Reconstruction of a spinor via Fierz identities

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We investigate to what extent a spinor is recovered from S, P, V_μ, A_μ, and $T_{\mu\nu}$ satisfying the Fierz identities. To do this, we introduce a parameter representation for the 16 tensor quantities, which takes care of all the Fierz identities, and set up a spinor-determining equation. This equation is solved. The solution contains 4 arbitrary complex numbers. These constants can be determined except an overall phase if we demand that the bilinear forms agree with the original 16 tensor quantities.

The close relation between a spinor and tensors has become an important subject of investigation in recent years, in connection with the supersymmetry theory, the Bose-Fermi equivalence theory, field theory of extended objects, etc. We examine in this note the reconstructabilities of a spinor from the Fierz identities.

It is well known that 16 bilinear quantities are formed by a given spinor u_α ($\alpha=1,2,3,4$). They are scalar, pseudoscalar, vector, axial vector, and tensor denoted, respectively, by

$$S \equiv \bar{u}u \quad , \tag{1a}$$

$$P \equiv i\bar{u}\gamma_5 u \quad , \tag{1b}$$

$$V_\mu \equiv i\bar{u}\gamma_\mu u \quad , \tag{1c}$$

$$A_\mu \equiv i\bar{u}\gamma_5\gamma_\mu u \quad , \tag{1d}$$

$$T_{\mu\nu} \equiv \bar{u}\sigma_{\mu\nu}u \quad . \tag{1e}$$

Now, we ask ourselves the following questions: To what extent can the spinor be recovered for given S, P, V_μ, A_μ, and $T_{\mu\nu}$? An associated question is how many data in terms of tensors should be given to recover the spinor.

To answer the above questions, we recall that a spinor can be constructed out of two constrained vectors[1] and that the very constraint condition plays an essential role.[2,3]

Let us first analyze the relations which exist among quantities (1). Such relations are provided by the Fierz identities

$$(\gamma_A)_{\alpha\beta}(\gamma_B)_{\lambda\rho} = (\tfrac{1}{4})^2 \sum_{C,D} \mathrm{Tr}(\gamma_A\gamma_C\gamma_B\gamma_D) \times (\gamma_C)_{\lambda\beta}(\gamma_D)_{\alpha\rho} \quad , \tag{2}$$

where γ_A, γ_B, γ_C, γ_D are 16 linearly independent components of the Dirac γ algebra. By multiplying the Fierz identities by $\bar{u}_\alpha u_\beta \bar{u}_\lambda u_\rho$ and summing over all indices (we assume all the u_α are c numbers) we obtain $16\times 16 = 256$ relations among 16 quantities in Eq. (1). Of course, these are highly redundant and only the following 9 relations are independent[4]:

$$T_{\mu\nu}V_\nu = -PA_\mu \quad , \tag{3a}$$

$$^*T_{\mu\nu}V_\nu = SA_\mu \quad , \tag{3b}$$

$$V_\mu V_\mu = -A_\mu A_\mu = -(S^2+P^2) \quad , \tag{3c}$$

where $^*T_{\mu\nu}$ is the dual of $T_{\mu\nu}$ defined by

$$^*T_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\lambda\rho}T_{\lambda\rho} \quad . \tag{4}$$

[Note that Eqs. (3a) and (3b) contain only 7 independent relations.] Hence, we can give at our disposal only 7 among 16 quantities in Eq. (1). For example, we can choose $\vec{V}$, $\vec{A}$, and P. It proves convenient, however, to use the following 7 parameters θ, ϕ, χ, ξ_1, ξ_2, S, and P to represent the quantities in (1):

$$A_\mu = (S^2+P^2)^{1/2}(C_1 a_i^{(1)}, iS_1) \quad , \tag{5a}$$

$$V_\mu = (S^2+P^2)^{1/2}(S_1C_2a_i^{(1)} + S_2a_i^{(2)}, iC_1C_2) \quad , \tag{5b}$$

$$\tfrac{1}{2}\epsilon_{ijk}T_{jk} = S(C_2a_i^{(1)} - S_1S_2a_i^{(2)}) + PC_1S_2a_i^{(3)} \quad , \tag{5c}$$

$$iT_{i4} = -P(C_2a_i^{(1)} - S_1S_2a_i^{(2)}) + SC_1S_2a_i^{(3)} \quad , \tag{5d}$$

where

$$\left.\begin{aligned} C_i &= \cosh\xi_i \\ S_i &= \sinh\xi_i \end{aligned}\right\} \quad i=1,2 \tag{6}$$

and $a_i^{(k)}$ ($k=1,2,3$) are an orthogonal triad represented by the Euler angles θ, ϕ, and χ as

$$\vec{a}^{(1)} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \quad , \tag{7a}$$

$$\vec{a}^{(2)} = \begin{pmatrix} \cos\theta\cos\phi\cos\chi - \sin\phi\sin\chi \\ \cos\theta\sin\phi\cos\chi + \cos\phi\sin\chi \\ -\sin\theta\cos\chi \end{pmatrix} \quad , \tag{7b}$$

$$\vec{a}^{(3)} = \begin{pmatrix} -\cos\theta\cos\phi\sin\chi - \sin\phi\cos\chi \\ -\cos\theta\sin\phi\sin\chi + \cos\phi\cos\chi \\ \sin\theta\sin\chi \end{pmatrix} \quad . \tag{7c}$$

We have assumed here that $S^2+P^2\neq 0$. The relations (5) are compatible with the relations (3).

From the Lorentz transformation property of the quantities in the left-hand side of Eqs. (5), we can determine that of the parameters θ, ϕ, χ, ξ_1, and ξ_2 as

$$\delta\xi_1=a_i^{(1)}\delta v_i \ , \tag{8a}$$

$$\delta\xi_2=\frac{1}{C_1}a_i^{(2)}\delta v_i \ , \tag{8b}$$

$$\delta\theta=(e_1\sin\phi-e_2\cos\phi)\delta\alpha+\frac{S_1}{C_1}(\cos\theta\cos\phi\delta v_1+\cos\theta\sin\phi\delta v_2-\sin\theta\delta v_3) \ , \tag{8c}$$

$$\delta\phi=(e_1\cot\theta\cos\phi+e_2\cot\theta\sin\phi-e_3)\delta\alpha-\frac{S_1}{C_1}\left[\frac{\sin\phi}{\sin\theta}\delta v_1-\frac{\cos\phi}{\sin\theta}\delta v_2\right] \ , \tag{8d}$$

$$\delta\chi=-\left[e_1\frac{\cos\phi}{\sin\theta}+e_2\frac{\sin\phi}{\sin\theta}\right]\delta\alpha+\left[\frac{C_2}{C_1S_2}a_1^{(3)}+\frac{S_1}{C_1}\cot\theta\sin\phi\right]\delta v_1+\left[\frac{C_2}{C_1S_2}a_2^{(3)}-\frac{S_1}{C_1}\cot\theta\cos\phi\right]\delta v_2+\frac{C_2}{C_1S_2}a_3^{(3)}\delta v_3 \ , \tag{8e}$$

where $\vec{e}$ is the axis of rotation, $\delta\alpha$ the infinitesimal angle of rotation, and $\delta\vec{v}$ the infinitesimal velocity of boost.

In order to construct a spinor, we consider a four-component function of the parameters, $u_\alpha(\theta,\phi,\chi,\xi_1,\xi_2)$. Under the infinitesimal Lorentz transformation, the function u_α undergoes the transformation

$$u_\alpha(\theta,\phi,\chi,\xi_1,\xi_2)\to u_\alpha(\theta',\phi',\chi',\xi_1',\xi_2')=u_\alpha(\theta,\phi,\chi,\xi_1,\xi_2)+\left[\delta\theta\frac{\partial}{\partial\theta}+\delta\phi\frac{\partial}{\partial\phi}+\delta\chi\frac{\partial}{\partial\chi}+\delta\xi_1\frac{\partial}{\partial\xi_1}+\delta\xi_2\frac{\partial}{\partial\xi_2}\right]u_\alpha \ . \tag{9}$$

On the other hand, if u_α is a spinor, it must transform as

$$u_\alpha\to u_\alpha'=\left[I+\frac{i}{4}\sigma_{\mu\nu}\epsilon_{\mu\nu}\right]_{\alpha\beta}u_\beta \ , \tag{10}$$

with

$$\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2i \tag{11}$$

and

$$\epsilon_{ij}=\epsilon_{ijk}e_k\delta\alpha \ , \tag{12a}$$

$$\epsilon_{i4}=-i\delta v_i \ . \tag{12b}$$

If we equate Eqs. (9) and (10), we obtain the spinor-determining equation which can be solved, giving

$$u\equiv\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=\begin{pmatrix}c_1\alpha_++d_1\alpha_-\\c_2\alpha_++d_2\alpha_-\end{pmatrix} \ , \tag{13}$$

with

$$\alpha_+=\begin{pmatrix}e^{-i\phi/2}\cos\frac{\theta}{2}\\e^{i\phi/2}\sin\frac{\theta}{2}\end{pmatrix}e^{-i\chi/2} \ , \tag{14a}$$

$$\alpha_-=\begin{pmatrix}-e^{-i\phi/2}\sin\frac{\theta}{2}\\e^{i\phi/2}\cos\frac{\theta}{2}\end{pmatrix}e^{i\chi/2} \ , \tag{14b}$$

and c_1 c_2, d_1, and d_2 are complex numbers involving further arbitrary complex numbers c, d, e, and f, i.e.,

$$c_1=ce^{(\xi_1+\xi_2)/2}+de^{(\xi_1-\xi_2)/2}+ee^{-(\xi_1-\xi_2)/2}+fe^{-(\xi_1+\xi_2)/2} \ , \tag{15a}$$

$$c_2=ce^{(\xi_1+\xi_2)/2}+de^{(\xi_1-\xi_2)/2}-ee^{-(\xi_1-\xi_2)/2}-fe^{-(\xi_1+\xi_2)/2} \ , \tag{15b}$$

$$d_1=-ee^{(\xi_1+\xi_2)/2}+fe^{(\xi_1-\xi_2)/2}+ce^{-(\xi_1-\xi_2)/2}-de^{-(\xi_1+\xi_2)/2} \ , \tag{15c}$$

$$d_2=ee^{(\xi_1+\xi_2)/2}-fe^{(\xi_1-\xi_2)/2}+ce^{-(\xi_1-\xi_2)/2}-de^{-(\xi_1+\xi_2)/2} \ . \tag{15d}$$

The arbitrary constants c, d, e, f are independent of θ, ϕ, χ, ξ_1, ξ_2, but can still contain S and P. Now, these constants can be fixed if we substitute the solution (13) into the right-hand side of Eqs. (1) and compare with the parameter representation (5). The calculation is very tedious, but the result is

$$c=d=\tfrac{1}{2}c_0e^{i\alpha/2} \ , \tag{16a}$$

$$e=f=\tfrac{1}{2}c_0e^{-i\alpha/2} \ , \tag{16b}$$

where

$$\alpha = \tan^{-1}(P/S) \ , \tag{17}$$

$$4|c_0|^2 = (S^2 + P^2)^{1/2} \ . \tag{18}$$

In other words, the spinor giving the representation (5) can be recovered all but for an arbitrary phase in c_0. As is seen above, the gauge freedom of the first kind is obtained in constructing the spinor from the gauge-invariant tensors satisfying the Fierz identities, whereas the gauge freedom of the second kind comes from the gauge-invariant field strengths satisfying the Bianchi identity. Such an analogy between the gauge transformations of the first and the second kinds would be further substantiated if we could interpret the Fierz identities as physical laws just as the Bianchi identity represents such physical laws as the nonexistence of the magnetic monopole and Faraday's law.

Note added. The author's attention was drawn to the series of papers by V. A. Zhelnorovich in which the spinor construction from tensors is discussed at length from a different viewpoint. See V. A. Zhelnorovich, Dok. Akad. Nauk SSSR 249, 85 (1979) [Sov. Phys. Dokl. 24, 899 (1979)].

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[1]M. A. Kramers, *Quantum Mechanics* (North-Holland, Amsterdam, 1957); E. T. Whittaker, Proc. R. Soc. London A158, 38 (1937).

[2]F. Bopp and R. Haag, Z. Naturforsch. 59, 644 (1950).

[3]O. Hara and T. Goto, Prog. Theor. Phys. 44, 1383 (1970). Other papers on spinor theory can be traced from here.

[4]See, for example, F. A. Kaempffer, Phys. Rev. D 23, 918 (1981); J. F. Dumais, M. Sc. thesis, University of Alberta, 1975 (unpublished).