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Relativistic form factors for hadrons with quark-model wave functions

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The relationship between relativistic form factors and quark-potential-model wave functions is examined using an improved version of an approach by Licht and Pagnamenta. Lorentz-contraction effects are expressed in terms of an effective hadron mass which varies as the square root of the number of quark constituents. The effective mass is calculated using the rest-frame wave functions from the mean-square momentum along the direction of the momentum transfer. Applications with the parameter-free approach are made to the elastic form factors of the pion, proton, and neutron using a Hamiltonian which simultaneously describes mesons and baryons. A comparison of the calculated radii for pions and kaons suggests that the measured kaon radius should be slightly smaller than the corresponding pion radius. The large negative squared charge radius for the neutron is partially explained via the quark model but a full description requires the inclusion of a small component of a pion "cloud" configuration. The problematic connection between the sizes of hadrons deduced from form factors and the "measured" values of average transverse momenta is reconciled in the present model.

I. INTRODUCTION

For a number of years simple quark potential models[1-10] have proven quite successful in describing the static properties of hadrons. Unfortunately, there is as yet no definitive way to use the model wave functions to describe dynamical properties such as those contained in the electromagnetic form factors of pions, protons, and neutrons. Although static properties of light hadrons are quite well described in their rest frames by nonrelativistic wave functions,[3-10] or more realistically by pseudorelativistic wave functions,[1,2] the electromagnetic form factors derived from them cannot be compared directly to data. For a completely satisfactory description of the data up to moderate momentum transfer (q) one would require a full-fledged relativistic theory of multiparticle systems. Such a theory appears to be currently intractable and for the foreseeable future there is probably no viable alternative to a basically three-dimensional approach with noncovariant wave functions. Nevertheless, one must give a relativistic meaning, however limited, to such a wave function before it can be used in calculations of nonstatic properties. Since the external kinematics of the system can be treated in a relativistically invariant manner it is the internal model wave function which requires a tractable scheme of relativization.

Several schemes[11-14] for boosting model wave functions to relativistic velocities have been proposed. Basically these suggestions allow for the Lorentz contractions which occur when the composite systems are in motion. However, the prescriptions proposed to date have all differed. It is the purpose of this work to find a reasonable and parameter-free prescription which can be consistently tested using model wave functions calculated from a Hamiltonian that is the same for mesons and baryons. The Hamiltonian and the wave functions derived from it for the systems of interest are discussed in Sec. II.

The relativistic "boost" prescription adopted here is based on the Licht and Pagnamenta approach[11] but modified in a manner similar to the suggestions of Mitra and Kumari[12] so that the resulting form factors scale correctly[15,16] at large q^2. Our particular prescription is discussed in Sec. III and involves an effective cluster "boost mass" which can be calculated directly from the model wave functions. The use here of an effective cluster mass M which varies as $\sqrt{n}$ for an n-particle cluster is in keeping with the earlier proposals by Brodsky and Chertok.[17]

Comparisons of our calculations with the available electromagnetic data on the pion, kaon, proton, and neutron are given in Sec. IV and provide an accurate description of the data. The present prescription suggests the interesting result that the measured kaon charge radius should be smaller in magnitude than the measured pion charge radius even though their "rest-frame" radii behave in just the opposite manner. It also gives some insight into the negative mean squared charge radius of the neutron and reconciles the "observed" sizes of hadrons with the "measured" values of the average transverse momentum squared $\langle p_\perp^2\rangle$ of quarks in a hadron.

II. MODEL WAVE FUNCTIONS AND FORM FACTORS

In the rest frame of a given composite system of n quarks we have found a nonperturbative quark model which yields a reasonably accurate description of both $q\bar{q}$ mesons[1] and q^3 baryons.[2] The Hamiltonian for particles of mass m_i and momentum

$\vec{p}_i$ is

$$H=\sum_i (p_i{}^2+m_i{}^2)^{1/2}+\sum_{i>j}\vec{F}_i\cdot\vec{F}_j v_{ij} \qquad (1)$$

with the constraint $\sum_i \vec{p}_i=0$. The interaction v_{ij} involves several components:

$$v_{ij}=\langle V_{\rm SR}\rangle+V_{\rm LR}+\langle V_A\rangle \qquad (2)$$

in which the long-range (LR) component is a linear confinement potential and the short-range (SR) component is associated with a one-gluon-exchange potential with strong spin-spin interactions. The term V_A applies only to $q\bar{q}$ interactions and allows for their mutual annihilation into two or three gluons. Details are given in Ref. 1.

Although the above Hamiltonian uses relativistic kinematics and effective quark sizes, the use of Pauli spin for each quark yields eigenvectors which are not invariant under Lorentz transformations on the center of momentum. The solutions are therefore limited to situations wherein the system is in its own rest frame. Form factors can be derived directly from such wave functions, e.g., the elastic charge form factor for a charged system of point quarks:

$$F_{\rm RF}(q^2)=\frac{1}{Z}\int\psi_\alpha^*(\tau)\sum_{i=1}^{n} e_i e^{i\vec{q}\cdot\vec{r}_i}\psi_\alpha(\tau)d\tau\,, \qquad (3)$$

in which Z is the total charge, e_i and $\vec{r}_i$ are the charge and position vector of the ith quark, and $\psi_\alpha(\tau)$ is the model eigenvector solution with eigenvalue α and with τ collectively denoting all internal coordinates. We refer to the above as a rest-frame (RF) form factor rather than a nonrelativistic form factor because we have included relativistic corrections in our basic Hamiltonian.

For $q\bar{q}$ mesons only one separation vector $\vec{r}$ is involved and solutions are obtained using

$$\psi_\alpha(\tau)=\sum_L [Y_L(\hat{r})\otimes\chi_S]_M^J\,U_{LSJ}^\alpha(r)\,, \qquad (4)$$

in which χ_S is a total spin state with eigenvalue S constructed from the quark and antiquark Pauli spins. The total angular momentum (J) is constructed by vector coupling the spin state χ_S to the orbital state represented by a spherical harmonic $Y_L(\hat{r})$. All the dynamics are in the radial solutions $U_{LSJ}^\alpha(r)$ which are conveniently transformed into a harmonic-oscillator basis via the expansion

$$U_{LSJ}^\alpha(r)=\sum_{n=1}^{N} a_{nLSJ}^\alpha R_{nL}(r)\,. \qquad (5)$$

Eigenvectors and eigenvalues are obtained for each S, J, and parity $\pi=(-)^{L+1}$ using a standard diagonalization procedure. For the pion and kaon considered in the next section $J^\pi=0^-$ and $S=0$ so that $L=0$ is also required. Elastic rest-frame form factors are then constructed using

$$F_{\rm RF}(q^2)=\sum_i \frac{e_i}{q_i}\int dr\,r\,U_{000}^{\alpha*}(r)\times\sin(q_i r)U_{000}^\alpha(r)f_i(q^2) \qquad (6)$$

in which $q_i=\sum_{j\neq i}m_j q/\sum_i m_i$ is the appropriate momentum transfer associated with each quark.

The extra factor $f_i(q^2)$ corresponds in our model to the use of quarks with an effective size[1] given by a Yukawa distribution, i.e.,

$$f_i(q^2)=(1+\beta_i{}^{-2}q^2)^{-1} \qquad (7)$$

with

$$\beta_i=\beta A^{1/3}m_i{}^{1/3} \qquad (8)$$

being given in terms of the scaled coupling constant A defined in Ref. 1 and a constant $\beta=1.659$ GeV$^{2/3}$. The use of a finite-size quark is essential to the success of the nonperturbative approach adopted here[1,2,5,9] and to our knowledge was first used by Kang and Sucher.[5]

For the charged pions we have $q_i=q/2$, $\sum_i e_i=\pm1$ and $f_i(q^2)$ is independent of i. The mean square radius defined by

$$F_{\rm RF}(q^2)=1-\langle r^2\rangle_{\rm RF}q^2/6+O(q^4) \qquad (9)$$

yields, for $\pi^\pm$,

$$\langle r^2\rangle_{\rm RF}=\tfrac{1}{4}\int dr\,r^4\,|U_{000}^\pi(r)|^2+6\beta_\pi{}^{-2}\,, \qquad (10)$$

with the last term arising from the quark finite size ($\beta_\pi\equiv\beta_i$ for the pion). As indicated in our earlier work[1] the model value of $\langle r^2\rangle_{\rm RF}{}^{1/2}$ obtained for pions is 0.425 fm which is significantly smaller than the measured values[18] $0.52\le\langle r\rangle^{1/2}\le 0.79$ fm. A corresponding discrepancy occurs in the form factor and as we shall see can be satisfactorily explained only when the effects of Lorentz contractions associated with the moving frame are included in the form factor of the $q\bar{q}$ system.

In the case of the kaon system we have $m_u=m_d=0.24$ GeV, $m_s=0.46$ GeV, and

$$\langle r^2\rangle_{\rm RF}=0.327\int dr\,r^4\,|U_{000}^K(r)|^2+4\beta_u{}^{-2}+2\beta_s{}^{-2}\,, \qquad (11)$$

which involves different coefficients from the pion due to the unequal masses of the quarks. We note that the value calculated in the rest frame

$$\langle r^2\rangle_{\rm RF}{}^{1/2}=0.444\ \text{fm}$$

is a little larger than the corresponding pion value as indeed one might expect due to the smaller potential "binding" effects in the kaon. Transforming to a moving frame inverts the above result as indicated in Sec. IV.

For baryons the model calculations are considerably more complicated. For three quarks we must have overall antisymmetry in the total space which includes color, flavor, Pauli spin, and orbital subspaces. Since we require a color singlet for physical baryons and for three quarks this can only be achieved by an antisymmetric color state (Young tableau $[1^3]$) the remaining subspaces must be constructed with overall symmetry [3]. The remaining subspaces are then described using a six-dimensional oscillator representation in the coordinates $\vec{\rho}$, $\vec{\lambda}$ which are given in terms of quark position vectors by

$$\vec{\rho} = (\tfrac{1}{2})^{1/2}(\vec{r}_1 - \vec{r}_2)\,, \quad \vec{\lambda} = (\tfrac{1}{6})^{1/2}[\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3]\,.$$

Appropriate orbital states with definite permutation symmetries (denoted by $[f]$) are constructed using the methods of Moshinsky and collaborators[19] which yield, for a total oscillator quanta $N = 2n_\rho + l_\rho + 2n_\lambda + l_\lambda$,

$$\Psi_{NLM_L}(\vec{\rho}, \vec{\lambda}, [f]\gamma) = \sum_{n_\rho l_\rho n_\lambda l_\lambda} a(n_\rho l_\rho n_\lambda l_\lambda, NL[f]\gamma) \times [R_{n_\rho l_\rho}(\vec{\rho}) \otimes R_{n_\lambda l_\lambda}(\vec{\lambda})]_{LM_L}\,. \tag{12}$$

The coefficients in Eq. (12) are the orbital coefficients of fractional parentage which allow us to use a convenient basis set wherein the particle labeled 3 is distinguished from the other two. We have also vector coupled the orbital angular momenta $\vec{l}_\rho$, $\vec{l}_\lambda$ to make the total orbital angular momentum $\vec{L}$ with projection M_L. For the nucleon we only use $L = 0$. The label γ is used to distinguish between degenerate states with the same quantum numbers which are linearly independent. We assume the basis is chosen to be orthonormal.

A completely symmetric *basis* state with definite total angular momentum J, isospin T, and hypercharge Y is constructed using appropriate products of SU(6) flavor and $O(3)$ orbital states:

$$\Phi^{\gamma}_{NSLJM}([3]) = \sum_f [\chi_{STYT_Z}([f]) \otimes \Psi_{NL}(\vec{\rho}, \vec{\lambda}, [f]\gamma)]_{JM}\,, \tag{13}$$

where for the nucleon $T = \frac{1}{2}$, $Y = +1$. In this case if $f = 3$ we have

$$\chi_{1/2,1/2,1,T_Z}([3]) = \frac{1}{\sqrt{2}}\,(\chi^S_\sigma \chi^S_\tau + \chi^A_\sigma \chi^A_\tau) \tag{14}$$

with

$$\chi_\sigma = [\chi_{12}(S_{12}) \otimes \chi_3(\tfrac{1}{2})]_{1/2\, S_Z} \tag{15}$$

and

$$\chi_\tau = [\chi_{12}(T_{12}) \otimes \chi_3(\tfrac{1}{2})]_{1/2\, T_Z} \tag{16}$$

given in terms of "parent" states of definite total spin (isospin) values S_{12} (T_{12}) for particles one and two which are symmetric (S) or antisymmetric (A) in particles 1 and 2 as S_{12} $(T_{12}) = 1$ or S_{12} $(T_{12}) = 0$, respectively. Similar parentage representations occur for the other symmetry states ([21] and $[1^3]$) and again we have a convenient basis in which particle 3 is distinguished from the other pair.

In the three-quark baryon the above parentage representations are convenient because they greatly facilitate the evaluation of matrix elements of one- or two-body operators. In particular, electric and magnetic operators given by

$$\mathcal{E} = \sum_i^n e_i e^{i\vec{q}\cdot\vec{r}_i}\,, \tag{17a}$$

$$\mathfrak{M} = \sum_i^n \mu_{iZ} e^{i\vec{q}\cdot\vec{r}_i} \tag{17b}$$

with $\mu_{iZ} = g_i e_i \sigma_{iZ}$ being the appropriate magnetic-moment operator in nuclear magnetons, can be written for identical particles as

$$\mathcal{E} = 3e_3 e^{i\vec{q}\cdot\vec{r}_3}\,, \tag{18a}$$

$$\mathfrak{M} = 3\mu_{3Z} e^{i\vec{q}\cdot\vec{r}_3}\,, \tag{18b}$$

which only act on the "last" particle. Matrix elements of $\mathcal{E}$ and $\mathfrak{M}$ are thereby reduced to one-body integrals since the dependence on $\vec{r}_1$, $\vec{r}_2$, etc., can be integrated out using orthonormality in the (12) subspaces.

For the nucleon we have only u and d quarks involved which have fractional charges $e_u = +\frac{2}{3}$, $e_d = -\frac{1}{3}$, respectively. If the quarks are Dirac point particles then we expect $g_u = g_d = M/m$, $g_s = M/m_s$, where M is the physical proton mass and $m = m_u = m_d$, m_s are the rest masses of each quark. Owing to our approximate treatment[1] wherein "small" components of the relativistic spinors are eliminated in favor of finite-size quarks we should expect g_i to be replaced by an effective g value given by

$$g_i^{\text{eff}} = M/m_i^{\text{eff}}\,, \tag{19}$$

in which m_i^{eff} is an effective quark mass. We introduce such a concept (rather than simply using an anomalous magnetic moment for quarks) because it appears that an effective mass m_i^{eff} can be calculated from the rest-frame Hamiltonian. Moreover, the same effective mass is needed in the boost prescription described in the next section.

Actual rest-frame baryon states $\psi_\alpha(\tau)$ are obtained using linear combinations of the states defined by Eq. (13) which diagonalize the baryon Hamiltonian. As discussed elsewhere[2] our baryon Hamiltonian is directly related to the meson Hamiltonian except for V_A becoming zero for baryons. For numerical simplicity we have also neglected spin-orbit and tensor interactions in the baryon calculations. In this work we also neglect the parametrized three-body effects considered in the earlier work.[2] In any event such effects are quite small and will not alter the major results obtained here. Rest-frame form factors for nucleons are then calculated from Eq. (3) for the electric form factor and from an analogous equation (but with μ_{iz} in place of e_i) for the magnetic form factor.

III. FORM FACTORS FOR MOVING FRAMES

We begin by recalling the essential assumptions of Licht and Pagnamenta[11] in making a transition from a rest-frame noncovariant matrix element to a relativistic matrix element in the Breit frame. The major assumption is that the interaction causing a transition from a given composite state $|A\rangle$ with four-momentum p to another composite state $|B\rangle$ with four-momentum p' can be regarded as instantaneous in the Breit frame [wherein $p=(E_A,\vec{p})$, $p'=(E_B,-\vec{p})$, and $q=(E_B-E_A,-2\vec{p})$ is the four-vector of momentum transfer]. A second assumption is that a given matter distribution can be described in each frame by a probability amplitude which is a function of three vectors only, i.e., $t=0$, corresponding to a single time formalism. If we place an observer in each frame then they see the same distribution but each would describe it in their frame coordinates. Licht and Pagnamenta identify

$$\psi_A^p(\{\vec{r}_i\})=\psi_A(\{\vec{r}_i^{\,0}\}), \tag{20}$$

where ψ_A^p is the probability amplitude for a moving observer and ψ_A is the wave function in the rest frame. The internal coordinates $\vec{r}_i$ and $\vec{r}_i^{\,0}$ in the respective frames are connected by Lorentz transformations. In particular, for elastic scattering $E_A=E_B$ and for the z axis chosen along $\vec{p}$ one obtains the following relationships for the internal coordinates:

$$x_i=x_i^0,\quad y_i=y_i^0,\quad z_i=\frac{M_A}{E_A}z_i^0. \tag{21}$$

The x and y directions are not affected by the boost and the z direction is contracted in the Breit frame by the inverse boost factor:

$$\frac{M_A}{E_A}=\frac{M_A}{(M_A{}^2+p^2)^{1/2}}\equiv(1+q^2/4M_A{}^2)^{-1/2}. \tag{22}$$

Here we interpret M_A as the *effective* mass of the composite system A and assume that it obeys a relativistic mass relation with respect to its constituents:

$$M_A{}^2=\sum_i(m_i^{\rm eff})^2=n(m^{\rm eff})^2 \tag{23}$$

for identical-mass quarks. This type of relationship has been suggested by Brodsky and Chertok[17] and appears to be more appealing than using the physical cluster mass. The physical mass of cluster A would be appropriate if the multiquark system behaved as a rigid body so that a momentum transfer imparted to a given constituent automatically is imparted to the entire cluster. However, the potential model considered here does not yield a rigid system and it seems more appropriate to regard each quark as having an effective mass $m^{\rm eff}$ which is primarily due to the total internal momentum of the quark in the direction $\hat{q}$. The momentum imparted to a quark with effective mass $m^{\rm eff}$ is then regarded as being transmitted to a cluster with effective mass M_A given by the relativistic sum over the internal momentum given by Eq. (23). The only important question remaining within the framework of such an approach is whether M_A can be calculated from the rest-frame states or not.

Presuming for the moment that M_A can be found, then elastic form factors could be generated from the rest-frame form factors of Sec. II using the simple substitution law[12]

$$F(q^2)=\left(\frac{M_A}{E_A}\right)^{2n-2}F_{\rm RF}\left(\frac{M_A{}^2}{E_A{}^2}q^2\right). \tag{24}$$

In adopting Eq. (24) or a modified version of it (see below) we deviate from Licht and Pagnamenta[11] according to the suggestions of Mitra and Kumari,[12] which correspond to the use of the factor $(M_A/E_A)^{2n-2}$ rather than $(M_A/E_A)^{n-1}$. The importance of using such a factor is that asymptotically Eq. (24) becomes

$$F(q^2\sim\infty)=\left(\frac{4M_A{}^2}{q^2}\right)^{n-1}F_{\rm RF}(4M_A{}^2), \tag{25}$$

which obeys the asymptotic $q^{-(2n-2)}$ power laws expected from field-theory considerations[15] as well as from experiment.[15,16]

The replacement of q^2 by $(M_A/E_A)^2q^2$ in the argument of $F_{\rm RF}(q^2)$ is a direct result of the coordinate relationship Eq. (21) being used to relate the Breit-frame form factor to the rest-frame form factor. The multiplicative factor $(M_A/E_A)^{2n-2}$ represents the Jacobian of the integration-volume transformation between the two frames according to the prescription of Mitra and Kumari. As pointed out by previous workers[11,12] these integration volume factors are needed because in the Breit frame the overlap integrals are less (due to the contraction in the Z direction) than in the rest frame. It is very important to note that the mean square radius is *increased* by these factors, i.e.,

$$\langle r^2\rangle=\langle r^2\rangle_{\rm RF}+\frac{3(n-1)}{2M_A{}^2} \tag{26}$$

involves an extra term added to the rest-frame value which depends upon the effective cluster mass M_A.

We now consider the evaluation of M_A for the systems of interest. To achieve a reasonable definition of M_A we need to improve the earlier formulations. In general the substitution law proposed by Licht and Pagnamenta implies a "rigid" system whereby $M_A \simeq$ physical cluster mass, which implies the internal motion of quarks is small. Since in present quark models of light hadrons the internal momenta of quarks are large compared to the rest mass it is more realistic to allow M_A to be a mass *operator* which depends upon the $n-1$ internal momenta of the quarks, i.e.,

$$M_A \to \hat{M}_A(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n)\,, \tag{27}$$

wherein $\vec{p}_1 = -\sum_{i=2}^{n} \vec{p}_i$ is the rest-frame constraint.

In changing from the rest frame to a moving frame in the direction of $\vec{p} = \frac{1}{2}\vec{q}$ we suggest the effective mass associated with such a one-dimensional transformation be related by a corresponding relativistic mass increase

$$\hat{M}_A = \left[M_A(0) + \sum_{i=1}^{n} p_{iq}{}^2/n\right]^{1/2} \tag{28}$$

wherein $p_{iq} = \vec{p}_i \cdot \hat{q}$ is the momentum of the ith quark in the direction of the momentum transfer or boost direction. Since the virtual photon operates on a single quark at a time we use the internal momentum squared contributed by just one quark. This is achieved in Eq. (28) by calculating the total $\sum_{i=1}^{n} p_{iq}{}^2$ and dividing by the number of quarks. The rest mass $M_A(0)$ is given by

$$M_A(0) = \sum_{i}^{n} m_i{}^2 \tag{29}$$

with m_i being the mass used in the rest-frame Hamiltonian of Eq. (1).

With the operator form [Eq. (28)] for $\hat{M}_A$ we can no longer assume that the Jacobian factors in Eq. (24) can be trivially factored from the matrix elements giving the rest-frame form factors. For $q\bar{q}$ and q^3 systems we will be dealing with a typical spatial matrix element

$$\mathfrak{M}^{\mathrm{RF}}_{NN'}(q^2) = \langle \Psi_{N0} | e^{i\vec{q}\cdot\vec{r}_n} | \Psi_{N'0}\rangle \tag{30}$$

by the substitution (we suppress the L index on the states since $L=0$ is used)

$$\begin{aligned}\mathfrak{M}_{NN'}(q^2) &= \left\langle \Psi_N \middle| \left(1 + \frac{q^2}{4\hat{M}_A{}^2}\right)^{n-1} \right.\\ &\qquad \left. \times \exp[i\vec{q}\cdot\vec{r}_n(1+q^2/4\hat{M}_A{}^2)^{-1/2}] \middle| \Psi_{N'} \right\rangle \\ &= \sum_{N''} \left\langle \Psi_N \middle| \left(1+\frac{q^2}{4\hat{M}_A{}^2}\right)^{n-1} \middle| \Psi_{N''}\right\rangle \mathfrak{M}^{\mathrm{RF}}_{N''N'}\left(\frac{\hat{M}_A{}^2}{\hat{E}_A{}^2}q^2\right).\end{aligned} \tag{31}$$

Equation (31) leads to Eq. (24) only if $\hat{M}_A$ is put equal to M_A (= constant) because then the term $(1 + q^2/4M_A{}^2)^{n-1}$ factors out and we obtain only the term $N'' = N$ due to orthogonality of the orbital basis used.

In our applications in Sec. IV we assume *diagonal* matrix elements are given by

$$\left\langle \Psi_N \middle| \left(1+\frac{q^2}{4\hat{M}_A{}^2}\right)^{1-n} \middle| \Psi_N \right\rangle = (1+q^2/4M_A{}^2)^{1-n}\,, \tag{32}$$

and off-diagonal elements ($n=3$ only) by

$$\langle \Psi_{N+2} | (1+q^2/4\hat{M}_A{}^2)^{-2} | \Psi_N\rangle = \frac{q^2}{2M_A{}^2}(1+cq^2/4M_A{}^2)^{-3}\,, \tag{33}$$

in which $c^3 = 2$ preserves the q^{-4} scaling at large q^2 and the leading term is correct for small q^2. Such off-diagonal terms occur only in the baryon case due to the inclusion in our baryon basis of different orbital symmetries. The mass M_A is calculated from Eq. (28) using

$$M_A = \left[M_A(0) + \sum_{i=1}^{n} \langle p_{iq}{}^2\rangle/n\right]^{1/2}\,, \tag{34}$$

in which $\langle p_{iq}{}^2\rangle$ is the expectation value of $p_{iq}{}^2$ for the rest-frame state of the system. For equal-mass constituents we can find m_A^{eff} from

$$m_A^{\mathrm{eff}} = \frac{1}{\sqrt{n}} M_A\,. \tag{35}$$

Clearly the above treatment of effective masses can be refined by allowing the masses in Eqs. (32)–(34) to become basis-state dependent. Such refinements are probably beyond the scope of the present treatment and as indicated in Sec. IV they only have small effects on the calculations of interest here.

IV. RESULTS

A. Pion form factor

The pion form factor calculated from Eqs. (24) and (6) with $M_\pi{}^2 = 2(m_\pi^{\mathrm{eff}})^2 = 0.407$ GeV^2 is in reasonable agreement with the photoproduction data[18] (full curve of Fig. 1). The dashed curve is the rest-frame form factor with no boost included ($M_\pi{}^2 \equiv \infty$), and as expected it is not a good description of the data. The wave function Ψ_π used to calculate $\langle p_{iq}{}^2\rangle$ is the calculated eigenvector of the Hamiltonian used to fit the entire meson spectrum in our earlier work.[1] The value of $\langle p_{iq}{}^2\rangle$ is 0.292 GeV^2 which in turn yields the value of $M_\pi{}^2$ from Eq. (34) with $M_\pi{}^2(0) = 2m_u{}^2 = 2m_d{}^2 = 0.115$ GeV^2. The value of m_u^{eff} for the pion is 0.451 GeV which is almost two times the rest mass m used in the Hamiltonian.

We also investigated the possibility of using other boost prescriptions than that described in Sec. III. All attempts using a physical pion mass m_π

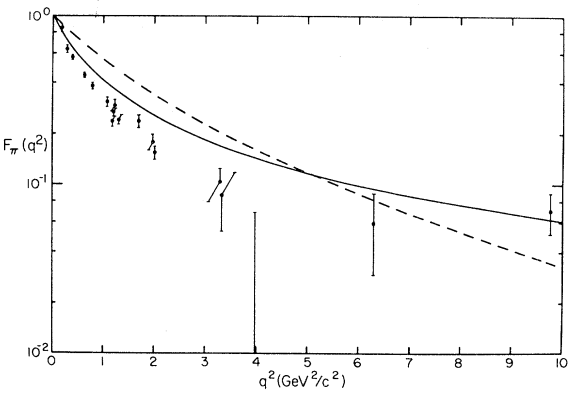

FIG. 1. Pion form factor calculated in the rest frame (dashed line) and moving frame (solid line). The data are from Ref. 18.

$=M_A$ yield far too rapid a falloff at large q^2. The inclusion of vector-meson propagators as used by Licht and Pagnamenta leads to an even faster falloff.

B. Pion and kaon radii

Using the wave functions of Ref. 1 we can easily calculate the root-mean-square radius for the charge distributions of the π and K mesons from Eq. (26) and the calculated values of M_π^2 and M_K^2. We find $\langle r^2\rangle_K^{1/2}$ is a little *smaller* than $\langle r^2\rangle_\pi^{1/2}$, i.e.,

$$\langle r^2\rangle_\pi^{1/2}=0.569 \text{ fm}, \quad \langle r^2\rangle_K^{1/2}=0.561 \text{ fm}, \tag{36}$$

wherein we use $M_\pi^2=0.407\ \mathrm{GeV}^2$, $M_K^2=0.497\ \mathrm{GeV}^2$, and $\langle r^2\rangle_{\mathrm{RF}}$ taken from Eqs. (10) and (11). These results are to be compared with the experimental values measured in π-e^-, K-e^- experiments[20,21]

$$\langle r^2\rangle_\pi^{1/2}=0.56\pm0.04 \text{ fm}, \quad \langle r^2\rangle_K^{1/2}=0.53\pm0.05 \text{ fm}, \tag{37}$$

which suggests that $\langle r^2\rangle_K^{1/2}\lesssim\langle r^2\rangle_\pi^{1/2}$ is possibly true. A more accurate determination of this result would be very useful as a test of the present ideology. It is the larger effective mass of the kaon which causes the moving-frame value for the kaon charge radius to be less than the pion value despite the fact that the opposite result holds in their rest frame. Assuming m_u^{eff} is the same for the pion and kaon allows us to extract a value for m_s^{eff} in the kaon. We find $m_s^{\mathrm{eff}}=0.542$ GeV which is significantly larger than the rest mass $m_s=0.460$ GeV used in the Hamiltonian.

C. Nucleon form factors

In the nucleon we are involved with two electromagnetic form factors denoted by $G_E(q^2)$ and $G_M(q^2)$ corresponding to the electric and magnetic form factors, respectively. The best rest-frame wave function of the nucleon is obtained by diagonalization of the three-quark Hamiltonian using up to $N=16$ quanta. For our purposes here we have used a simpler rest-frame solution which contains the important ingredients of the more exact solution. This simpler wave function has the following linear combination of its components:

$$\Psi(123)=a_0\Psi_0+a_2\Psi_2, \tag{38}$$

where omitting the overall color-singlet function

$$\Psi_0=R_{00}(\vec{\rho})R_{00}(\vec{\lambda})\chi_{1/2,1/2,1,T_Z}([3]) \tag{39}$$

and

$$\Psi_2=2^{-1}[R_{00}(\vec{\rho})R_{10}(\vec{\lambda})-R_{10}(\vec{\rho})R_{00}(\vec{\lambda})]\chi_{1/2,1/2,1,T_Z}([21+]) + 2^{-1/2}R_{01}(\vec{\rho})R_{01}(\vec{\lambda})\chi_{1/2,1/2,1,T_Z}([21-]), \tag{40}$$

and the $R_{nl}(\vec{r})$ are the three-dimensional oscillator solutions[22] with $R_{00}(\vec{\rho})$ being of the form $e^{-\nu\rho^2/2}$. The spin-flavor solutions $\chi_{STYT_Z}([f])$ are constructed with definite permutation symmetry. The fully symmetric state [3] was discussed above. The other two of interest here are given by

$$\chi_{1/2,1/2,1,T_Z}([21+])=\frac{1}{\sqrt{2}}(\chi_\sigma^A\chi_\tau^A-\chi_\sigma^S\chi_\tau^S), \quad \chi_{1/2,1/2,1,T_Z}([21-])=\frac{1}{\sqrt{2}}(\chi_\sigma^A\chi_\tau^S+\chi_\sigma^S\chi_\tau^A). \tag{41}$$

The component Ψ_0 is the conventional space symmetry wave function used by early workers which assigns the proton to the symmetric $\underline{56}$ spin-flavor multiplet. Owing to our strong spin-spin interactions we have allowed for the admixture of the orbital mixed symmetry states in Ψ. The coefficients a_0 and a_2 are chosen to represent the sign and magnitude of the overall mixing between the [3] and [21+] symmetries calculated in the more exact solution. The oscillator strength parameter is chosen in our simpler solution so that it yields a rest-frame radius which is the same as the exact solution. With this type of simplified solution we can use the boost prescriptions given by Eqs. (30)–(35). Detailed results for the nucleon form factors are given in the Appendix.

The values of $\langle p_{iq}^2\rangle$ in Eq. (34) for the proton and neutron are obtained using the simple wave function of Eq. (39) with $a_0=0.9878$, $a_2=-0.1560$, and $\nu=0.45\ \mathrm{GeV}^2$. This yields

$$\sum_{i=1}^{n} \langle p_{iq}^{2}\rangle /3 = 0.152\ \text{GeV}^2\,,$$

$M_A{}^2 \equiv M_p{}^2 =$ 0.32[illegible] GeV^2,

and (42)

$$m_p^{\text{eff}} = 0.329\ \text{GeV}\,.$$

Rest-frame form factors are calculated for the electric and magnetic operators [Eqs. (17)–(19)] using the wave functions of Eqs. (38)–(40) with the (electric, magnetic) results being shown by the dashed curves in Figs. 2 and 3 for protons and in Figs. 4 and 5 for neutrons. Applying Eqs. (30)–(35) as the modification to Eq. (24) for the moving-frame form factors we obtain the full curves shown in Figs. 2–5. The effect of making the mass M_p state dependent was investigated. Only the neutron electric form factors shows any sensitivity and this arises because it is dominated by the $N=0 \rightarrow N=2$ matrix element with a boost factor given by Eq. (33).

All moving-frame nucleon form-factor calculations appear to provide a reasonably accurate description of the data[23] over a wide range of q^2. The rest-frame form factors are seen to be inadequate descriptions for all q^2 and at large q^2 the results are diverging exponentially from the data. The magnetic form factors in the rest frame have been normalized using experimental moments[24] (i.e., for the proton $\mu_p = 2.79$ and for the neutron $\mu_n = -1.91$) rather than their calculated values with $g_i = M/m_i = M/(0.24\ \text{GeV})$. The moving-frame magnetic form factors on the other hand are normalized according to their *calculated* magnetic moments using $g_i = M/m_i^{\text{eff}}$ for each matrix element. We obtained the values shown in Table I. These values are in close agreement with the experimental magnetic moments[24] whereas the calculated rest-frame values with $g_i = M/(0.24\ \text{GeV})$ are 40% too large. Thus the effective-mass approach appears to provide a reliable estimate of the anomalous magnetic moments of our quarks as well as a boost mechanism for the form of the electromagnetic form factors. The effective mass of a quark in a nucleon is somewhat smaller than it is in a pion.

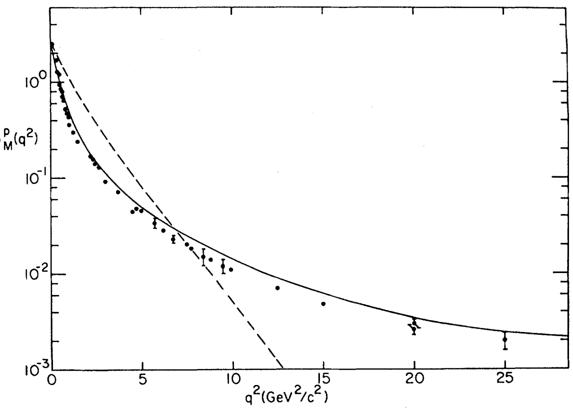

FIG. 3. The same as Fig. 2 except for the proton magnetic form factor.

At large q^2 the form of the q^2 dependence is dominated by the q^{-4} scaling behavior, i.e.,

$$G_{E,M} \sim C_{E,M}/q^4\,, \tag{43}$$

wherein the constants depend upon the fourth power of the boost mass as well as the value of the rest-frame form factor at $q^2 = 4M_p{}^2$. In view of the sensitivity of $C_{E,M}$ to the value of $M_p{}^2$ it is gratifying to find that the present prescription yields a reasonable result. For baryons (as for pions) the use of a physical nucleon mass for M_A leads to moving-frame form factors which are inadequate descriptions of the data particularly at large q^2 where the falloff rate is too slow.

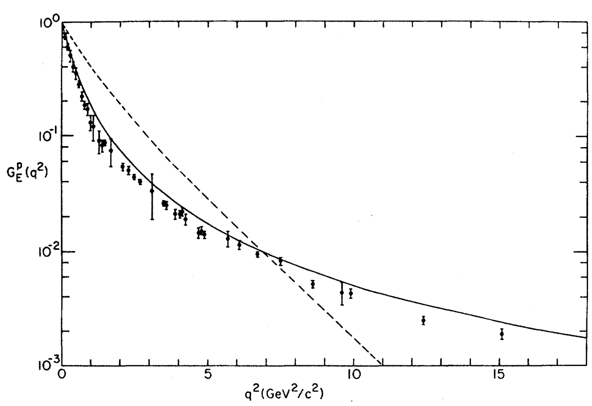

FIG. 2. Proton electric form factor calculated in the rest frame (dashed line) and moving frame (solid line). The data are from Ref. 23.

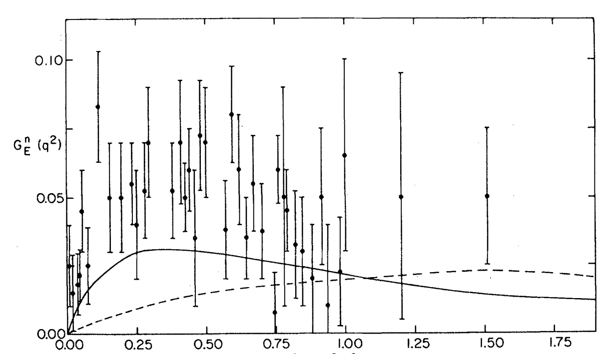

FIG. 4. The same as Fig. 2 except for the neutron electric form factor.

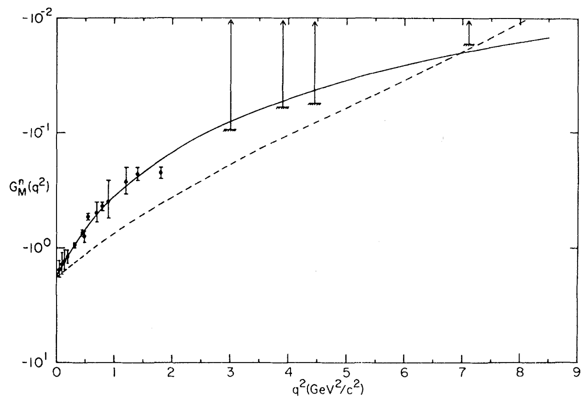

FIG. 5. The same as Fig. 2 except for the neutron magnetic form factor.

D. Proton and neutron radii

Results for the calculated squared charge radii for both nucleons are compared with the measured values[23-26] in Table I. The results for the moving frame are significantly larger than the rest-frame results. These large increases arise because of the boost terms such as those shown in Eq. (26) and are needed to provide radii similar to the experimental values. This is particularly true in the case of the neutron charge radius where the off-diagonal term of Eq. (36) yields a boost term in $\langle r^2\rangle_{ch}$ of -1.41 GeV^{-2} (-0.055 fm^2).

Additional corrections to the radii are expected because we are only calculating results using "primitive" configurations (q^3). The most important configurations which need to be studied are $(q^3)(q\bar{q})$ wherein the $(q\bar{q})$ as a virtual pion is expected to be dominant at large radii. A simple but naive "cutoff" calculation of the self-energy of a pion "cloud" which yields the correct nucleon mass and the correct one-pion-exchange tail yields small percentage changes in the proton radius but larger changes in the neutron charge radius as shown in Table I. Since these pion production calculations are oversimplified we are satisfied to note that the order-of-magnitude effects which arise from such sources are not unreasonable. A more detailed investigation of the role of more complicated configurations is now in progress and will be reported on at a later time. In the case of baryons we also need to include spin-orbit and tensor interactions, at which point a more precise comparison with data will be appropriate.

V. SUMMARY

The prescription adopted here allows the rest-frame wave functions of quark models to be used (without any new parameters being introduced) to calculate moving-frame electromagnetic form factors. Using a pseudorelativistic Hamiltonian which fits mesons and baryons, we find the boost prescription to be remarkably successful in providing accurate form factors for the pion, proton, and neutron over a wide range of q^2. The results given here are limited to the case of elastic form factors. Extension of the present approach to inelastic form factors should be feasible using analogous extensions to the suggestions of Mitra and Kumari. Improvements of the present approach will involve the use of more complicated configurations than the simple "primitive" configurations adopted here. However, it is clear that massive amounts of meson clouds around nucleons are not needed in the rest-frame wave functions. Small amounts of πN and $\pi\Delta$ configurations are expected ($\lesssim 5\%$ in magnitude) and appear to be necessary to provide a complete explanation of the neutron charge radius.

In concluding we note two other points of current interest. The first is the apparent discrepancy between the size of a hadron observed in electron scattering (~0.8 fm) and the average transverse

TABLE I. Proton and neutron charge radii and magnetic moments.

	Rest frame	Moving frame	Moving frame including virtual pions	Experiment
$\langle r_p^2\rangle$ (fm^2)	0.247	0.631	0.646	0.648 ± 0.018[b]
$\langle r_n^2\rangle$ (fm^2)	-0.011	-0.066	-0.097	-0.116 ± 0.002[c]
μ_p $(e\hbar/2M_pc)$	2.79[a]	2.807	2.824	2.793[d]
μ_n $(e\hbar/2M_pc)$	-1.91[a]	-1.867	-1.885	-1.913[d]

[a]Rest-frame magnetic

[b]This is the more commonly accepted value from Ref. 25. Note that Borkowski *et al.* (Ref. 26) determined a much larger value, $\langle r_p^2\rangle = 0.77 \pm 0.05$ fm^2, from their low-q^2 data.

[c]Reference 23.

[d]Reference 24.

momentum squared derived[27] from large-mass lepton-pair production in proton-proton or proton-nucleus collisions.[28] According to our present approach the size of a hadron observed in electron scattering is enhanced because of the Lorentz factors in the longitudinal components. Calculating the average transverse momenta squared $\langle p_\perp^2\rangle$ for our model Hamiltonian we obtain the values (all in GeV^2) of 0.584, 0.456, and 0.304 for the pion, kaon, and nucleon, respectively. These values appear to be in good agreement with those derived from experiment. We also note that the contribution to $\langle r^2\rangle$ for the matter distribution of nucleons arising from just the relative motion of quarks in the rest frame is 2.51 GeV^{-2}, corresponding to $\langle r^2\rangle^{1/2} = 0.31$ fm from this source. It is the finite quark size and boost mechanism which leads to the observed radius of 0.8 fm. Such a small intrinsic size is what is needed to provide large enough transverse momenta within hadrons.

The second point of interest is the importance of moving-frame considerations for hadron-hadron interactions. In particular, attempts[29,30] to derive the nucleon-nucleon interaction from quark models will have to be improved because the quark-exchange mechanism involves moving frames. Attempts to calculate nucleon-nucleon interactions including such effects are now under investigation by us and preliminary results show that moving-frame effects are important. Details of this latter work will be reported upon in the near future.

ACKNOWLEDGMENT

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APPENDIX

For completeness we give algebraic forms for the nucleon form factors arrived at using the wave function given by Eqs. (38) and (39). The form factors are given by

$$G_E^n(q^2) = -a_0 a_2[\alpha^0\beta_3^{02} + \tfrac{1}{2}\alpha^1(\beta_3^{00}+\beta_3^{22})]\,,$$

$$\begin{aligned}G_E^p(q^2) = {} & a_0^2[\alpha^0\beta_3^{00}+\alpha^1\beta_3^{02}] + \tfrac{1}{2}a_2^2\alpha^0\beta_3^p \\ & + \tfrac{1}{2}a_2^2[\alpha^1\beta_3^{02}+\tfrac{1}{2}\alpha^0(\beta_3^{00}+\beta_3^{22})] \\ & + a_0 a_2[\alpha^0\beta_3^{02}+\tfrac{1}{2}\alpha^1(\beta_3^{00}+\beta_3^{22})]\,,\end{aligned}$$

$$\begin{aligned}G_M^n(q^2) = -\frac{2}{3}\,\frac{M_N}{m_u^{\mathrm{eff}}}\,\{ & a_0^2[\alpha^0\beta_3^{00}+\alpha^1\beta_3^{02}] - \tfrac{1}{2}a_2^2\alpha^0\beta_3^p \\ & + \tfrac{1}{2}a_2^2[\alpha^1\beta_3^{02}+\tfrac{1}{2}\alpha^0(\beta_3^{00}+\beta_3^{22})] \\ & + \tfrac{1}{2}a_0 a_2[\alpha^0\beta_3^{02}+\tfrac{1}{2}\alpha^1(\beta_3^{00}+\beta_3^{22})]\}\,,\end{aligned}$$

$$\begin{aligned}G_M^p(q^2) = \frac{M_N}{m_u^{\mathrm{eff}}}\,\{ & a_0^2[\alpha^0\beta_3^{00}+\alpha^1\beta_3^{02}] - \tfrac{1}{6}a_2^2\alpha^0\beta_3^p \\ & + \tfrac{1}{2}a_2^2[\alpha^1\beta_3^{02}+\tfrac{1}{2}\alpha^0(\beta_3^{00}+\beta_3^{22})] \\ & + a_0 a_2[\alpha^0\beta_3^{02}+\tfrac{1}{2}\alpha^1(\beta_3^{00}+\beta_3^{22})]\}\,.\end{aligned}$$

In these equations the Jacobian terms are given by

$$\alpha^0 = (1+q^2/4M_A^2)\,,$$

$$\alpha^1 = \frac{q^2}{2M_A^2}(1+cq^2/4M_A^2)^{-3}\,,$$

with $c^3 = 2$ as given by Eqs. (32) and (34). In terms of the effective momentum transfer

$$q_e^2 = q^2/(1+q^2/4M_A^2)\,,$$

the internal-form-factor terms are given by

$$\beta_3^{00} = e^{-q_e^2/6\nu} f_3(q_e^2)\,,$$

$$\beta_3^{02} = (\tfrac{2}{3})^{1/2}\frac{q_e^2}{6\nu}\,e^{-q_e^2/6\nu} f_3(q_e^2)\,,$$

$$\beta_3^{22} = \left(1-\frac{2}{9}\,\frac{q_e^2}{\nu}+\frac{q_e^4}{54\nu^2}\right)e^{-q_e^2/6\nu} f_3(q_e^2)\,,$$

$$\beta_3^p = \left(1-\frac{q_e^2}{9\nu}\right)e^{-q_e^2/6\nu} f_3(q_e^2)\,,$$

where the effective-size form factor as given by Eq. (7) is

$$f_3(q_e^2) = (1+\beta_3^{-2}q_e^2)^{-1}\,.$$

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