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Variational calculation of π^+-p spin-flip cross section with three constraints

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Total cross section, elastic cross section, and the forward slope of the imaginary part of the scattering amplitude are the three constraints in a numerical calculation of the spin-flip cross section with variational calculus. In addition, the unitarity of partial waves is used as inequality constraints.

I. INTRODUCTION

A method to apply the variational calculus to scattering problems with spin was given in Refs. 1–3. The spinless case was treated in Refs. 4 and 5. In particular, the systematics of the inequality constraints was discussed in Ref. 5.

The basic idea in generalizing the theory of inequality constraints to the spin case is the definition of a complete set of classes such that the partial waves of a process with spin belong to one and only one of these classes according to their elasticity or inelasticity. Thus for the π^+p scattering there are two partial waves l_+ and l_- for a fixed l. Hence one can define four classes such that the two partial waves in one of these classes are both elastic, both inelastic, f_{l+} elastic and f_{l-} inelastic, and vice versa. This approach makes it possible to find the forms of partial waves in each class since the Lagrange multipliers introduced for the unitarity constraints have well defined forms in each class given by the theory of inequality constraints. For the theory we refer the reader to Refs. 1–5.

We have applied this method recently to π^+p and K^+p scattering[6,7] to find numerical upper bounds on the spin-flip cross section G. Only two constraints were used in these calculations, the total cross section and the forward slope of the imaginary part of the spin-nonflip amplitude. Considering the complexity of the problem the results were satisfactory, in that upper bounds could be found rigorously by taking the spin and full unitarity into account. Also the output was a considerable improvement over the input. One other interesting feature which emerged was the large discrepancy in the predictions of different phase-shift sets for the spin-flip cross section. Apparently this quantity had not been calculated before (at least to our knowledge). This demonstrates once more the need for measuring the rotation parameters or for theoretical criteria to distinguish between different phase shifts. From a practical point of view the bounds we found were too large even though the predictions of phase shifts in some cases differed by more than a factor 6. Apparently the total cross section and the forward slope alone do not constrain the spin-flip cross section well enough. For this reason we added a third constraint[3] which is the elastic section. Above the inelastic threshold we hoped that three constraints would give a better bound on G. The results confirm this. With two constraints the best bound was larger than a factor 8, whereas with three constraints it is less than a factor 2.

Mathematically, the problem is as follows. We have one quantity to be maximized which is G. There are three constraints we shall call A_0, S, and E. With these three constraints are associated three Lagrange parameters α, β, and γ. G, A_0, S, and E are functions of partial waves which have well defined forms in terms of α, β, and γ in each of the four classes we mentioned before. The problem consists of fitting the three constraints and thus determining the three Lagrange parameters with these three equations of constraint. With two constraints the relations were linear in α and β and solving them was not difficult. With three constraints the third multiplier γ introduces nonlinearity and one has to solve three nonlinear equations for α, β, and γ. In practice one has to solve the problem with a fairly complicated computer program. The computer picks up for a given number

(number of partial waves) in all possible ways partial waves from each class and fits the constraints and calculates the multipliers α, β, and γ. The unitarity and the maximum condition impose some inequality conditions on the solutions. The computer tests the solutions against those conditions and rejects them if they are not satisfied. The rest it prints out. If there is more than one solution we choose the largest one. Again we refer the reader for theory to Refs. 1–3 and for applications to (6) and (7).

To make this paper self-contained we shall repeat a minimum number of equations and the new features introduced by the third constraint.

In Sec. II, we give the expressions for G and the constraints. We also give the forms of the partial waves in different classes along with some examples of inequality conditions.

In Sec. III, we calculate from different phase-shift sets all physical quantities and present them in tables. We also find the solutions in this section and give them in tables for different energies.

In Sec. IV, we discuss and summarize our results.

II. VARIATIONAL PROBLEM

We want to maximize the spin-flip cross section $\sigma_{\rm SF}$ which is the integral of the square of the spin-flip amplitude g over all angles. The three constraints to be satisfied are the total cross section σ^T, the elastic cross section σ^E, and the forward slope of the imaginary part of the spin-nonflip amplitude f. Further, we require the partial waves to satisfy the unitarity (elastic or inelastic). Since the maximum problem is considered at a fixed energy, we define new quantities G, A_0, E, S so as not to carry energy-dependent factors in our calculations. These are

$$G=\frac{k^2}{2\pi}\sigma_{\rm SF}=\sum\frac{2l(l+1)}{2l+1}[(a_{l+}-a_{l-})^2+(r_{l+}-r_{l-})^2]\,, \tag{1}$$

$$A_0=\frac{k^2}{4\pi}\sigma^T=\sum[(l+1)a_{l+}+la_{l-}]\,, \tag{2}$$

$$E=\frac{k^2}{4\pi}\sigma^E=\sum[(l+1)(a_{l+}{}^2+r_{l+}{}^2)+l(a_{l-}{}^2+r_{l-}{}^2)]\,, \tag{3}$$

$$S=4k^2\frac{k}{s^{1/2}}\frac{dA}{dt}\bigg|_{t=0}=\sum l(l+1)[(l+1)a_{l+}+la_{l-}]\,. \tag{4}$$

Here A is the imaginary part of the spin-nonflip amplitude. a_{l+}, a_{l-}, r_{l+}, and r_{l-} are the imaginary and real parts of the partial waves. k is the c.m. momentum, s the c.m. total energy squared. Equations (2)–(4) are the equality constraints. The inequality constraints are given by the unitarity of the partial waves and are

$$u_l=a_{l+}-a_{l+}{}^2-r_{l+}{}^2\geq 0\,, \tag{5}$$

$$v_l=a_{l-}-a_{l-}{}^2-r_{l-}{}^2\geq 0\,. \tag{6}$$

The Lagrange function has the form

$$L=G+\alpha A_0+\beta S+\gamma E+\sum(l+1)\lambda_l u_l+\sum l\mu_l v_l\,. \tag{7}$$

Here λ_l and μ_l are the l-dependent multipliers which have well defined forms in each of the following four classes.

(I^+I^-) Both f_{l+} and f_{l-} are inelastic in this class and have the forms

$$r_{l+}=r_{l-}=0\,,$$

$$a_{l+}=a_{l-}=-\frac{1}{2\gamma}[\alpha+l(l+1)\beta]\,.$$

It was found that in the two-constraint case this class did not contribute[1]; but now it can contribute.[3]

(I^+B^-) In this class f_{l+} is inelastic but f_{l-} is elastic. The forms of the real and imaginary parts are

$$r_{l+}=r_{l-}=0\,.$$

$$(1)\quad a_{l+}=\frac{1}{B+\gamma}\{B-\tfrac{1}{2}[\alpha+l(l+1)\beta]\}\,,$$

$$a_{l-}=1$$

or

$$(2)\quad a_{l+}=-\frac{1}{2(B+\gamma)}[\alpha+l(l+1)\beta]\,,$$

$$a_{l-}=0\,.$$

Here B stands for $B=2l/(2l+1)$.

(I^-B^+) In this class f_{l+} is elastic and f_{l-} inelastic. The real and imaginary parts have the forms

TABLE I. G and the constraints as constructed with π^+p phase shifts of eight different groups at 2025 MeV c.m. total energy. Aside from kinematical factors, A_0, S, G, and E are the total cross section, forward slope, spin-flip cross section, and elastic cross section as defined by Eqs. (2), (4), (1), and (3).

	A_0	S	G	E	E/A_0	G/A_0
Berkeley Boone 21	4.2129	33.3678	0.3634	1.6607	0.3942	0.0863
Berkeley path 1	4.2455	32.6431	0.2092	1.6554	0.3895	0.0493
Berkeley path 2	4.1954	30.2232	0.3416	1.6587	0.3954	0.0814
Saclay (2021 MeV)	4.1155	28.0632	0.1739	1.5855	0.3852	0.0423
CERN Kirsopp	4.0726	30.1279	0.4769	1.5914	0.3908	0.1171
CERN Experimental (2024 MeV)	4.1611	27.9800	0.4681	1.7725	0.4260	0.1125
CERN Theoretical	4.1634	28.0034	0.4867	1.7952	0.4312	0.1169
Helsinki-Karlsruhe	4.2220	38.4210	0.4285	1.6593	0.3930	0.1015

$r_{l+}=r_{l-}=0$.

(1) $a_{l+}=1$,

$$a_{l-}=\frac{1}{D+\gamma}\{D-\tfrac{1}{2}[\alpha+l(l+1)\beta]\} ,$$

or

(2) $a_{l+}=0$,

$$a_{l-}=-\frac{1}{2(D+\gamma)}[\alpha+l(l+1)\beta] .$$

Here D stands for $D=2(l+1)/(2l+1)$.

(B^+B^-) In this class both partial waves are elastic and there are five different possibilities:

(1) $r_{l+}=r_{l-}=a_{l+}=a_{l-}=0$,

(2) $r_{l+}=r_{l-}=0$, $a_{l+}=0$, $a_{l-}=1$,

(3) $r_{l+}=r_{l-}=0$, $a_{l+}=1$, $a_{l-}=0$,

(4) $r_{l+}=r_{l-}=0$, $a_{l+}=1$, $a_{l-}=1$,

(5) $r_{l+}\neq 0$, $r_{l-}\neq 0$,

$a_{l+}\neq 0$, $a_{l-}\neq 0$.

In this case the forms of the partial waves are complicated and we shall not repeat them here but will refer the reader to Eqs. (97)–(100) of Ref. 3.

Unitarity and the maximum condition impose in all classes on the expression $\alpha+l(l+1)\beta+\gamma$ inequality constraints. Again we shall not repeat them here and will be content by giving a few examples. Thus in class B^+B^- for the case (4) relations

$$\alpha+l(l+1)\beta+\gamma\geq B$$

and

$$\alpha+l(l+1)\beta+\gamma\geq D$$

have to be satisfied. In other classes there are similar inequalities.

III. CONSTRAINTS

The three constraints A_0, S, and E are calculated from different phase-shift solutions. For A_0 and E to be different, that is for the third constraint to be effective, we have to be in the inelastic region. As references we used Refs. 8 and 9. Reference 8 goes only up to 2189 MeV c.m. total energy. For higher energies we used Ref. 9. Since the data for different groups do not have the same normalization, we calculated in addition to A_0, S, G, and E also E/A_0 and G/A_0. The energies we choose

TABLE II. G and the constraints as constructed with π^+p phase shifts of four different groups at 2189 MeV c.m. total energy. Aside from kinematical factors, A_0, S, G, and E are the total cross section, forward slope, spin-flip cross section, and elastic cross section as defined by Eqs. (2), (4), (1), and (3).

	A_0	S	G	E	E/A_0	G/A_0
CERN Kirsopp	4.6519	46.6031	0.3450	1.4624	0.3144	0.0742
CERN experimental	4.7751	46.6685	0.3255	1.5600	0.3267	0.0682
CERN theoretical	4.2413	32.8115	0.3084	1.4204	0.3349	0.0727
Helsinki-Karlsruhe	4.6850	51.9000	0.3299	1.5242	0.3253	0.0704

TABLE III. G and the constraints as constructed with π^+p Helsinki-Karlsruhe phase shifts at five different energies between 2223 MeV and 2350 MeV c.m. total energy. Aside from kinematical factors, A_0, S, G, and E are the total cross section, forward slope, spin-flip cross section, and elastic cross section as defined by Eqs. (2), (4), (1), and (3).

	A_0	S	G	E	E/A_0	G/A_0
2223 MeV	4.9160	57.3160	0.3282	1.5461	0.3145	0.0668
2244 MeV	5.1050	61.9400	0.3337	1.5708	0.3077	0.0654
2277 MeV	5.3780	70.1100	0.3348	1.6058	0.2986	0.0623
2302 MeV	5.6360	76.2900	0.3397	1.6434	0.2916	0.0603
2350 MeV	6.0470	86.5360	0.3307	1.7174	0.2840	0.0547

from Ref. 8 are 2025 MeV and 2189 MeV c.m. total energy. Helsinki-Karlsruhe phase shifts of Ref. 9 go up to 4435 MeV and we choose the energies 2025, 2189, 2223, 2244, 2277, 2302, and 2350 MeV. The results are given in Tables I–III. As can be seen from the data, in the predictions of different groups for G or normalized $G(G/A_0)$ there are big differences. At 2025 MeV, for example, they differ almost by a factor 3.

IV. SOLUTIONS

We search for solutions in the following way. In each class r_{l+}, r_{l-}, a_{l+}, a_{l-} have well defined forms as functions of the Lagrange multipliers α, β, γ and the partial-wave index l. Also in each class there may be more than one type for the partial waves which we discussed in Sec. II. We label all these different forms of partial waves as follows:

Class	Type	Label
I^+I^-	(1)	9
I^+B^-	(1)	1
	(2)	2
I^-B^+	(1)	3
	(2)	4
B^+B^-	(2)	5
	(3)	6
	(4)	7
	(5)	8

In addition we use the label 10 for zero amplitudes. As was discussed in Ref. 6 partial waves in all classes except 8 have a minimum value imposed by unitarity and maximum conditions. Therefore the constraints are satisfied with a small number of partial waves. For the two-constraint case we had searched solutions with up to eight partial waves and had found only two, three, or four wave solutions. Since the computer cost becomes prohibitive, we started in the three-constraint case with a four-wave solution search. After this we increased the number of waves to five, six, and seven to see whether new solutions could be found. In most cases no new solutions were found. In a few instances five- and six-wave solutions were found and these did not change the bounds, found with four waves, appreciably. In no case was a seven-wave solution found.

As indicated before, once the selected wave number is given, the computer picks up partial waves in all possible ways from all classes. It fits with those waves (which are functions of the Lagrange multipliers α,β,γ and the wave index l) the constraints A_0, S, and E. In comparison with the two-constraint case,[6] the addition of the third constraint E introduces the third Lagrange multiplier γ, which in turn makes the three constraint equations nonlinear. The computer solves these three equations for α, β, and γ by iteration, and finds the numerical values of the partial waves which fit the constraints. However not all waves are acceptable, since they must also satisfy the inequality conditions imposed by unitarity and maximality. The computer tests the solutions against those conditions and rejects the ones which violate the inequalities. With the remaining ones it constructs the spin-flip cross section G and prints it out. This is the required upper bound. In general there will be more than one solution; in this case we choose the largest. The values of α, β, and γ also are printed out.

We showed the solutions as a sequence of numbers, each indicating a particular label. Thus

2 9 4 2 4 10 10

means that
$l=1$ wave has the form of the class $I^+B^-(2)$,
$l=2$ wave has the form of the class $I^+I^-(1)$,
$l=3$ wave has the form of the class $I^-B^+(2)$,
$l=4$ wave has the form of the class $I^+B^-(2)$,
$l=5$ wave has the form of the class $I^-B^+(2)$,

TABLE IV. Partial-wave solutions at 2025 MeV c.m. total energy which maximize the spin-flip cross section (G) and satisfy the constraints. The first column gives the names of the phase-shift groups. The six-number code in the second column indicates the labels of the classes for partial waves from $l=1$ to $l=6$. α, β, and γ are the Lagrange multipliers. G_i is the value of G as constructed from phase shifts. G_0 is the bound found with the variational calculus.

	Maximum solution	α	β	γ	G_i	G_0
Berkeley Boone 21	2 9 4 4 10 10	1.5022	−0.0689	−1.8949	0.3634	1.1316
Berkeley path 1	9 2 2 4 10 10	1.6340	−0.0759	−1.9597	0.2092	1.0102
Berkeley path 2	2 9 2 4 10 10	1.4565	−0.0709	−1.7421	0.3416	0.9051
Saclay (2021 MeV)	9 9 4 8 10 10	1.6601	−0.0929	−1.7819	0.1739	0.7173
CERN Kirsopp	9 9 4 8 10 10	1.6544	−0.0796	−1.9762	0.4769	0.9781
CERN experimental (2024 MeV)	2 9 2 8 10 10	1.7212	−0.0921	−1.8542	0.4681	0.9976
CERN theoretical	1 9 9 4 10 10	1.8103	−0.0903	−1.9545	0.4867	1.0448
Helsinki-Karlsruhe	2 4 9 2 10 10	1.3782	−0.0543	−1.9930	0.4285	1.4406

$l=6$ wave is zero, and
$l=7$ wave is zero.
We give the solutions in Tables IV, V, and VI. As remarked before, the results show a definite improvement over the two-constraint case. The best result is for the CERN theoretical-fit phase shifts at 2189 MeV for which the bound for G is 0.55 as opposed to the value of G constructed from phase shifts, which is 0.31. The ratio is only 1.78. In the worst case we have a factor of 4.8 and in most cases the bounds differ from constructed values of G by a factor of order 2 or 3.

IV. SUMMARY

The application of the variational method was extended to problems with spin.[1–3] Unitarity is taken fully into account. For π^+p scattering upper bounds are calculated at different inelastic energies for the spin-flip cross section G. In addition to the unitarity of partial waves which represent inequality constraints, three global constraints are used. These are the total cross section, elastic cross section, and the forward slope of the imaginary part of the spin-nonflip amplitude. The results show a definite improvement over the two-constraint case studied before[6] in which the elastic cross section was not used.

The technique is rigorous and general. In principle it can be applied to arbitrary spins. In practice the number of classes for amplitudes becomes large with larger spins. But this can still be handled with computers in applications. Also the constraints must have the single series form in partial waves. In the case studied here two of the constraints (A_0,S) were linear in partial waves and one (E) was quadratic. Almost all of the solutions found start with $l=1$ wave either in class 2 or 9. When comparing the results of this paper with those of the two-constraint case,[6] we would like to remark that the labeling of classes has not been changed. Only, since the class I^+I^- did not contribute in the two-constraint case, it was not used. Here it can contribute and is labeled 9 in order not to change the label numbers of the following classes. The label of the zero-amplitude class is

TABLE V. Partial-wave solutions at 2189 MeV c.m. total energy which maximize the spin-flip cross section (G) and satisfy the constraints. The first column gives the names of the phase-shift groups. The six-number code in the second column indicates the labels of the classes for partial waves from $l=1$ to $l=6$. α, β, and γ are the Lagrange multipliers. G_i is the value of G as constructed from phase shifts. G_0 is the bound found with variational calculus.

	Maximum solution	α	β	γ	G_i	G_0
CERN Kirsopp	2 2 9 9 8 10	1.2993	−0.0445	−1.9496	0.3450	0.8650
CERN experimental	9 9 4 9 8 10	1.3316	−0.0476	−1.8978	0.3255	0.8934
CERN theoretical	2 9 9 4 10 10	1.2463	−0.0581	−1.5768	0.3084	0.5503
Helsinki-Karlsruhe	2 9 4 2 4 10	1.1704	−0.0369	−1.9374	0.3299	1.1672

TABLE VI. Partial-wave solutions between 2223 MeV and 2350 MeV c.m. total energy which maximize the spin-flip cross section (G) and satisfy the constraints. Only Helsinki-Karlsruhe phase shifts have been used. The six-number code in the second column indicates the labels of the classes for partial waves from $l=1$ to $l=6$. α, β, and γ are the Lagrange multipliers. G_i is the value of G as constructed from phase shifts. G_0 is the bound found with variational calculus.

	Maximum solution	α	β	γ	G_i	G_0
2223 MeV	2 9 9 2 8 10	1.2106	−0.0379	−1.9497	0.3282	1.1317
2244 MeV	9 9 4 4 9 10	1.2173	−0.0355	−1.9912	0.3337	1.1183
2277 MeV	2 9 2 4 9 10	1.0963	−0.0295	−1.8970	0.3348	1.1334
2302 MeV	9 9 2 4 4 10	1.1070	−0.0288	−1.8944	0.3397	1.0938
2350 MeV	2 9 9 4 2 8	1.1841	−0.0280	−1.9900	0.3307	1.0499

changed from 9 to 10.

The number of partial waves needed in the solution increases with energy as one would expect. Only at the highest energy we used, 2350 MeV, had the variational solution six waves. At all lower energies four or five waves were sufficient.

Finally, we would like to remark that the values of G constructed from different phase shifts differed at some of the energies we looked at by as much as a factor 2.8. (In other cases it was even larger.) This difference is a relative value between the predictions of phase shifts and not between the unknown experimental value of G and the phase-shift predictions. The latter could be better or worse.

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