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Application of current algebra and partial conservation of axial-vector current to proton-antiproton annihilations

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A soft-pion study using current algebra and the partially conserved axial-vector current (PCAC) hypothesis is made of the proton-antiproton annihilation process $\bar{p}p \to K^+K^-3\pi^0$. Of the three neutral pions only two are considered as soft with the third pion forming part of the external final state. The differential rate for the process is normalized to the differential rate of the corresponding process without the soft pions. Theoretical predictions for the branching ratios at various antiproton momenta are compared with the existing experimental data.

I. INTRODUCTION

Successes of the current-algebra—PCAC (partially conserved axial-vector current) formalism have established that the study of strong-interaction processes is much simpler using current operators. In recent years, among the strong interactions studied with the formalism are proton-antiproton annihilations[1–4] and kaon-proton interactions.[5,6] In all these applications, the maximum number of soft pions involved is two. In processes like $\bar{p}p \to KK\pi\pi$, the pions can have momenta too large for them to be treated as soft. Also in the $\bar{p}p$ annihilation processes at vanishing momentum, the number of pions emitted can be as many as four.[7]

In applying the current-algebra—PCAC formalism to multiple-pion-emission processes, one finds that the manipulation of derivatives through time-ordered products becomes increasingly difficult as the pion number increases. However, in dealing with such processes, it is not necessary to treat all the pions as soft. One can regard some of them as being in the external final state. Therefore, in our attempt towards decreasing the pion momenta and also increasing the number of emitted pions, we consider the process $\bar{p}p \to K^+K^-3\pi^0$, and treat only two of the pions as soft. This way we avoid double commutators between A_μ and $\partial_\mu A_\mu$, which are present when all the three pions are treated soft. Taking all the pions in the neutral mode has the added advantage that one needs to consider gradient coupling only as pointed out by Weinberg.[8] Like Uritam,[1] we too subscribe to van Hove's point of view that the theoretical expressions should be in the most differential form available, rather than have some or all the variables integrated over. So we present the differential rate for the process in all the relevant kinematic variables and normalize it to the differential rate for the process $\bar{p}p \to K^+K^-\pi^0$. In order to make a comparison with experimental data, we finally integrate the resulting expressions numerically over the independent variables to obtain the branching ratios for various antiproton momenta. Owing to lack of experimental results for $\bar{p}p \to K^+K^-3\pi^0$, the branching ratios thus obtained are, however, compared with data on final products in other charge states. The comparison is quite reasonable since the results of Greenhut and Intemann[4] indicate that the branching ratios for pions in different charge states have nearly the same order of magnitude, especially at low center-of-mass energies.

In Sec. II, we derive the amplitude and differential rate for the process $\bar{p}p \to K^+K^-\pi^0 + 2\pi^0$ and normalize it to the differential rate for the same process without the soft pions. In Sec. III, we present the branching ratios for various center-of-mass energies obtained after integrating over the relevant kinematic variables, and compare them with the available experimental results.

II. $\bar{p}p \to K^+K^-\pi^0 + 2\pi^0$ AMPLITUDE AND DIFFERENTIAL RATE

According to the Lehmann-Symanzik-Zimmermann reduction formula, the matrix element for the emission of two pions in the reaction

$$i \to f + \pi^{\alpha}(k_1) + \pi^{\beta}(k_2) \tag{2.1}$$

is proportional to

$$\int d^4x\, d^4y\, e^{-ik_1x} e^{-ik_2y} \times \langle f \mid T(\phi_\pi^\alpha(x),\, \phi_\pi^\beta(y)) \mid i \rangle \,. \tag{2.2}$$

i and f are the initial and final states containing particles such as $\bar{p}$, p, K^+, K^-, and π^0. k_1 and k_2 are the pion momenta and α and β are their isospin indices. We have the PCAC equation

$$\partial_\mu A_\mu^\alpha = (C_\pi/\sqrt{2})\phi_\pi^\alpha \,, \tag{2.3}$$

where $C_\pi = \sqrt{2} G_A M_N \mu^2 / g_r(0)$; $G_A (\simeq 1.18)$ is the axial-vector coupling constant, g_r is the rationalized, renormalized pion-nucleon coupling constant ($g_r^2/4\pi \simeq 14.6$), ϕ_π^α is the renormalized pion field operator, and μ and M_N are the pion and nucleon masses. Through Eq. (2.3), Eq. (2.2) can be rewritten in terms of the derivatives of the axial-vector currents A_μ. The derivatives can be brought outside the time-ordered product using the standard identity. The resulting commutators like

$$\delta(x_0 - y_0)[A_0^\alpha(x),\, \partial_\mu A_\mu^\beta(y)] = \delta_{\alpha\beta}\sigma(x)\delta(x - y) \tag{2.4}$$

are dropped on the basis that they are of the same order as the PCAC correction terms. Integrating the resulting expression by parts, introducing Klein-Gordon operators and through Eq. (2.3) replacing $\partial_\mu A_\mu$ with ϕ_π, we have

$$k_1^\mu k_2^\nu R_{\mu\nu}^{\alpha\beta} = \tfrac{1}{2} i \frac{C_\pi^2}{(\mu^2 + k_1^2)(\mu^2 + k_2^2)} R_{2\pi}^{\alpha\beta} + \tfrac{1}{2} i \epsilon_{\alpha\beta\gamma}(k_2 - k_1)^\lambda R_\lambda^\gamma \,, \tag{2.5}$$

where we have used the relation

$$\delta(x_0 - y_0)[A_0^\alpha(x), A_\mu^\beta(y)] = i\delta(x - y)\epsilon_{\alpha\beta\gamma} V_\mu^\gamma(x) \,, \tag{2.6}$$

and

$$R_{\mu\nu}^{\alpha\beta} = i \int d^4x\, d^4y\, e^{-ik_1x} e^{-ik_2y} \langle f \mid T(A_\mu^\alpha(x), A_\nu^\beta(y)) \mid i \rangle \tag{2.7}$$

is the matrix element for the emission of two axial-vector currents in the process $i \to f$. The factor of $\frac{1}{2}$ on the right-hand side of Eq. (2.5) comes because of the pulling out of derivatives symmetrically from the time-ordered product.

$$R_{2\pi}^{\alpha\beta} = -\int d^4x\, d^4y\, e^{-ik_1x} e^{-ik_2y} (\mu^2 - \Box_x)(\mu^2 - \Box_y) \langle f \mid T(\phi_\pi^\alpha(x), \phi_\pi^\beta(y)) \mid i \rangle \tag{2.8}$$

is the matrix element for the emission of two pions of momenta k_1 and k_2 and isospins α and β in the process $i \to f$.

$$R_\lambda^\gamma = \int d^4x\, e^{-i(k_1 + k_2)x} \langle f \mid V_\lambda^\gamma(x) \mid i \rangle \tag{2.9}$$

is the matrix element for the emission of an isovector photon in the process $i \to f$. This term can be dropped from Eq. (2.5) if all the pions emitted have the same isospin index. We have

$$S_{ab} = \delta_{ab} + iR_{ab} = \delta_{ab} + i(2\pi)^4 \delta(p_a - p_b) M_{ab} / N_{ab}^{1/2} \,. \tag{2.10}$$

Equation (2.5) now becomes

$$k_1^\mu k_2^\nu M_{\mu\nu}^{\alpha\beta} = \tfrac{1}{2} i \frac{C_\pi^2}{(\mu^2 + k_1^2)(\mu^2 + k_2^2)} M_{2\pi}^{\alpha\beta} + \tfrac{1}{2} i \epsilon_{\alpha\beta\gamma}(k_2 - k_1)^\lambda M_\lambda^\gamma \,. \tag{2.11}$$

If we now set $k_1 = \xi K_1$ and $k_2 = \xi K_2$, so that $\xi \to 0$ corresponds to the soft-pion limit, for a process like $\alpha \to \beta + m\pi$ where α and β are different hadronic states, the matrix element is of zeroth order[9] in ξ. Therefore, in evaluating the expression (2.11) in the soft-pion limit, we look for pole terms that go as k^{-2} in $M_{\mu\nu}^{\alpha\beta}$. The matrix element for the emission of an isovector photon of momentum $k = (k_1 + k_2)$ also must have pole terms that go as k^{-1}. In evaluating the isovector term, we can use the low-energy theorem for photon emission due to Low.[10] With the help of the theorem, we can express the amplitude for the emission of the photon in the process $i \to f$ in terms of the corresponding process where the photon is absent. Setting k_1, $k_2 \to 0$ and considering two-neutral-pion emission, $\alpha=\beta=3$, we have

$$k_1^\mu k_2^\nu M^{33}_{\mu\nu} = \tfrac{1}{2} i \frac{C_\pi{}^2}{\mu^4} M^{33}_{2\pi} \ . \tag{2.12}$$

Pole terms of the order of k^{-1} arise when an axial-vector vertex is attached to a nonterminating external line. Parity forbids insertion of A_μ into pseudoscalar-meson lines. Therefore, we need to consider only the diagrams given in Fig. 1.

The contribution from the diagrams can be written in the usual way. The central interaction in these diagrams can be written as

$$\mathscr{M} = A + B'\not{Q}_1 + B''\not{Q}_2 + C\sigma_{\mu\nu} Q_1^\mu Q_2^\nu \ , \tag{2.13}$$

where $Q_1 = q_1 - q_2 + k_3$, $Q_2 = q_1 - q_2 - k_3$, and q_1, q_2 are the kaon momenta and k_3 is the pion momentum. If the kaon production is considered to be at threshold,

$$\mathscr{M} = A + B\not{Q} \ , \tag{2.14}$$

where $Q = k_3$. Retaining only the zeroth-order terms in pion momenta in $k_1^\mu k_2^\nu M^{33}_{\mu\nu}$, we have

$$k_1^\mu k_2^\nu M^{33}_{\mu\nu} = \frac{G_A{}^2}{k_1^0 k_2^0 (k_1^0 + k_2^0)} \bar{v}_s(p_2)(AF_A + BF_B) u_r(p_1) \ . \tag{2.15}$$

Considering proton-antiproton annihilation at vanishing laboratory momentum, we have $p_1 = p_2 = (\vec{0}, iM_N)$ and

$$F_A = -4(k_1^0 + k_2^0)(\vec{k}_1 \cdot \vec{k}_2) \ , \tag{2.16}$$

$$\begin{aligned} F_B = {} & k_1^0(-\not{Q}\not{k}_1\not{k}_2 + \not{k}_1\not{Q}\not{k}_2 - \not{k}_2\not{k}_1\not{Q} + \not{k}_2\not{Q}\not{k}_1) + k_2^0(-\not{Q}\not{k}_2\not{k}_1 + \not{k}_2\not{Q}\not{k}_1 - \not{k}_1\not{k}_2\not{Q} + \not{k}_1\not{Q}\not{k}_2) \\ & + 2ik_1^0 k_2^0(\not{Q}\not{k}_2 - \not{k}_2\not{Q} + \not{Q}\not{k}_1 - \not{k}_1\not{Q}) \ . \end{aligned} \tag{2.17}$$

Replacing the Dirac spinors with Pauli spinors we find

$$\bar{v}_2(p_2) A F_A u_r(p_1) = 0 \ . \tag{2.18}$$

Therefore, the matrix element for the emission of two pions in the process $\bar{p}p \to K^+K^-\pi^0$ is given as

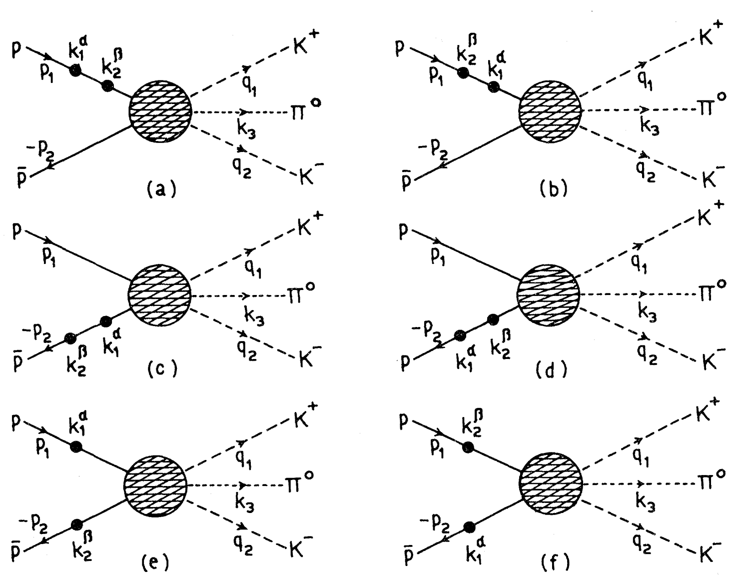

FIG. 1. Diagrams of order k^{-2} contributing to $M^{\alpha\beta}_{\mu\nu}$.

$$M_{2\pi}^{33}\equiv M_{sr}^{00}=-i\frac{2G_A{}^2\mu^4}{C_\pi{}^2k_1^0k_2^0(k_1^0+k_2^0)}\bar{v}_s(p_2)BF_Bu_r(p_1)\ , \tag{2.19}$$

where r and s are spin indices of the proton and antiproton, respectively. The matrix element thus obtained is valid in the unphysical region k_1, $k_2\to 0$. Using PCAC as a "smoothness condition", we assume that the matrix element can be extrapolated on to the pion mass shell.[8,11] We find the $|M_{sr}^{00}|^2$ averaged over the initial and summed over the final spin states as

$$\langle |M_{sr}^{00}|^2\rangle=8B^2\left[\frac{2G_A{}^2\mu^4}{C_\pi{}^2k_1^0k_2^0(k_1^0+k_2^0)}\right]^2\{k_1^{0^2}[\vec{k}_1{}^2\vec{k}_2{}^2\vec{Q}^2-\vec{k}_2{}^2(\vec{k}_1\cdot\vec{Q})^2-(\vec{k}_2\times\vec{k}_1\cdot\vec{Q})^2]$$
$$+k_2^{0^2}[\vec{k}_2{}^2\vec{k}_1{}^2\vec{Q}^2-\vec{k}_1{}^2(\vec{k}_2\cdot\vec{Q})^2-(\vec{k}_1\times\vec{k}_2\cdot\vec{Q})^2]$$
$$+2k_1^0k_2^0[(\vec{k}_1\cdot\vec{k}_2)^2\vec{Q}^2-(\vec{k}_2\cdot\vec{k}_1)(\vec{k}_2\cdot\vec{Q})(\vec{k}_1\cdot\vec{Q})]\}\ . \tag{2.20}$$

The differential rate for the reaction $\bar{p}p\to K^+K^-\pi^0+2\pi^0$ is given by the expression

$$d^{15}w^{00}=\frac{(2\pi)^4}{(2\pi)^{15}}\frac{M_N{}^2}{p_1^0p_2^0}\delta(p_1+p_2-q_1-q_2-k_1-k_2-k_3)\langle |M_{sr}^{00}|^2\rangle\frac{d^3\vec{q}_1}{2q_1^0}\frac{d^3\vec{q}_2}{2q_2^0}\frac{d^3\vec{k}_1}{2k_1^0}\frac{d^3\vec{k}_2}{2k_2^0}\frac{d^3\vec{k}_3}{2k_3^0}\ . \tag{2.21}$$

We consider the $\bar{p}p$ system as a single particle decaying into R $(=K^+K^-\pi^0)$ and K $(=2\pi^0)$. The system R decays further into S $(=K^+\pi^0)$ and T $(=K^-)$. K itself decays into two pions while S decays into a kaon and a pion. Out of the 15 variables in Eq. (2.21), 9 get integrated over trivially. The remaining six variables are chosen to be the following: $m_{KK\pi}{}^2=-R^2$, $m_{\pi\pi}{}^2=-K^2$, $m_{K\pi}{}^2=-S^2$, θ_K (the angle between R and S), θ_π (the angle between K and one of the decayed pions), and ϕ [the relative azimuthal angle between the dipion (K) and R decay planes]. We have

$$d^6w^{00}=dm_{KK\pi}{}^2dm_{K\pi}{}^2dm_{\pi\pi}{}^2d(\cos\theta_K)d(\cos\theta_\pi)d\phi\frac{1}{(2\pi)^8}\langle |M_{sr}^{00}|^2\rangle\frac{1}{2^{10}E^2m_{KK\pi}{}^2m_{K\pi}{}^2m_{\pi\pi}}$$
$$\times[(E^2+m_{KK\pi}{}^2-m_{\pi\pi}{}^2)^2-4E^2m_{KK\pi}{}^2]^{1/2}[(m_{KK\pi}{}^2+m_{K\pi}{}^2-m_K{}^2)^2-4m_{KK\pi}{}^2m_{K\pi}{}^2]^{1/2}$$
$$\times[(m_{K\pi}{}^2+m_K{}^2-\mu^2)^2-4m_{K\pi}{}^2m_K{}^2]^{1/2}(m_{\pi\pi}{}^2-4\mu^2)^{1/2}\ , \tag{2.22}$$

where E^2 is the square of the center-of-mass energy of the $\bar{p}p$ system.

$\langle |M_{sr}^{00}|^2\rangle$ given in Eq. (2.20) has to be expressed in terms of $m_{KK\pi}{}^2$, $m_{K\pi}{}^2$, $m_{\pi\pi}{}^2$, $\cos\theta_K$, $\cos\theta_\pi$, and ϕ. For this we first express the variables in Eq. (2.20) in terms of Lorentz invariants and then in terms of the chosen variables. The relevant equations are tabulated in the Appendix.

In Eq. (2.22) the unknown term B^2 in $\langle |M_{sr}^{00}|^2\rangle$ can be eliminated if we normalize the rate d^6w^{000} to the rate of the process $\bar{p}p\to K^+K^-\pi^0$ where the soft pions are absent. Considering the annihilation at rest and kaon production at threshold, the matrix element for the reaction $\bar{p}p\to K^+K^-\pi^0$ becomes

$$M_{sr}=\bar{v}_s(p_2)BQu_r(p_1)\ . \tag{2.23}$$

One finds

$$\langle |M_{sr}|^2\rangle=\tfrac{1}{2}B^2\left[E^2-2m_{K\pi}{}^2-2m_K{}^2+\frac{(m_K{}^2-m_{K\pi}{}^2)^2}{E^2}\right]\ . \tag{2.24}$$

The corresponding expression for the differential rate is

$$dw = dm_{K\pi}^2 \frac{11}{16(2\pi)^3} \langle |M_{sr}|^2 \rangle [(-m_{K\pi}^2 + m_K^2 + \mu^2)^2 - 4m_K^2 \mu^2]^{1/2}$$
$$\times \frac{1}{E^2 m_{K\pi}^2} [(E^2 + m_{K\pi}^2 - m_K^2)^2 - 4E^2 m_{K\pi}^2]^{1/2} . \tag{2.25}$$

The ratio of Eq. (2.22) to Eq. (2.25) gives the differential rate for the process $\bar{p}p \to K^+K^-\pi^0 + 2\pi^0$, normalized to the differential rate for the same process without the soft pions.

III. COMPARISON OF THEORETICAL AND EXPERIMENTAL BRANCHING RATIOS

Our expression for the differential rate for the process $\bar{p}p \to K^+K^-3\pi^0$ presented in its fully differential form in Eq. (2.22), affords us a great deal of information, predicting a spectrum in the six independent kinematic variables. The simplest comparison we can make with the available experimental data is, of course, through the branching ratio,

$$R = w(\bar{p}p \to K^+K^-3\pi^0)/w(pp \to K^+K^-\pi^0) ,$$

obtained by integrating the expressions in Eqs. (2.22) and (2.25) over the kinematic variables within their limits. We perform the integration numerically for different energies by generalizing our expressions to hold at nonzero laboratory momenta of the antiproton. The theoretical predictions thus obtained for the branching ratio R are tabulated in Table I. Since experimental results for the particular reaction $\bar{p}p \to K^+K^-3\pi^0$ are not available, comparison could be made only with data[12] on the final products in other charge states. Although the theoretical branching ratios are consistently lower, they show the same energy dependence as the experimental ratios as displayed in Fig. 2.

IV. REMARKS

It is known experimentally[13] that $\bar{p}p$ annihilation at rest occurs in the S state. That state can be either a singlet ^1S state or a triplet ^3S state; in general it is a mixture of these two states. In the past current-algebra predictions have been in good agreement with experiment for S waves. At higher energies more angular momentum states come in and current algebra is not reliable. When annihilation takes place at rest, there is enough center-of-

TABLE I. Ratio of calculated reaction rates $R = w(\bar{p}p \to K^+K^-3\pi^0)/w(\bar{p}p \to K^+K^-\pi^0)$ at various center-of-mass energies.

$\bar{p}$ laboratory momentum (GeV/c)	Center-of-mass energy (GeV)	R
0.0	1.877	0.008
0.5	1.938	0.015
1.0	2.082	0.044
1.2	2.149	0.067
1.36	2.205	0.091
1.5	2.254	0.115
1.61	2.293	0.158
1.80	2.360	0.229
1.95	2.412	0.292
2.0	2.430	0.314
2.5	2.602	0.599
2.7	2.669	0.745
3.0	2.767	1.011
3.5	2.926	1.561
3.7	2.987	1.822
4.0	3.077	2.258

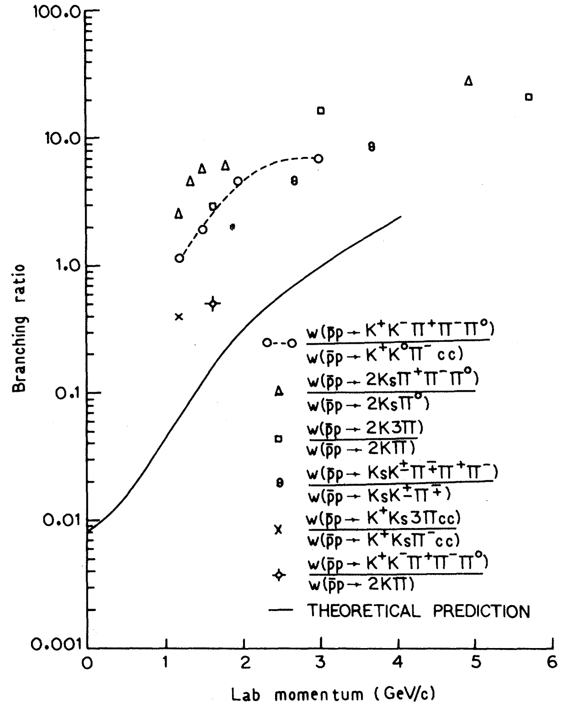

FIG. 2. Comparison of theoretical prediction of $R = w(\bar{p}p \to K^+K^-3\pi^0)/w(\bar{p}p \to K^+K^-\pi^0)$ with experiment at various laboratory momenta.

mass energy available to produce the desired final-state products, namely, $KK3\pi$. Until now little theoretical attention has been given to $\bar{p}p$ annihilation with more than two pions in the final state. This is partially due to the computational difficulties involved as the pion number increases beyond two. In our case we overcome the difficulty by treating only two of the pions as soft. Considering annihilation at rest introduces its own simplifications.

Retaining the isovector photon term M_λ^γ and considering two oppositely charged pions as soft might have brought the theoretical predictions closer to the experimental results. However, in such a case only half the diagrams given in Fig. 1 contribute to $k_1^\mu k_2^\nu M_{\mu\nu}^{\alpha\beta}$.

Our theoretical predictions when compared with the available experimental data are consistently lower by an order of magnitude even at small center-of-mass energies. Even in the past discrepancies of this kind have been reported.[14,4] The predictions of Grant, Schillaci, and Silbar[15] for the process $pp \to np\pi^+$ are smaller by an order of magnitude compared to the experimental results. Schillaci and Silbar[14] felt that higher-order terms, present because $\mu \neq 0$, dominate the resonance effects and that the resonance contribution due to the production of intermediate isobar $\Delta(1236)$ was substantial enough to account for the observed discrepancies. Similar "off-shell" resonance effects were suspected to be the reason behind the discrepancy between soft-pion calculations[4] and experimental results in $\bar{p}p \to K\bar{K}2\pi$. In a recent soft-pion study of the process $NN \to NN\pi$, Dubach, Silbar, and Kloet[16] show that the postemission Δ-resonance contribution is significant. However, inclusion of Δ-postemission and nucleon-preemission diagrams removed the discrepancy with experiment only partially.

It is well known that the annihilations yielding K mesons produce an intermediate resonance state of $K^*(890)$ meson in large fractions. For example, at 1.2 GeV/c laboratory momentum of the antiproton, the rate of resonance production,[17] $K^*K\pi\pi$ $(K\bar{K}^*\pi\pi)$ in the process $\bar{p}p \to K_1^0K^\pm\pi^\mp\pi^+\pi^-$ is about 80%. In addition, pion resonances ρ and ω are known to be produced along with the kaons. A look at Table II reveals the abundance of resonance production in two different reactions of the type $\bar{p}p \to KK3\pi$ at 3.7 GeV/c.[18] This definitely influences the amplitude for reactions such as production of the Δ resonance in the case of $NN \to NN\pi$.

The other reason for the discrepancy could be that we have treated only two of the pions as soft

TABLE II. Percentage of resonance production in $\bar{p}p$ annihilation into two kaons and three pions at 3.7 GeV/c.

	Percentage of resonance production		
Final state	K^*	ρ	ω
$K_1^0K^\pm\pi^\mp\pi^+\pi^-$	75±15	25±10	
$K_1^0K_1^0\pi^+\pi^-\pi^0$	30±15	40±15	5±5

with the third forming part of the final state. With three-pion emission, one has to consider π-π effects[19] and these would have come out of the soft-pion formalism automatically if all the three pions were treated soft. This is now being investigated and is to be shortly reported elsewhere.

ACKNOWLEDGMENTS

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APPENDIX

The terms appearing in Eq. (2.20) must be expressed in terms of the chosen kinematic variables $m_{KK\pi}{}^2$, $m_{K\pi}{}^2$, $m_{\pi\pi}{}^2$, $\cos\theta_K$, $\cos\theta_\pi$, and ϕ. For this purpose we first define

$$P=R+K,\quad R=S+T,\quad K=k_1+k_2, \Delta=k_1-k_2,\quad Q=S-T,\quad S=q_1+k_3,\quad T=q_2\ . \tag{A1}$$

In terms of these, the variables in Eq. (2.20) are given as

$$k_i^0=-P\cdot k_i/E\ , \tag{A2}$$

$$\vec{k}_i{}^2=k_i{}^2+(P\cdot k_i)^2/E^2\ , \tag{A3}$$

$$\vec{Q}^2=Q^2+(P\cdot Q)^2/E^2\ , \tag{A4}$$

$$\vec{k}_i\cdot\vec{Q}=(k_i\cdot Q)+(P\cdot k_i)(P\cdot Q)/E^2\ , \tag{A5}$$

$$\vec{k}_1\cdot\vec{k}_2=k_1\cdot k_2+(P\cdot k_1)(P\cdot k_2)/E^2\ , \tag{A6}$$

with $i=1,2$. We can form the following invariants from the variables in (A1):

$$K^2,\ Q\cdot K,\ R\cdot K,\ \Delta^2,\ \Delta\cdot R,\ \Delta\cdot Q,\ Q^2,\ Q\cdot R,\ R^2,\ S^2,\ S\cdot R\ . \tag{A7}$$

We write the terms occurring in (A2) to (A6) in terms of those in (A7):

$$P\cdot k_1=\tfrac{1}{2}(R\cdot K+\Delta\cdot R+K^2)\ , \tag{A8}$$

$$P\cdot k_2=\tfrac{1}{2}(R\cdot K-\Delta\cdot R+K^2)\ , \tag{A9}$$

$$P\cdot Q=Q\cdot R+Q\cdot K\ , \tag{A10}$$

$$k_1\cdot Q=\tfrac{1}{2}(Q\cdot K+\Delta\cdot Q)\ , \tag{A11}$$

$$k_2\cdot Q=\tfrac{1}{2}(Q\cdot K-\Delta\cdot Q)\ , \tag{A12}$$

$$k_1\cdot k_2=\tfrac{1}{4}(K^2-\Delta^2)\ , \tag{A13}$$

where

$$K^2=-m_{\pi\pi}{}^2\ , \tag{A14}$$

$$R^2=-m_{KK\pi}{}^2\ , \tag{A15}$$

$$S^2=-m_{K\pi}{}^2\ , \tag{A16}$$

$$\Delta^2=m_{\pi\pi}{}^2-4\mu^2\ , \tag{A17}$$

$$Q^2=m_{KK\pi}{}^2-2m_{K\pi}{}^2-2m_K{}^2\ , \tag{A18}$$

$$R\cdot K=\tfrac{1}{2}(m_{KK\pi}{}^2+m_{\pi\pi}{}^2-E^2)\ , \tag{A19}$$

$$Q\cdot R=m_K{}^2-m_{K\pi}{}^2\ , \tag{A20}$$

$$S\cdot R=\tfrac{1}{2}(m_K{}^2-m_{KK\pi}{}^2-m_{K\pi}{}^2)\ , \tag{A21}$$

$$\Delta\cdot R=-\cos\theta_\pi(\Delta^2)^{1/2}\left[\frac{R^2K^2-(R\cdot K)^2}{K^2}\right]^{1/2}, \tag{A22}$$

$$Q\cdot K=-2\cos\theta_K\left[\frac{S^2R^2-(S\cdot R)^2}{R^2}\right]^{1/2}\left[\frac{R^2K^2-(R\cdot K)^2}{R^2}\right]^{1/2}+\frac{(Q\cdot R)(R\cdot K)}{R^2}, \tag{A23}$$

$$\Delta\cdot Q=\frac{\cos\phi(L_1L_2)^{1/2}-[(Q\cdot R)K^2-(R\cdot K)(Q\cdot K)](\Delta\cdot R)}{(R\cdot K)^2-R^2K^2}. \tag{A24}$$

Also,

$$L_1=[R^2K^2-(R\cdot K)^2]\Delta^2-(\Delta\cdot R)^2K^2, \tag{A25}$$

$$L_2=[R^2K^2-(R\cdot K)^2]Q^2+[2(Q\cdot K)(R\cdot K)-K^2(Q\cdot R)](Q\cdot R)-(Q\cdot K)^2R^2. \tag{A26}$$

We also have

$$(\vec{k}_1\times\vec{k}_2\cdot\vec{Q})^2=(\vec{k}_2\times\vec{k}_1\cdot\vec{Q})^2=\Sigma, \tag{A27}$$

$$\begin{aligned}\Sigma=-\frac{1}{4E^2}\{&K^2[R^2Q^2\Delta^2-\Delta^2(Q\cdot R)^2-R^2(\Delta\cdot Q)^2-Q^2(\Delta\cdot R)^2+2(\Delta\cdot R)(Q\cdot R)(\Delta\cdot Q)]\\&+\Delta^2[2(Q\cdot K)(Q\cdot R)(R\cdot K)-R^2(Q\cdot K)^2-Q^2(R\cdot K)^2]\\&+(Q\cdot K)^2(\Delta\cdot R)^2+(R\cdot K)(\Delta\cdot Q)^2-2(Q\cdot K)(R\cdot K)(\Delta\cdot R)(\Delta\cdot Q)\}.\end{aligned} \tag{A28}$$

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