to the SHO wave function due to the K/q term in the potential, we obtain, $\xi\simeq 0.53$ GeV/c, remarkably close to the above value for ξ. But both of these values are quite close to the value we have adopted for ξ, namely, $\xi\simeq 0.56$ GeV/c. In fact detailed calculations show that changing ξ from 0.56 to 0.52 GeV/c will change our numerical estimates of the decay rates by less than 3%. This situation comes about because decreasing ξ increases the nonrelativistic limit of the decay rate as well as its relativistic corrections which have the opposite sign to the nonrelativistic term, so that the corrected decay rate remains almost the same.

Another important thing to mention is that some authors[18,23] take the spin-orbit term in the expression for $h^{(1)}$ to be

$$V_{LS}=\frac{2}{m^2c^2}\vec{1}\cdot\vec{s}\left[\frac{1}{q}\frac{d}{dq}\right]\times\left[U_{\text{AF}}^{(0)}(q)-\tfrac{1}{4}U_{\text{C}}^{(0)}(q)\right]. \tag{88}$$

In Eqs. (33) and (37) we had neglected the contribution from $U_{\text{C}}^{(0)}(q)$ to V_{LS}, due to the effect of Thomas precession.[24] But inclusion of this contribution will not make any change in our calculations in the previous sections; when $U_{\text{C}}^{(0)}(q)$ is the SH potential $\frac{1}{2}\mu\omega_0^2q^2$, its contribution to V_{LS} given by Eq. (88) is independent of q, and therefore its matrix elements between states whose radial wave functions are orthogonal simply vanish. Of course this modification of the LS term in the potential will affect the fine and the hyperfine structure splittings of the energy levels—presumably in the favorable direction.[18]

We should also remark that some authors[17,25] attribute a large anomalous magnetic moment to the quarks. We do not see much theoretical justification for this point of view. We had assumed throughout this paper that the quarks are point Dirac particles and hence have either zero or very small anomalous magnetic moments. If the quarks indeed have large anomalous magnetic moments we should modify our starting Eq. (1). There will be additional spin-dependent terms in the interaction Hamiltonian of Eq. (1).

TABLE I. Numerical estimate of the decay rate for $\psi' \to \chi_j + \gamma$ given by Eq. (66). r = (correction/nonrelativistic value).

Decay	$\psi' \to \chi_0 + \gamma$	$\psi' \to \chi_1 + \gamma$	$\psi' \to \chi_2 + \gamma$
Photon energy ω_{BA}^I (GeV)	0.27	0.18	0.13
Nonrelativistic value of $1/\tau_{BA}$ (keV)	31	27	19
Corrections			
Corrections due to the modification of the wave function			
Correction due to the k/q term $(\sqrt{5}\gamma_j^{\text{Coul}} - \sqrt{6}\beta^{\text{Coul}})$ (r_1)	+9%	+9%	+9%
Correction due to relativistic modification (r_2)	−65%	−46%	−34%
Total correction $(r_1 + r_2)$	−56%	−37%	−25%
Correction due to recoil term $(-k_0/2Mc)$ (r_3)	−4%	−3%	−2%
Correction due to the relativistic term $(-\frac{1}{2}k_0^2/\xi^2)$ (r_4)	−12%	−5%	−3%
Correction due to the relativistic term $(\frac{3}{10}\omega_{BA}^I/\mu c^2)$ (r_5)	+10%	+6%	+5%
Total relativistic correction $(r_R = r_2 + r_3 + r_4 + r_5)$	−71%	−48%	−34%
Total correction $(r_1 + r_R)$	−62%	−39%	−25%
Predicted decay rate including the corrections (keV)	12	16	14
Experimental value of decay rate (Ref. 31) (keV)	15.5±5	15.3±4.1	15.0±4.3

TABLE II. Numerical estimate of the decay rate for $\chi_j+\psi+\gamma$ given by Eq. (71). r = (correction/nonrelativistic value).

Decay Photon energy ω_{BA}^I (GeV)	$\chi_0\to\psi+\gamma$ 0.315	$\chi_1\to\psi+\gamma$ 0.410	$\chi_2\to\psi+\gamma$ 0.450
Nonrelativistic value of $(1/\tau_{BA})_{\rm NR}$ (keV)	214	475	628
Corrections			
Corrections due to modification of wave function			
Correction due to k/q term $[(4/\sqrt{6})\beta^{\rm Coul}]$ (r_1)	−17%	−17%	−17%
Correction due to relativistic modification (r_2)	−7%	−7%	−7%
Total correction (r_1+r_2)	−24%	−24%	−24%
Correction due to recoil term $(-k_0/2Mc)$ (r_3)	−5%	−6%	−7%
Correction due to relativistic term $(-\frac{1}{4}k_0{}^2)$ (r_4)	−8%	−13%	−16%
Correction due to relativistic term $(\frac{3}{20}\omega_{BA}^I/\mu c^2)$ (r_5)	−6%	−7%	−8%
Total relativistic correction $(r_R=r_2+r_3+r_4+r_5)$	−26%	−33%	−38%
Total correction (r) (r_1+r_R)	−43%	−50%	−55%
Predicted decay rate including the corrections $[(1/\tau_{BA})_{\rm NR}(1+r)]$ (keV)	122	238	283

Next we should also make a comment about another assumption in our calculations. We took the unperturbed energy eigenfunctions to be the SHO wave functions and treated the K/q term together with $h^{(1)}$ (of order $1/c^2$) as the perturbation. Treating the K/q term as part of the perturbation, even though it is of zeroth order in $1/c^2$, can be justified in our present problem, because, in the calculation of the $E1$ decay rates we are interested in the matrix elements of $\vec{q}$, $q^2\vec{q}$, etc., which are not very sensitive to the short-distance behavior of the wave functions. In fact, the first-order correction due to the K/q term to the quantity $|\langle A|\vec{q}|B\rangle|^2$, comes out to be only about $+9\%$ for $\psi'\to\chi_j+\gamma$ decays and -17% for $\chi_j\to\psi+\gamma$ decays, which result may be taken as a justification of our assumption. On the other hand, when we calculate the fine and the hyperfine structure splittings we are calculating essentially the expectation values of the $1/q^3$ term in the unperturbed energy eigenstates. So they should be sensitive to the short-distance behavior of the wave function which is governed to a great extent by the K/q term. In fact the hyperfine structure splittings—calculated in the first-order perturbation theory using the SHO wave functions as the unperturbed wave functions—turn out to be too small by a factor of about 2. We expect that the effect of the K/q term in the potential will increase these splittings substantially, because the K/q term is attractive (K is negative) and so its inclusion should increase the probability of finding the c and $\bar{c}$ quarks at small separations (small values of q), which in turn will increase the expectation value of the $1/q^3$ terms since most of the contributions to these matrix elements come from a small region surrounding the point $q=0$. In fact detailed numerical calculations[17,18] confirm this expectation.

Sucher,[26] Karl, Meshkov, and Rosner,[27] and McClary and Byers[28] have also done work on the relativistic corrections to the radiative transitions of charmonium. Although Sucher has explicit formulas[26] for the relativistic corrections of the $M1$ transitions of charmonium and somewhat implicit formulas for the $E1$ transitions, he does not make any numerical calculation for the $E1$ transitions using a specific model. In comparing his work with ours, it should be noted that his so-called "pair terms" are all buried in the first term of our Eq. (1) which involves the commutator of $\vec{r}_\mu$ and H. Also he does not make use of the relativistic c.m. variables. In his approach the dynamical recoil effects are in principle contained in the dependence[26] of the final-state internal wave function on the total momentum of the composite system. Since the use of relativistic c.m. variables does not contribute anything for the $E1$ transitions of an electrically neutral system such as charmonium our results should agree with those of Sucher even if he neglects the dependence of the internal wave function on the recoil momentum. On the other hand, for $M1$ transitions,[29] we expect agreement only if this dependence is taken into account in Sucher's approach. Karl, Meshkov, and Rosner[27] were only interested in the spin-dependent corrections to the $E1$ transitions, coming from the possible anomalous magnetic moments of the quarks. In comparing our work with that of McClary and Byers[28] it should be borne in mind that all spin-dependent operator corrections to the $E1$ transition operator are buried in the commutator term, involving $[\vec{r}_\mu,H]$ of Eq. (1). In other words, when the $E1$ transition operator is expressed in terms of the commutator between the position operator and the Hamiltonian, there are no spin-dependent corrections if the quarks are point Dirac particles with no anomalous magnetic moments. McClary and Byers have by now also carried out a complete numerical calculation[30] for a specific potential.

Finally, we should mention a word of caution regarding our numerical results. We have shown that if we take into account the first-order relativistic corrections for $\psi'\to\chi_j+\gamma$ decay rates, we could get better agreement with experiment. On the other hand, since the corrections are of the order of 50%, some may question the reliability of the final results.

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