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Equivalence of the four-point interaction and the Yukawa interaction. II. A fermion model

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It is explicitly demonstrated that, just as in the case of the four-point bosonic interaction, the compositeness conditions by themselves do not ensure a complete equivalence of the Yukawa interaction with the corresponding four-fermion interaction. We study the corresponding situation in the context of the Lee model versus the separable-potential model, with the N and θ fields quantized as fermions and V as a boson. We find that the spectrum of the Lee model consists of that of the separable potential together with some other contributions. It is shown that in the limit in which the bare coupling constant of the Yukawa theory becomes infinite, these additional contributions move to infinity. It is further shown that the finite-energy wave functions calculated from the two theories coincide in the strong-coupling limit (this limit ensures the compositeness condition). It is demonstrated, in the same manner as for the bosonic case, that in order to make the two theories essentially identical, in addition to taking the strong-coupling limit, all the spectral contributions at infinity in the Yukawa theory have to be removed. It is, thus, suggested that the proofs of equivalence of the four-fermion interaction and the Yukawa-type interaction may need to be reexamined.

I. INTRODUCTION

Ever since the possibility of a close connection between the nonrenormalizable four-point interaction and the renormalizable Yukawa interaction was pointed out in the early sixties,[1] there have been several attempts to derive both Abelian[2] and non-Abelian[3] gauge theories starting from one single quartically self-coupled spinor field. The *modus operandi* of all these authors is to directly compare the forms of the renormalized Lagrangians for the four-fermion and Yukawa-type interactions and demonstrate thereby that there is a one-to-one correspondence between the Feynman graphs of the two theories provided certain renormalization constants of the Yukawa theory vanish.[4] If these compositeness conditions are satisfied, the two theories, these authors argue, are equivalent; the gauge bosons of the Yukawa theory are then regarded as composites of the fermions. Moreover, this alleged equivalence is used to further argue that the four-fermion interaction is renormalizable.

This led us to investigate[5] the corresponding situation for two soluble models, the Lee model[6] (Yukawa-type theory) and the separable-potential model (four-point interaction), and study the sense in which these two distinct theories could be regarded as equivalent. We proceeded by first showing that it was possible to realize the compositeness condition $Z=0$ in the strong-coupling (SC) limit, defined by first introducing an ultraviolet cutoff on the momentum integrals and then taking the bare coupling to infinity. The cutoff could then be taken to infinity. We found, however, that in the cutoff theory the spectrum of the Lee model developed additional spectral contributions beyond the cutoff, which did not appear in the separable-potential model. These moved to infinity in the SC limit. Furthermore, we showed that the contributions of these to the finite-energy scattering amplitudes and S matrix vanish in the SC limit, so that in the SC limit the scattering amplitudes and S matrix calculated from the Lee model and the separable-potential model coincide at finite energies. We also demonstrated that if we explicitly removed all contributions to the Hamiltonian matrix elements from the part of the spectrum at infinity the Hamiltonians of the Lee model and the separable-potential model become effectively the same in the SC limit. We emphasize, though, that unless this is done the spectrum of the Lee model differs from that of the separable-potential model, and the two theories are clearly distinct. It is only when the spectral contributions that occur in the Lee model but not in the separable-potential model are explicitly omitted that the Lee model is transmuted into the model of the separable potential.

This demonstration led us to suggest that if the same scenario prevails in the case of fully relativistic field theories a mere demonstration of the coincidence of all the Green's functions (or equivalently, the scattering amplitudes) of the four-fermion and Yukawa-type theories is not sufficient to ensure their equivalence, at variance with what has been implied by other authors.[2,3] It may be argued, though, that the demonstration in Ref. 5 is restricted to the case of Bose fields. To strengthen the analogy between our model study and the real case of interest, we are led to the comparison of two theories of fermions, one in which fermions interact by a contact interaction, and the other in which the interaction is mediated by a boson, with a view to study the situation when the compositeness conditions are satisfied, i.e., when the intermediate boson is regarded as a composite of the fermions.

To this end, we consider, once again, the comparison of the Lee model and the separable-potential model, with the important difference in that the N and θ fields are quantized as fermions and the V as a boson. We demonstrate that just as in the case of the bosonic Lee model[5] the compositeness condition, by itself, does not ensure the equivalence of the two theories. Once again, we find that the Lee model develops additional spectral contributions beyond the cutoff; as before,[5] these do not affect the finite-energy scattering amplitudes or S-matrix elements in the SC limit. We show, just as in the case of the bosonic Lee model, that in the SC limit the two theories become effectively identical if, and only if, all effects of those states that have their origin in the cutoff are explicitly removed. Hence, the transmutation mechanism discussed in Ref. 5 is not characteristic of Bose fields.

The plan of this paper is as follows. In Sec. II we show that the solutions of the fermionic separable-potential model in the $N\theta$ sector are the same as in the bosonic case. The effects of the altered statistics of the particles, it is shown, manifest themselves in the $N\theta\theta$ sector. In Sec. III, we discuss the solutions to the Lee model in the $N\theta$ and the $N\theta\theta$ sectors. Once again, we find that the solutions in the lower sector are unaltered from those of the bosonic Lee model. In Sec. IV we demonstrate the transmutation of the fermionic Lee model to the fermionic separable potential. On account of the simpler structure of the solutions in the present case, this demonstration can be done much more directly than in the case of the Lee model for bosons. We conclude in Sec. V with some general remarks. In Appendix A, we list the important formulas used in deriving the results in the text. In Appendix B, we explicitly show that the physical states in the $N\theta\theta$ sector of the Lee model are, indeed, complete with respect to the usual inner product. In Appendix C, we show that the finite-energy wave functions of the Lee model reduce to those of the separable-potential model in the strong-coupling limit.

II. THE FERMIONIC SEPARABLE-POTENTIAL MODEL

The system described by the separable-potential model consists of two fields, associated with the N and θ particles, interacting via a four-point interaction. In order to mimic the features of the fully relativistic four-fermion interaction as closely as possible, we quantize these fields as fermions. The Hamiltonian for the system is given by

$$H=\int d^3k\, k a^\dagger(k)a(k) - \int d^3k\, h(k)a^\dagger(k)N^\dagger \int d^3l\, h(l)Na(l) \; . \quad (2.1)$$

As in Ref. 5, all integrals are assumed to be cut off at $|\vec{k}|=L$.

Just as in the bosonic case, the theory decomposes into a countable number of disconnected sectors labeled by the eigenvalues of the operators $\mathcal{N}_\theta$ and $\mathcal{N}_N$ defined by

$$\mathcal{N}_\theta=\int d^3k\, a^\dagger(k)a(k)$$

and

$$\mathcal{N}_N=N^\dagger N \; . \quad (2.2)$$

The lowest nontrivial sector is characterized by $\mathcal{N}_N=1$ and $\mathcal{N}_\theta=1$.

In what follows, we use the notation of Ref. 5. Thus, the eigenstates of the free Hamiltonian are denoted by $|\;\rangle$ whereas those of the total Hamiltonian are denoted by $|\;\rangle\rangle_\lambda$, λ being the corresponding eigenvalue. The phases of the states are defined so that

$$| N\theta_k \rangle = a^\dagger(k) N^\dagger | 0\rangle$$

and

$$| N\theta_k \theta_l \rangle = a^\dagger(k) a^\dagger(l) N^\dagger | 0\rangle \ . \tag{2.3}$$

In the $N\theta$ sector, the equation of motion can be written as

$$(\lambda - k)\rho_\lambda(k) = -\int h(k)h(l)\rho_\lambda(l)d^3l$$

with

$$\rho_\lambda(k) = \langle N\theta_k | N\theta \rangle\rangle_\lambda \ . \tag{2.4}$$

Equation (2.4) is identical to its counterpart in the case of the bosonic separable-potential model.[5] Thus, at least in the $N\theta$ sector, the separable-potential model with fermions does not differ from that with bosons. This is to be expected since in this sector there is no possibility for identical-particle effects to manifest themselves. The spectrum, therefore, consists of a continuum of scattering states for $0 \le \lambda \le L$ and a discrete state, $| B \rangle\rangle_M$, at the isolated point $M < 0$.[7] The corresponding solutions can be written as[5]

$$| N\theta \rangle\rangle_\lambda = \int d^3k \, \rho_\lambda(k) | N\theta_k \rangle \ , \tag{2.5a}$$

with

$$\rho_\lambda(k) = \delta(\vec{\lambda} - \vec{k}) - \frac{h(k)h(\lambda)}{\lambda - k + i\epsilon} \frac{1}{D^+(\lambda)} \tag{2.5b}$$

for the scattering states and

$$| B \rangle\rangle_M = \int d^3k \, \rho_M(k) | N\theta_k \rangle \tag{2.6a}$$

with

$$\rho_M(k) = [-D'(M)]^{-1/2} \frac{h(k)}{M - k} \ . \tag{2.6b}$$

The function $D(z)$ that appears in Eqs. (2.5) and (2.6) is given by

$$D(z) = 1 + \int d^3k \frac{h^2(k)}{z - k} \ , \tag{2.7}$$

with

$$D^\pm(\lambda) = D(\lambda \pm i\epsilon) \ .$$

We now turn to the solutions of the $N\theta\theta$ sector of the separable-potential model. Since $\mathcal{N}_\theta$ and $\mathcal{N}_N$ are conserved by the Hamiltonian, the eigenstates in this sector can be expanded in terms of the free eigenstates as

$$| \ \rangle\rangle_\lambda = \tfrac{1}{2} \int d^3k \, d^3l \, \Xi_\lambda(k,l) | N\theta_k \theta_l \rangle \ . \tag{2.8}$$

As in Ref. 5, the functions $\Xi_\lambda(k,l)$ are most conveniently evaluated in terms of the amplitudes

$$\Omega_\lambda(k,l) = {}_k\langle\langle N\theta | a(l) | \ \rangle\rangle_\lambda \tag{2.9a}$$

and

$$\Omega_\lambda(l) = {}_M\langle\langle B | a(l) | \ \rangle\rangle_\lambda \ . \tag{2.9b}$$

From the completeness of physical states in the $N\theta$ sector, we obtain

$$\Xi_\lambda(k,l) = \int d^3p \, \rho_p(l) \Omega_\lambda(p,k) + \rho_M(l)\Omega_\lambda(k) \ . \tag{2.10}$$

By considering the action of the Hamiltonian on the states $a^\dagger(l) | N\theta \rangle\rangle_k$ and $a^\dagger(l) | B \rangle\rangle_M$, we find that the amplitudes $\Omega_\lambda(k,l)$ and $\Omega_\lambda(l)$ obey the equations of motion

$$(\lambda - k - l)\Omega_\lambda(k,l) = -h(l) \int d^3p \, h(p) \Omega_\lambda(k,p) \tag{2.11a}$$

and

$$(\lambda - M - l)\Omega_\lambda(l) = -h(l) \int d^3p \, h(p) \Omega_\lambda(p) \ . \tag{2.11b}$$

It is amusing to note that these equtaions are the same as those for the separable-potential model for bosons. The important difference, of course, is that in the present case we are looking for solutions to $\Xi_\lambda(k,l)$ that are antisymmetric under the interchange of k and l. As is clear from Eq. (2.8), the symmetric solutions lead to null eigenvectors on account of the fermionic nature of the θ particles.

As in the case of the separable-potential model with bosons, we find that the spectrum of the fermion separable-potential model also admits $N\theta\theta$ and $B\theta$ continuum eigenstates for $0 \le \lambda \le 2L$ and $M \le \lambda \le M + L$, respectively. The corresponding solutions take the form

$$\Omega_\lambda(k,l) = \delta(\vec{\zeta}_2 - \vec{k})\delta(\vec{\zeta}_1 - \vec{l}) - \delta(\vec{\zeta}_1 - \vec{k})\delta(\vec{\zeta}_2 - \vec{l}) + \frac{h(l)h(\zeta_2)}{(\zeta_2 - l)D^+(\zeta_2)} \delta(\vec{\zeta}_1 - \vec{k}) - \frac{h(l)h(\zeta_1)}{(\zeta_1 - l)D^+(\zeta_1)} \delta(\vec{\zeta}_2 - \vec{k}) \tag{2.12a}$$

with

$$\lambda=\zeta_1+\zeta_2$$

and

$$\Omega_\lambda(l)=0 \tag{2.12b}$$

for the $N\theta\theta$ scattering states, and

$$\Omega_\lambda(k,l)=-\frac{h(l)}{\lambda-k-l}\frac{1}{[-D'(M)]^{1/2}}\delta(\vec{\xi}-\vec{k}) \tag{2.13a}$$

and

$$\Omega_\lambda(l)=\delta(\vec{\xi}-\vec{l})-\frac{h(l)h(\xi)}{\lambda-M-L}\frac{1}{D^+(\xi)} \tag{2.13b}$$

with

$$\lambda=M+\xi$$

for the $B\theta$ scattering states.

Although the system of Eqs. (2.11) admits a nontrivial solution for $\lambda=2M$, the corresponding $\Xi_{2M}(k,l)$ is symmetric.[5(b)] As a result, we are led to conclude that the fermionic separable potential admits no bound state in the $N\theta\theta$ sector. We will subsequently show that in the SC limit the spectrum and the wave functions of the Lee model at finite energies reduce to those of the separable-potential model.

We now turn our attention to the Hamiltonian matrix elements in the $N\theta$ and the $N\theta\theta$ sectors. From Eq. (2.1), these can be easily seen to be

$$\langle N\theta_k\,|\,H\,|\,N\theta_l\rangle=k\delta(\vec{k}-\vec{l})-h(k)h(l) \tag{2.14}$$

and

$$\langle N\theta_k\theta_l\,|\,H\,|\,N\theta_p\theta_q\rangle=(k+l)[\delta(\vec{k}-\vec{p})\delta(\vec{l}-\vec{q})-\delta(\vec{l}-\vec{p})\delta(\vec{k}-\vec{q})]$$
$$-[h(k)h(p)\delta(\vec{l}-\vec{q})-(\vec{k}\leftrightarrow\vec{l})-(\vec{p}\leftrightarrow\vec{q})+(\vec{k}\leftrightarrow\vec{l})(\vec{p}\leftrightarrow\vec{q})]\ . \tag{2.15}$$

As expected, the Hamiltonian matrix element in the lower sector is unchanged from the Bose case, whereas the effects of the statistics of the fermions clearly manifest themselves in the antisymmetric structure of the matrix element in the $N\theta\theta$ sector. We will show in Sec. IV that, in the SC limit, the Hamiltonian matrix elements of the "transmuted" Lee model essentially reduce to those presented in Eqs. (3.14) and (3.15).

III. THE FERMIONIC LEE MODEL

In addition to the fields N and θ, the Lee model involves yet another field V interacting with the N and θ fields via a Yukawa interaction. The Hamiltonian for the system is given by

$$H=m_0V^\dagger V+\int d^3k\, ka^\dagger(k)a(k)+g_0\int d^3k\, f(k)[V^\dagger Na(k)+a^\dagger(k)N^\dagger V]\ . \tag{3.1}$$

We have, for simplicity, considered the θ particle to be massless and have, once again,[5] ignored any recoil of the N and the V particles. As discussed in previous sections, the fields N and θ are quantized as fermions. In analogy with Yukawa field theories, the field V which mediates the $N\theta$ scattering process is quantized as a boson. The quantization rules are, therefore, written as

$$\begin{aligned}
&\{N,N^\dagger\}=[V,V^\dagger]=1\ ,\\
&\{a(k),a^\dagger(l)\}=\delta(\vec{k}-\vec{l})\ ,\\
&\{N,N\}=[V,V]=\{a(k),a(l)\}=0\ ,\\
&\{N,a(k)\}=\{N,a^\dagger(k)\}=[N,V]=[N,V^\dagger]\\
&\qquad=[a(k),V]=[a(k),V^\dagger]=0\ ,
\end{aligned} \tag{3.2}$$

together with their Hermitian conjugates.

The fermionic Lee model also decomposes into a countable number of disconnected sectors labeled by the eigenvalues of the operators $\mathcal{N}_1$ and $\mathcal{N}_2$ defined by

$$\mathcal{N}_1=\mathcal{N}_V+\mathcal{N}_N$$

and

$$\mathcal{N}_2=\mathcal{N}_V+\mathcal{N}_\theta \tag{3.3}$$

with

$$\mathcal{N}_V=V^\dagger V\ .$$

The lowest nontrivial sector, the $N\theta$ or V sector, is characterized by $\mathcal{N}_1=\mathcal{N}_2=1$ while the next sector, the $N\theta\theta$ or $V\theta$ sector, is labeled by $\mathscr{N}_1=1$, $\mathscr{N}_2=2$. In this section, we first show that the lower-sector solutions are unchanged by the altered statistics of the N and θ particles. This is in keeping with our earlier arguments. We now proceed to study the spectrum and the solutions of the fermionic Lee model in the $V\theta$ sector.

The $N\theta$ sector

In the $N\theta$ sector, the equations of motion are

$$(\lambda-m_0)\sigma_\lambda=g_0\int d^3k\, f(k)\rho_\lambda(k) \tag{3.4a}$$

and

$$(\lambda-k)\rho_\lambda(k)=g_0 f(k)\sigma_\lambda\ , \tag{3.4b}$$

where

$$\sigma_\lambda=\langle V \mid \ \rangle\rangle_\lambda \tag{3.5a}$$

and

$$\rho_\lambda(k)=\langle N\theta_k \mid \ \rangle\rangle_\lambda\ . \tag{3.5b}$$

These equations are identical to the equations of motion for the $N\theta$ sector of the bosonic Lee model.[5] The spectrum consists of an $N\theta$ continuum for $0\le\lambda\le L$ and discrete states $\mid V_1\rangle\rangle$ and $\mid V_2\rangle\rangle$ at M_1 and M_2, respectively, with $M_1<0$ and $M_2>L$. In the notation of Ref. 5, the corresponding solutions can be written as

$$\mid V_i\rangle\rangle=\sqrt{Z_i}\mid V\rangle+\int d^3k\, F^i(k)\mid N\theta_k\rangle \tag{3.6a}$$

and

$$\mid N\theta\rangle\rangle_\lambda=g_\lambda^*\mid V\rangle+\int d^3k\, G_{\lambda k}\mid N\theta_k\rangle\ , \tag{3.6b}$$

where

$$Z_i=\frac{1}{\alpha'(M_i)}\ , \tag{3.7a}$$

$$F^i(k)=\sqrt{Z_i}\frac{g_0 f(k)}{M_i-k}\ , \tag{3.7b}$$

$$g_k^*=\frac{g_0 f(k)}{\alpha^+(k)}\ , \tag{3.7c}$$

with

$$\alpha(z)=z-m_0-{g_0}^2\int d^3k\frac{f^2(k)}{z-k} \tag{3.7d}$$

and

$$G_{kl}=\delta(\vec{k}-\vec{l})+\frac{g_0 g_k^* f(l)}{k-l+i\epsilon}\ . \tag{3.7e}$$

We now consider the SC limit of these solutions, with the proviso that M_1 remain fixed and finite[5]: we find, since the fermionic nature of the N and θ particles is irrelevant in the $N\theta$ sector, as before[5] that the finite-energy eigenstates $\mid V_1\rangle\rangle$ have no contributions from the bare state $\mid V\rangle$ whereas the state $\mid V_2\rangle\rangle$ which moves to infinity in the SC limit becomes identical to $\mid V\rangle$. The compositeness condition $Z_1=0$ is satisfied in the SC limit; $\mid V_1\rangle\rangle$ may, therefore, be regarded as a composite of the N and θ particles. In fact, for the purpose of comparison with the separable-potential model, $\mid V_1\rangle\rangle$ is identified with the state $\mid B\rangle\rangle_M$ in the separable-potential model. The correspondence between the separable-potential model of fermions and the fermionic Lee model is completed by noting that the finite-energy scattering amplitudes and S matrix calculated from the two theories coincide in the SC limit.[5] This follows since the solutions for both the separable-potential model and the Lee model, in the $N\theta$ sector, are unaffected by the altered statistics of the N and θ particles.

We conclude the discussion of the lower sector by pointing out that the Hamiltonian matrix elements in the bare state basis,

$$\langle V\mid H\mid V\rangle=m_0\ , \tag{3.8a}$$

$$\langle V\mid H\mid N\theta_k\rangle=g_0 f(k)\ , \tag{3.8b}$$

$$\langle N\theta_k\mid H\mid N\theta_l\rangle=k\delta(\vec{k}-\vec{l})\ , \tag{3.8c}$$

differ from those of the separable-potential model on account of the difference in the spectra of the two theories. As in Ref. 5, in order to make the two theories identical, in addition to taking the SC limit so as to satisfy the compositeness condition, all spectral contributions at infinity have to be removed; then, exactly as in the case of the bosonic Lee model, the fermionic Lee model is transmuted[5] into the model of the fermionic separable potential.

The $V\theta$ ($N\theta\theta$) sector

As we have discussed earlier, it is in this sector that the effects of the fermionic character of the N and θ particles are expected to become evident. We, therefore, solve this sector in some detail, comparing and contrasting with the corresponding situations for the bosonic Lee model. Again, since $\mathscr{N}_1$ and $\mathscr{N}_2$ defined by Eqs. (3.3) are conserved by the Hamiltonian, the eigenstates in this sector can be expanded in terms of the bare states as

$$| \, \rangle\rangle_\lambda = \int d^3k\, \psi_\lambda(k) \,|\, V\theta_k \rangle + \tfrac{1}{2} \int d^3k\, d^3l\, \psi_\lambda(k,l) \,|\, N\theta_k\theta_l \rangle \ . \tag{3.9}$$

The equations of motion for the Källén-Pauli components

$$\psi_\lambda(k) = \langle V\theta_k \,|\, \rangle\rangle_\lambda \tag{3.10a}$$

and

$$\psi_\lambda(k,l) = \langle N\theta_k\theta_l \,|\, \rangle\rangle_\lambda \tag{3.10b}$$

are

$$(\lambda - m_0 - l)\psi_\lambda(l) = g_0 \int d^3k\, f(k)\psi_\lambda(l,k) \tag{3.11a}$$

and

$$(\lambda - k - l)\psi_\lambda(k,l) = g_0 f(l)\psi_\lambda(k) - g_0 f(k)\psi_\lambda(l) \ . \tag{3.11b}$$

The solution to this system is, once again, most conveniently obtained by introducing an overcomplete set of basis vectors,[8] $a^\dagger(l)\,|\,V_i\rangle\rangle$, $a^\dagger(l)\,|\,N\theta\rangle\rangle_k$, $V^\dagger N\,|\,V_i\rangle\rangle$, and $V^\dagger N\,|\,N\theta\rangle\rangle_k$. We define the amplitudes

$$\langle\langle V_i \,|\, a(l) \,|\, \rangle\rangle_\lambda \equiv \phi_\lambda{}^i(l) \ , \tag{3.12a}$$

$$_k\langle\langle N\theta \,|\, a(l) \,|\, \rangle\rangle_\lambda = \phi_\lambda(k,l) \ , \tag{3.12b}$$

$$\langle\langle V_i \,|\, N^\dagger V \,|\, \rangle\rangle_\lambda = \chi_\lambda{}^i \ , \tag{3.12c}$$

and

$$_k\langle\langle N\theta \,|\, N^\dagger V \,|\, \rangle\rangle_\lambda = \chi_\lambda(k) \ . \tag{3.12d}$$

For brevity of notation, we have suppressed the degeneracy index on the $N\theta\theta$ scattering states. The equations of motion for these amplitudes can be obtained by considering the action of the Hamiltonian on each element of the aforementioned overcomplete basis. These can be written in the form

$$(\lambda - M_i - l)\phi_\lambda{}^i(l) = g_0 f(l)\chi_\lambda{}^i \ , \tag{3.13a}$$

$$(\lambda - k - l)\phi_\lambda(k,l) = g_0 f(l)\chi_\lambda(k) \ , \tag{3.13b}$$

$$(\lambda - M_i - m_0)\chi_\lambda{}^i = g_0 \int d^3k\, f(k)\phi_\lambda{}^i(k) \ , \tag{3.13c}$$

and

$$(\lambda - k - m_0)\chi_\lambda(k) = g_0 \int d^3l\, f(l)\phi_\lambda(k,l) \ . \tag{3.13d}$$

Equations (3.13a) and (3.13b) are identical to their counterparts for the bosonic Lee model, whereas Eqs. (3.13c) and (3.13d) differ from the corresponding equations for the bosonic case in that only the first of the three driving terms [see Eqs. (3.22c) and (3.22d) of Ref. 5(b)] that occur in the boson case persist for the present case. This simplicity of structure of these latter equations can be directly traced to the fact that the fields N and V obey opposite statistics.

The amplitudes $\chi_\lambda{}^i$, $\chi_\lambda(k)$ can, as in the case of the bosonic Lee model, be written in terms of the amplitudes $\phi_\lambda{}^i(l)$ and $\phi_\lambda(k,l)$ as

$$\chi_\lambda{}^i = -\int d^3p\, F^i(p) \sum_j \phi_\lambda^j(p)\sqrt{Z_j} - \int d^3p\, d^3q\, F^i(p) g_q^* \phi_\lambda(q,p) \tag{3.14a}$$

and

$$\chi_\lambda(k) = -\int d^3p\, G_{kp}^* \sum_j \sqrt{Z_j}\phi_\lambda{}^j(p) - \int d^3p\, d^3q\, G_{kp}^* g_q^* \phi_\lambda(q,p) \ . \tag{3.14b}$$

This follows since the vectors $a^\dagger(l)\,|\,V_i\rangle\rangle$ and $a^\dagger(l)\,|\,N\theta\rangle\rangle_k$ form a(n) (oblique) basis. The full content of the fermionic Lee model is realized by solving the equations of motion (3.13), subject to the constraints (3.14).

The solution to the system of equations (3.13) is, as before, obtained by first eliminating the functions $\phi_\lambda{}^i(k)$ and $\phi_\lambda(k,l)$ using the first two equations, and then solving for the remaining functions using the latter two. We then check whether the solutions thus obtained satisfy the constraint equations.

$V\theta$ scattering states

We define $\xi_i = \lambda - M_i$. Then, from Eqs. (3.13a) and (3.13b)

$$\phi_\lambda{}^i(l) = \delta(\vec{\xi}_i - \vec{l}) + \frac{g_0 f(l)\chi_\lambda{}^i}{\xi_i - l + i\epsilon} \tag{3.15a}$$

and

$$\phi_\lambda(k,l) = \frac{g_0 f(l)\chi_\lambda(k)}{\lambda - k - l + i\epsilon} \ . \tag{3.15b}$$

From Eqs. (3.13c) and (3.13d) we readily obtain

$$\alpha^{+}(\xi_i)\chi_\lambda{}^i = g_0 f(\xi_i)$$

and

$$\alpha^{+}(\lambda-k)\chi_\lambda(k)=0 .$$

The solutions can thus be written as

$$\chi_\lambda(k) = -\sqrt{Z_1}\delta(\vec{\xi}_1-\vec{k}) - \sqrt{Z_2}\delta(\vec{\xi}_2-\vec{k}) , \tag{3.16a}$$

$$\chi_\lambda{}^i = \tilde{g}^*_{\xi_i} , \tag{3.16b}$$

$$\phi_\lambda{}^i(l) = \tilde{G}_{\xi_i l} , \tag{3.16c}$$

and

$$\phi_\lambda(k,l) = -\frac{g_0 f(l)}{\lambda-k-l+i\epsilon} \times [\sqrt{Z_1}\delta(\vec{\xi}_1-\vec{k})+\sqrt{Z_2}\delta(\vec{\xi}_2-\vec{k})] \tag{3.16d}$$

with

$$\tilde{g}_{\xi_i} = g_{\xi_i}\theta(L-\xi_i)\theta(\xi_i) ,$$

$$\tilde{G}_{\xi_i l} = G_{\xi_i l}\theta(L-\xi_i)\theta(\xi_i) .$$

$N\theta\theta$ scattering states

We define $\lambda=\zeta_1+\zeta_2$, $0\le\zeta_1, \zeta_2\le L$. Then from Eqs. (3.13a) and (3.13b) we have

$$\phi_\lambda{}^i(l) = \frac{g_0 f(l)\chi_\lambda{}^i}{\lambda-M_i-l+i\epsilon}$$

and

$$\phi_\lambda(k,l) = \delta(\vec{\zeta}_1-\vec{l})\delta(\vec{\zeta}_2-\vec{k}) - \delta(\vec{\zeta}_2-\vec{l})\delta(\vec{\zeta}_1-\vec{k}) + \frac{g_0 f(l)\chi_\lambda(k)}{\lambda-k-l+i\epsilon} .$$

Using Eqs. (3.13c) and (3.13d) we obtain

$$\alpha(\lambda-M_i)\chi_\lambda{}^i=0$$

and

$$\alpha^{+}(\lambda-k)\chi_\lambda(k) = g_0 f(\zeta_1)\delta(\vec{\zeta}_2-\vec{k}) - g_0 f(\zeta_2)\delta(\vec{\zeta}_1-\vec{k}) .$$

The solution to the system (3.10) for $N\theta\theta$ scattering states can, therefore, be written as

$$\chi_\lambda{}^i = 0 , \tag{3.17a}$$

$$\phi_\lambda{}^i(l) = 0 , \tag{3.17b}$$

$$\chi_\lambda(k) = g^*_{\zeta_1}\delta(\vec{\zeta}_2-\vec{k}) - g^*_{\zeta_2}\delta(\vec{\zeta}_1-\vec{k}) , \tag{3.17c}$$

and

$$\phi_\lambda(k,l) = \delta(\vec{\zeta}_1-\vec{l})\delta(\vec{\zeta}_2-\vec{k}) - \delta(\vec{\zeta}_1-\vec{k})\delta(\vec{\zeta}_2-\vec{l}) + \frac{g_0 f(l)}{\lambda-k-l+i\epsilon}\chi_\lambda(k) . \tag{3.17d}$$

The homogeneous terms in Eqs. (3.16) and (3.17) have been adjusted so as to reproduce a δ-function normalization (with unit coefficient) for the scattering states.

Discrete states

For a discrete state with eigenvalue Λ, it follows from Eqs. (3.13a) and (3.13b) that

$$\phi_\Lambda{}^i(l) = \frac{g_0 f(l)}{\Lambda-M_i-l}\chi_\Lambda{}^i \tag{3.18a}$$

and

$$\phi_\Lambda(k,l) = \frac{g_0 f(l)\chi_\Lambda(k)}{\Lambda-k-l} . \tag{3.18b}$$

These, together with Eqs. (3.13c) and (3.13d), readily yield

$$\alpha(\Lambda-M_i)\chi_\Lambda{}^i=0 \tag{3.18c}$$

and

$$\alpha(\Lambda-k)\chi_\Lambda(k)=0 . \tag{3.18d}$$

Thus, from Eq. (3.18d), for a discrete point in the spectrum we have

$$0=\chi_\Lambda(k) = {}_k\langle\langle N\theta \,|\, N^\dagger V \,|\; \rangle\rangle_\lambda = -\int d^3p\, G^*_{kp}\psi_\Lambda(p) . \tag{3.19}$$

It is clear that Eq. (3.18c) admits nontrivial solutions for $\Lambda=2M_1$, $2M_2$, and M_1+M_2. As in the bosonic case, we have to check that these satisfy the constraint conditions. Equivalently, by writing the Källén-Pauli component $\psi_\lambda(l)$ as

$$\psi_\lambda(l) = -\sum_i F^i(l)\chi_\Lambda{}^i - \int G^*_{pl}\chi_\Lambda(p) ,$$

we obtain from Eq. (3.19) using Eq. (A9) of Appendix A that any discrete state satisfies the condition

$$\sum_j \sqrt{Z_j}\chi_\Lambda{}^j = 0 \; .$$

Thus if any $\chi_\Lambda{}^i = 0$, a null solution results. We are therefore led to conclude from Eq. (3.18c) that there is a single discrete point in the spectrum at $\Lambda = M_1 + M_2$. The solution to Eqs. (3.13) for this state can be obtained by requiring the condition obtained above be satisfied. We find[9]

$$\chi^1_{M_1+M_2} = \sqrt{Z_2} \; , \tag{3.20a}$$

$$\chi^2_{M_1+M_2} = -\sqrt{Z_1} \; , \tag{3.20b}$$

$$\phi^1_{M_1+M_2}(l) = F^2(l) \; , \tag{3.20c}$$

$$\phi^2_{M_1+M_2}(l) = -F^1(l) \; , \tag{3.20d}$$

$$\chi_\Lambda(k) = 0 \; , \tag{3.20e}$$

and

$$\phi_\Lambda(k,l) = 0 \; . \tag{3.20f}$$

The normalization in Eq. (3.20) has been fixed by the requirement

$$\langle\langle M_1 + M_2 \mid M_1 + M_2 \rangle\rangle = 1 \; .$$

By writing the Källén-Pauli components in terms of the overcomplete-basis components as

$$\psi_\lambda(l) = \sum_j \sqrt{Z_j}\phi_\lambda{}^j(l) + \int d^3k \, g_k^* \phi_\lambda(k,l) \tag{3.21a}$$

and

$$\psi_\lambda(l,k) = \sum_j F^j(k)\phi_\lambda{}^j(l) + \int d^3p \, G_{pk}\phi_\lambda(p,l) \; , \tag{3.21b}$$

we can easily verify that our solutions satisfy the equations of motion (3.11). Moreover, the solutions also satisfy the constraint equations (3.14).

The spectrum of the fermionic Lee model in this sector consists of (i) an $N\theta\theta$ scattering continuum from $\lambda = 0$ to $\lambda = 2L$, (ii) a $V_1\theta$ scattering continuum from $\lambda = M_1$ to $\lambda = M_1 + L$, (iii) a $V_2\theta$ scattering continuum from $\lambda = M_2$ to $\lambda = M_2 + L$, and (iv) a discrete state at $\lambda = M_1 + M_2$.[10] (Note that at least in the SC limit, $M_1 + M_2 >> 2L$.) This spectrum is illustrated in Fig. 1.

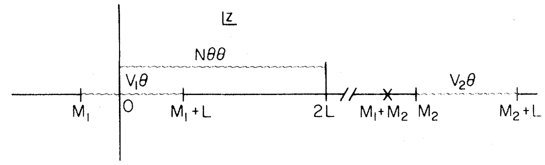

FIG. 1. The spectrum of the fermionic Lee model in the $N\theta\theta$ sector.

We see that, in the SC limit, the finite-energy part of the Lee-model spectrum coincides with the spectrum of the separable-potential model. Furthermore, we have explicitly demonstrated in Appendix C that in the SC limit, the finite-energy wave functions of the Lee model reduce to corresponding ones of the separable-potential model. As a result, in the SC limit, the scattering amplitudes and the S matrix for the two theories are the same at finite energies, i.e., the agreement between the fermion separable-potential model and the fermionic Lee model persists even in the $V\theta$ sector where the effects of the fermionic nature of the particles manifest themselves in a rather transparent way.

We now turn our attention to the matrix elements of the Hamiltonian of the fermionic Lee model. Direct computation from Eq. (3.1) using Eqs. (3.2) yields

$$\langle V\theta_k \mid H \mid V\theta_l \rangle = (m_0 + l)\delta(\vec{k} - \vec{l}) \; , \tag{3.22a}$$

$$\langle V\theta_k \mid H \mid N\theta_p\theta_q \rangle = g_0[f(q)\delta(\vec{p} - \vec{k}) - f(p)\delta(\vec{q} - \vec{k})] \; , \tag{3.22b}$$

$$\langle N\theta_k\theta_l \mid H \mid N\theta_p\theta_q \rangle = (k + l)[\delta(\vec{k} - \vec{p})\delta(\vec{l} - \vec{q}) - \delta(\vec{k} - \vec{q})\delta(\vec{l} - \vec{p})] \; . \tag{3.22c}$$

Once again, these differ from those of the separable-potential model. The reason for the difference, of course, lies in the difference in the spectra of the two theories. As with the bosonic case, in order to make the fermionic separable-potential model and the fermionic Lee model identical, we shall see that the contributions to the Lee-model Hamiltonian matrix elements that arise from the $\mid V_2\theta\rangle\rangle$ scattering state and the state $\mid M_1 + M_2\rangle\rangle$ (these are absent in the spectrum of the separable-potential model) have to be explicitly removed. This transmutation of the Lee model into the separable-potential model forms the subject of Sec. IV.

IV. THE TRANSMUTATION OF THE FERMIONIC LEE MODEL

In previous sections, we have shown that in the SC limit, the finite-energy scattering amplitudes and the S matrix calculated from the fermionic Lee model and the fermionic separable-potential model are essentially the same. The spectra of the two theories, however, are quite distinct. The difference comes from the fact that in the Lee model, additional spectral contributions arise due to the presence of an ultraviolet cutoff. Thus, as with the bosonic case,[5] the compositeness condition $Z_1=0$ does not ensure a complete equivalence of the two theories. In this section, we show that in order to ensure the identity of the two theories, in addition to satisfying the compositeness conditions, all contributions to the Hamiltonian coming from the part of the Lee-model spectrum at infinity have to be removed. Then, as in Ref. 5, the Lee model with the reduced spectrum becomes essentially equivalent to the separable-potential model.

Dynamical rearrangement of the Hamiltonian

We denote the Källén-Pauli basis vectors by $|x\rangle$ and the eigenvectors of the total Hamiltonian by $|z\rangle\rangle$. Then from the closure relations, we obtain

$$\langle x'|H|x\rangle=\sum_{|z\rangle\rangle} z\langle x'|z\rangle\rangle\langle\langle z|x\rangle , \qquad (4.1)$$

with z denoting the energy eigenvalue of the state $|z\rangle\rangle$. The wave functions $\langle x|z\rangle\rangle$ for the $N\theta$ sector are the same as for the case of the bosonic Lee model and can readily be read off from Eqs. (3.6). Those for the $N\theta\theta$ sector differ from those in Ref. 5 on account of the fermionic nature of the particles. These have been explicitly written out in Appendix B.

We have explicitly verified that the right-hand side of Eq. (4.1) indeed reproduces the matrix elements in Eqs. (3.8) and (3.22) for the $N\theta$ and $N\theta\theta$ sectors, respectively. This verification for the lower sector is exactly identical to the calculation in Ref. 5 since the wave functions are unchanged from those of the bosonic Lee model in the same sector.

The transmutation of the Lee model

We have seen that the dynamical rearrangement of the Hamiltonian does not alter the content of the Lee model. We now show that if we take the SC limit and explicitly exclude from the sum in Eq. (4.1) all contributions from those states that move to infinity in the SC limit, the Lee-model Hamiltonian essentially reduces to that of the separable potential, again with the identification of the form factors $h(k)$ and $g_0 f(k)/\sqrt{M_2}$. The demonstration of this, for the case of the $N\theta$ sector, proceeds exactly as in the case of the bosonic Lee model. We merely present the results here, referring the reader to Ref. 5 for details of the calculation. In the SC limit, we can write

$$\langle V|H|V\rangle=m_0-M_2+[M_2] , \qquad (4.2a)$$

$$\langle V|H|N\theta_k\rangle=0+[g_0 f(k)] , \qquad (4.2b)$$

and

$$\langle N\theta_k|H|N\theta_l\rangle=k\delta(\vec{k}-\vec{l})-\frac{g_0{}^2 f(k)f(l)}{M_2}+\left[\frac{g_0{}^2 f(k)f(l)}{M_2}\right] . \qquad (4.2c)$$

In Eqs. (4.2), the terms in the square brackets denote the contributions to Eq. (4.1) from the additional states that arise due to the presence of a cutoff, i.e., from the single state $|V_2\rangle\rangle$ in the $N\theta$ sector. We see from Eqs. (4.2) that if we omit the spectral contributions from this state, the bare V particle decouples from the N and the θ particles. Moreover, also in the SC limit, the Hamiltonian matrix elements with the $|V_2\rangle\rangle$ contributions removed become identical to those of the separable-potential model if the form factors $h(k)$ and $g_0 f(k)/\sqrt{M_2}$ are identified. [See Eq. (2.8).]

We now turn to the transmutation of the fermionic Lee model in the $V\theta$ sector. To this end, we calculate the contributions to the Hamiltonian matrix elements in the Källén-Pauli basis from the states $|V_2\theta\rangle\rangle$ and $|M_1+M_2\rangle\rangle$. We remind the reader that these states arise due to the presence of the cutoff and are responsible for the difference in the spectra of the two theories. We first focus our attention on the matrix element $\langle V\theta_k|H|V\theta_l\rangle$. We have

$$\langle V\theta_k|H|V\theta_l\rangle=\sum_{\lambda}\lambda\psi_\lambda(k)\psi_\lambda^*(l) , \qquad (4.3)$$

where λ runs over the complete spectrum. The contribution to the $|V_2\theta\rangle\rangle$ scattering states can be written using our solution Eqs. (3.15) together with Eqs. (3.21) as

$$\int_0^L d\xi_2[\sqrt{Z_2}G_{\xi_2 k}-g_{\xi_2}^*F^2(k)][\sqrt{Z_2}G_{\xi_2 l}^*-g_{\xi_2}F^2(l)](M_2+\xi_2) .$$

This can be evaluated using the compendium of integrals in Appendix A. In the SC limit, we find it reduces to $M_2[\delta(\vec{k}-\vec{l})-F^1(k)F^1(l)]$. The contribution of the discrete state $|M_1+M_2\rangle\rangle$ to this matrix element is, from Eqs. (3.20) and (3.21),

$$(M_1+M_2)\langle V\theta_k \mid M_1+M_2\rangle\rangle\langle\langle M_1+M_2 \mid V\theta_l\rangle$$
$$=(M_1+M_2)[\sqrt{Z_1}F^2(k)-\sqrt{Z_2}F^1(k)][\sqrt{Z_1}F^2(l)-\sqrt{Z_2}F^1(l)] \overset{s}{=} M_2\delta(\vec{k}-\vec{l}) .$$

Here $\overset{s}{=}$ denotes equality in the SC limit. We see, therefore, that the contribution of the states $|V_2\theta\rangle\rangle$ and $|M_1+M_2\rangle\rangle$ to the matrix element $\langle V\theta_k \mid H \mid V\theta_l\rangle$ is just $M_2\delta(\vec{k}-\vec{l})$ in the SC limit. In the notation introduced earlier for the lower sector we can, therefore, write

$$\langle V\theta_k \mid H \mid V\theta_l\rangle=(m_0-M_2)\delta(\vec{k}-\vec{l})+[M_2\delta(\vec{k}-\vec{l})] . \tag{4.4a}$$

The contribution of the states $|V_2\theta\rangle\rangle$ and $|M_1+M_2\rangle\rangle$ to the matrix elements $\langle V\theta_k \mid H \mid N\theta_p\theta_q\rangle$ and $\langle N\theta_k\theta_l \mid H \mid N\theta_p\theta_q\rangle$ can be calculated in a similar manner; these can be written as

$$\langle V\theta_k \mid H \mid N\theta_p\theta_q\rangle=0+[g_0 f(q)\delta(\vec{p}-\vec{k})-g_0 f(p)\delta(\vec{q}-\vec{k})] \tag{4.4b}$$

and

$$\langle N\theta_k\theta_l \mid H \mid N\theta_p\theta_q\rangle=(k+l)\{\delta(\vec{k}-\vec{p})\delta(\vec{l}-\vec{q})-\delta(\vec{k}-\vec{q})\delta(\vec{l}-\vec{p})\}$$
$$-\left\{\frac{g_0{}^2 f(k)f(p)}{M_2}\delta(\vec{l}-\vec{q})-(\vec{k}\leftrightarrow\vec{l})-(\vec{p}\leftrightarrow\vec{q})+(\vec{k}\leftrightarrow\vec{l})(\vec{p}\leftrightarrow\vec{q})\right\}$$
$$+\left[\frac{g_0{}^2 f(k)f(p)}{M_2}\delta(\vec{l}-\vec{q})-(\vec{k}\leftrightarrow\vec{l})-(\vec{p}\leftrightarrow\vec{q})+(\vec{k}\leftrightarrow l)(\vec{p}\leftrightarrow q)\right] . \tag{4.4c}$$

We remind the reader that in Eqs. (4.4), the terms in the square brackets are the contributions from that part of the Lee-model spectrum that arises due to the presence of the cutoff, i.e., from the $|V_2\theta\rangle\rangle$ and $|M_1+M_2\rangle\rangle$ states. We see, once again, that it is only when these contributions are removed that the bare V particle decouples from the N and θ particles and the Lee-model Hamiltonian essentially reduces to that of the separable potential. We emphasize that unless this is done, the two theories have distinct spectra and are, therefore, inequivalent.

The role of the ultraviolet cutoff in the transmutation of the fermionic Lee model into the fermionic separable potential is the same as for the corresponding bosonic case. Irrespective of the existence of any cutoff, the finite-energy scattering amplitudes and S-matrix elements coincide when the coupling constant moves to infinity. The Hamiltonian matrix elements of the two theories are, however, different. The two theories, therefore, are clearly distinct and there appears to be no natural way of transforming one into the other. The introduction of the cutoff, however, divides the spectrum of the Yukawa theory into two parts, one of which coincides with the spectrum of the four-point interaction model, and the other consisting exclusively of contributions that arise because of the presence of the cutoff. The latter move to infinity in the SC limit. By removing these contributions, we are able to make the spectra of the two theories identical, thereby ensuring an equivalence of the theories when the compositeness conditions are satisfied. We note here that the equivalence proofs existing in the literature confine themselves to the demonstration of the Green's functions (or equivalently, the scattering amplitudes) without any reference to the Hamiltonian. Hence, it is suggested that these proofs should not be regarded as complete.

To summarize, we have explicitly shown that the compositeness condition, $Z_1=0$, by itself does not ensure the equivalence of the fermionic Lee model and the corresponding separable-potential model. In order to see how the equivlance could be realized, we introduced an ultraviolet cutoff on both theories. As a result, there appeared additional states in the spectrum of the Lee model but not in the separable-potential model. We showed that it was when the spectral contributions of these states were explicitly removed, that the two theories become effectively identical in the SC limit. (This limit was required to ensure the compositeness condition $Z_1=0$.) Thus, the transmutation of a Yukawa-type interaction to a four-point interaction discussed in Ref. 5 can also be done for Fermi fields, i.e., the transmutation mechanism is not characteristic of Bose fields. Its role in fully relativistic field theories, however, merits further investigation.

V. CONCLUDING REMARKS

In the last few years, several authors[11] have suggested the renormalizability of the four-fermion interaction, basing their arguments on the alleged equivalence of the four-fermion theory and the corresponding Yukawa theory. As discussed in Sec. I, these "proofs" of equivalence are based on the direct comparison of the renormalized Lagrangians of the two theories and the demonstration that there is a one-to-one correspondence between the respective Feynman graphs (or equivalently, scattering amplitudes) provided the compositeness conditions are satisfied. This had led us to investigate[5] the corresponding situation for two soluble models, the Lee model (Yukawa-type theory) and the separable-potential model (four-point interaction). We showed that if we considered the SC limit of the Yukawa theory so that the compositeness condition $Z_1=0$ is satisfied, the finite-energy wave functions, and hence the scattering amplitudes calculated from the two theories, indeed became numerically the same. The spectra of the two theories, however, were quite distinct in the presence of an ultraviolet cutoff. This reflected itself in the fact that the "bare-basis" Hamiltonian matrix elements of the two theories were different. We showed that in order to ensure essential identity of the two theories, in addition to satisfying the compositeness conditions, all contributions from the part of the Lee-model spectrum that were a consequence of the cutoff (these are absent in the separable-potential model) had to be explicitly removed. Then the Lee model was transmuted into the separable-potential model.

In order to strengthen the analogy with the alleged equivalence[3] of the gauge interaction and the four-fermion interaction, in this paper, we restudied the corresponding situation for the Lee model and the separable-potential model, but this time quantized the N and θ particles as fermions and the V particle as a boson. We found that the fermionic separable potential indeed could give rise to a collective bosonic state in the $N\theta$ sector. In the same sector, for the Lee model with a cutoff there were two discrete states $|V_1\rangle\rangle$ and $|V_2\rangle\rangle$. We showed that the V_1 particle could, in the strong-coupling limit, be identified with the bosonic bound state of the separable-potential model. In this limit, the state $|V_2\rangle\rangle$ moved to infinity. We showed that just as for the bosonic case, although the finite-energy wave functions calculated from the two theories were the same in the SC limit, in order to make the Hamiltonians the same, the spectral contributions due to the state $|V_2\rangle\rangle$ had to be explicitly removed. Then the Yukawa theory was transmuted into the four-fermion theory. We further checked this transmutation property for the $N\theta\theta$ sector of the models.

To conclude, our model study shows that although the compositeness conditions ensure the equivalence of the finite-energy scattering amplitudes and the S matrix of the four-fermion and the Yukawa-type theories, it does not follow that the two theories are identical as implied by many authors.[2,3] In order to make the two theories equivalent, the spectrum of the Yukawa theory has to be truncated, i.e., the contributions from the part of the spectrum of the Yukawa theory that is not common with the spectrum of the four-fermion theory have to be explicitly removed. The truncated Yukawa theory is then equivalent to the four-fermion theory. In conclusion, we state that if a similar scenario persists in the case of fully relativistic field theories, the present proofs of equivalence and the conclusion that the four-fermion interaction is renormalizable may need further examination.

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APPENDIX A

In this appendix, we merely list some formulas which are of use in deriving some of the results in the text. The scattering states are normalized as

$$\langle N\theta_k \,|\, N\theta_l \rangle = \delta(\vec{k}-\vec{l}) \,, \tag{A1}$$

$$\langle V\theta_k \,|\, V\theta_l \rangle = \delta(\vec{k}-\vec{l}) \,, \tag{A2}$$

$$\langle N\theta_k\theta_l \,|\, N\theta_p\theta_q \rangle = \delta(\vec{k}-\vec{p})\delta(\vec{l}-\vec{q}) - \delta(\vec{k}-\vec{q})\delta(\vec{l}-\vec{p}) \,. \tag{A3}$$

The completeness relations take the form

$$|V\rangle\langle V| + \int d^3k \,|N\theta_k\rangle\langle N\theta_k| = I \tag{A4}$$

and

$$\int d^3k \,|V\theta_k\rangle\langle V\theta_k| + \tfrac{1}{2}\int d^3k\, d^3l \,|N\theta_k\theta_l\rangle\langle N\theta_k\theta_l| = I \,, \tag{A5}$$

in the $N\theta$ and $V\theta$ sectors, respectively.

We list below certain integrals which have been extensively used in the text.

$$\int d^3k\, F^i(k)F^j(k) = \delta_{ij} - (Z_iZ_j)^{1/2} \,, \tag{A6}$$

$$\int d^3k\, |g_k|^2 = 1 - Z_1 - Z_2 \,, \tag{A7}$$

$$\int d^3k\, G_{pk}G^*_{qk} = \delta(\vec{p}-\vec{q}) - g_pg^*_q \,, \tag{A8}$$

$$\int d^3k\, F^i(k)G^*_{pk} = -\sqrt{Z_i}\,g_p \,, \tag{A9}$$

$$\int d^3k\, g_kG_{kp} = -\sqrt{Z_1}F^1(p) - \sqrt{Z_2}F^2(p) \,, \tag{A10}$$

$$\int d^3k\, G^*_{kp}G_{kq} = \delta(\vec{p}-\vec{q}) - F^1(p)F^1(q) - F^2(p)F^2(q) \,, \tag{A11}$$

$$\int d^3k\, g^*_kG^*_{kp}(k+l) = -\sum_i (M_i+l)\sqrt{Z_i}F^i(p) + g_0f(p) \,, \tag{A12}$$

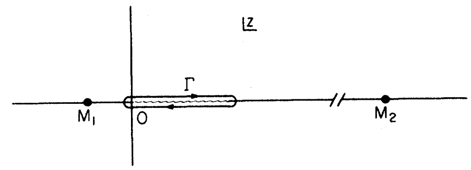

FIG. 2. The contour Γ that occurs in Eq. (A16) of Appendix A.

$$\int d^3k\, G_{kp}G^*_{kq}(k+l) = (p+l)\delta(\vec{p}-\vec{q}) - \sum_i (M_i+l)F^i(p)F^i(q) \,, \tag{A13}$$

$$\int d^3k\, |g_k|^2(k+l) = -\sum_i [(M_i+l)Z_i] + m_0 + l \,. \tag{A14}$$

The discontinuity of the function $\alpha(Z)$ defined in Eq. (3.7) can be written as

$$\alpha^+(k) - \alpha^-(k) = 2\pi i \bar{f}^2(k) \,,$$

where

$$\int d^3k\, f^2(k) \cdots = \int dk\, \bar{f}^2(k) \cdots \,. \tag{A15}$$

Finally, we note that many momentum integrals occurring in the text can be written as contour integrals in the cut k plane as

$$\int_0^L d^3k\, |g_k|^2\Psi(k) = -\frac{1}{2\pi i}\int_\Gamma \frac{dk}{\alpha(k)}\Psi(k) \,, \tag{A16}$$

where Γ is the contour shown in Fig. 2.

APPENDIX B

In this appendix we show that the physical states in the $V\theta$ sector of the Lee model form a complete set, i.e.,

$$\sum_i \int d\xi_i \,|V_i\theta\rangle\rangle_{\xi_i\xi_i}\langle\langle V_i\theta| + \tfrac{1}{2}\int d\zeta_1 d\zeta_2 \,|N\theta\theta\rangle\rangle_{\zeta_1\zeta_2\zeta_1\zeta_2}\langle\langle N\theta\theta| + |M_1+M_2\rangle\rangle\langle\langle M_1+M_2| = I \,. \tag{B1}$$

In terms of the Källén-Pauli components defined in the text, Eq. (B1) can be equivalently written as

$$S_\lambda \psi_\lambda(l)\psi^*_\lambda(l') = \delta(\vec{l}-\vec{l}') \,, \tag{B2a}$$

$$S_\lambda \psi_\lambda(k,l)\psi^*_\lambda(k') = 0 \,, \tag{B2b}$$

and

$$S_\lambda \psi_\lambda(k,l)\psi^*_\lambda(k',l') = \delta(\vec{k}-\vec{k}')\delta(\vec{l}-\vec{l}') - \delta(\vec{k}-\vec{l}')\delta(\vec{l}-\vec{k}') \,. \tag{B2c}$$

In Eqs. (B2), the symbol S_λ denotes an integration over the continuum eigenstates with a proper weight [see Eq. (B1)] and a sum over the discrete eigenstates.

The Källén-Pauli components can be easily calculated using Eqs. (3.21) and the solutions in terms of the overcomplete basis presented in the text. We have

$$\psi_\lambda(l)=\sum_i [\sqrt{Z_i}\widetilde{G}_{\xi_i l}-\widetilde{g}^*_{\xi_i}F^i(l)] \,, \tag{B3a}$$

$$\psi_\lambda(k,l)=\sum_i [F^i(l)\widetilde{G}_{\xi_i k}-(\vec{k}\leftrightarrow\vec{l}\,)] \tag{B3b}$$

for the $|V_i\theta\rangle\rangle$ eigenstates,

$$\psi_\lambda(l)=g^*_{\zeta_2}G_{\zeta_1 l}-g^*_{\zeta_1}G_{\zeta_2 l}\,, \tag{B4a}$$

$$\psi_\lambda(k,l)=G_{\zeta_2 l}G_{\zeta_1 k}-(\vec{k}\leftrightarrow\vec{l}\,) \tag{B4b}$$

for the $|N\theta\theta\rangle\rangle_{\zeta_1\zeta_2}$ eigenstates, and

$$\psi_\lambda(l)=\sqrt{Z_1}F^2(l)-\sqrt{Z_2}F^1(l)\,, \tag{B5a}$$

$$\psi_\lambda(k,l)=F^2(k)F^1(l)-(k\leftrightarrow l) \tag{B5b}$$

for the discrete state $|M_1+M_2\rangle\rangle$.

We now proceed to demonstrate Eq. (B2a). We have

$$\begin{aligned}S_\lambda\psi_\lambda(l)\psi^*_\lambda(l')=&\sum_i\int_0^L d\xi_i[\sqrt{Z_i}G_{\xi_i l}-g^*_{\xi_i}F^i(l)][\sqrt{Z_i}G^*_{\xi_i l'}-g_{\xi_i}F^i(l')]\\&+\tfrac{1}{2}\int d\zeta_1 d\zeta_2(g^*_{\zeta_2}G_{\zeta_1 l}-g^*_{\zeta_1}G_{\zeta_2 l})(g_{\zeta_2}G_{\zeta_1 l'}-g_{\zeta_1}G^*_{\zeta_2 l'})\\&+[\sqrt{Z_1}F^2(l)-\sqrt{Z_2}F^1(l)][\sqrt{Z_1}F^2(l')-\sqrt{Z_2}F^1(l')]\,.\end{aligned}$$

The integrals can be easily worked out using Eqs. (A10) and (A11) presented in the text. We then find

$$S_\lambda\psi_\lambda(l)\psi_\lambda(l')=\delta(\vec{l}-\vec{l}\,')$$

which is the required result. Equations (B2b) and (B2c) can be demonstrated in an identical fashion.

APPENDIX C

In this appendix, we show that the Källén-Pauli wave functions for the states $|V_1\theta\rangle\rangle_\lambda$ and $|N\theta\theta\rangle\rangle_\lambda$ reduce to the corresponding wave functions for the states $|B\theta\rangle\rangle_\lambda$ and $|N\theta\theta\rangle\rangle_\lambda$ of the separable-potential model. To this end, we first compute the functions $\Xi_\lambda(k,l)$ for these states. From Eq. (3.10), and the solutions presented in Sec. II, we find

$$\begin{aligned}\Xi_\lambda(k,l)=&\,\delta(\vec{\zeta}_2-\vec{l}\,)\delta(\vec{\zeta}_1-\vec{k})-\frac{h(\zeta_2)h(l)}{(\zeta_2-l)D^+(\zeta_2)}\delta(\vec{\zeta}_1-\vec{k})+\frac{h(\zeta_1)h(l)}{(\zeta_1-l)D^+(\zeta_1)}\delta(\vec{\zeta}_2-\vec{k})\\&-\frac{h(k)h(l)h(\zeta_1)h(\zeta_2)}{(\zeta_1-l)(\zeta_2-k)D^+(\zeta_1)D^+(\zeta_2)}-(\vec{k}\leftrightarrow\vec{l}\,)\end{aligned} \tag{C1}$$

for the $|N\theta\theta\rangle\rangle_\lambda$ state and

$$\Xi_\lambda(k,l)=\frac{1}{[-D(M)]^{1/2}}\left[\frac{h(l)\delta(\vec{\xi}-\vec{k})}{M-l}-\frac{h(l)h(k)h(\xi)}{(M-l)(\xi-k)D^+(\xi)}\right]-(\vec{k}\leftrightarrow\vec{l}\,) \tag{C2}$$

for the $|B\theta\rangle\rangle_\lambda$ state.

We proceed to show that the finite-energy Lee-model wave functions reduce to those of the separable potential. We recall[5] that in the SC limit,

$$Z_1 \sim \frac{1}{{g_0}^2}\ , \quad G_{\lambda k} \sim (g_0)^0\ , \quad g_\lambda^* \sim \frac{1}{g_0}\ , \quad \text{and} \quad F^1(l) \sim (g_0)^0$$

for all finite λ. It then follows from Eqs. (B3a) and (B4a) that $\psi_\lambda(l) \overset{s}{=} 0$ for the $|V_1\theta\rangle\rangle_\lambda$ and the $|N\theta\theta\rangle\rangle_\lambda$ states.

We now turn our attention to the functions $\psi_\lambda(k,l)$. For the $|V_1\theta\rangle\rangle_\lambda$ state, we have from Eq. (B3a)

$$\begin{aligned}\psi_\lambda(k,l) &= \sqrt{Z_1}\,\frac{g_0 f(l)}{M_1 - l}\left[\delta(\vec{\xi}_1 - \vec{k}) + \frac{g_0 g^*_{\xi_1} f(k)}{\xi_1 - k}\right] - (k \leftrightarrow l) \\ &= \frac{1}{[\alpha(M_1)]^{1/2}}\,\frac{g_0 f(l)}{M_1 - l}\left[\delta(\vec{\xi}_1 - \vec{k}) + \frac{{g_0}^2 f(k) f(\xi_1)}{\alpha^+(\xi_1)(\xi_1 - k)}\right] - (k \leftrightarrow l) \\ &\overset{s}{=} \frac{1}{[-D'(M_1)]^{1/2}}\,\frac{g_0 f(l)}{\sqrt{M_2}}\,\frac{1}{M_1 - l}\left[\delta(\vec{\xi}_1 - \vec{k}) - \frac{{g_0}^2 f(k) f(\xi_1)}{M_2 D^+(\xi_1)(\xi_1 - k)}\right] - (\vec{k} \leftrightarrow \vec{l})\ ,\end{aligned} \tag{C3}$$

where in the last step, we have used[5]

$$\alpha(z) \overset{s}{=} -M_2 D(z)\ .$$

Identification of the state $|V_1\rangle\rangle$ with the state $|B\rangle\rangle_M$ of the separable-potential model so that $M_1 = M$, together with the identification of the form factors $h(k)$ and $g_0 f(k)/\sqrt{M_2}$, leads us to conclude from Eqs. (C2) and (C3) that

$$\psi_\lambda(k,l) \overset{s}{=} \Xi_\lambda(k,l)$$

for the $|V_1\theta\rangle\rangle_\lambda$ state.

Similarly, from Eq. (B4b), we have for the $|N\theta\theta\rangle\rangle_\lambda$ state of the Lee model

$$\begin{aligned}\psi_\lambda(k,l) &= G_{\xi_2 l} G_{\xi_1 k} - (\vec{k} \leftrightarrow \vec{l}) \\ &= \left[\delta(\vec{\xi}_2 - \vec{l}) + \frac{g_0 g^*_{\xi_2} f(l)}{\xi_2 - l}\right]\left[\delta(\vec{\xi}_1 - \vec{k}) + \frac{g_0 g^*_{\xi_1} f(k)}{\xi_1 - k}\right] - (\vec{k} \leftrightarrow \vec{l}) \\ &= \delta(\vec{\xi}_2 - \vec{l})\delta(\vec{\xi}_1 - \vec{k}) + \frac{{g_0}^2 f(\xi_2) f(l)}{\alpha^+(\xi_2)(\xi_2 - l)}\delta(\vec{\xi}_1 - \vec{k}) \\ &\quad + \frac{{g_0}^2 f(\xi_1) f(k)}{\alpha^+(\xi_1)(\xi_1 - k)}\delta(\vec{\xi}_2 - \vec{l}) + \frac{{g_0}^4 f(l) f(k) f(\xi_1) f(\xi_2)}{(\xi_2 - l)(\xi_1 - k)\alpha^+(\xi_2)\alpha^+(\xi_1)} - (\vec{k} \leftrightarrow \vec{l}) \\ &\overset{s}{=} \delta(\vec{\xi}_2 - \vec{l})\delta(\vec{\xi}_1 - \vec{k}) - \frac{{g_0}^2 f(\xi_2) f(l)}{M_2 D^+(\xi_2)}\,\frac{1}{(\xi_2 - l)}\delta(\vec{\xi}_1 - \vec{k}) \\ &\quad - \frac{{g_0}^2 f(\xi_1) f(k)}{M_2 D^+(\xi_1)}\,\frac{1}{(\xi_1 - k)}\delta(\vec{\xi}_2 - \vec{l}) + \frac{{g_0}^4 f(l) f(k) f(\xi_1) f(\xi_2)}{{M_2}^2 D^+(\xi_1) D^+(\xi_2)}\,\frac{1}{(\xi_2 - l)(\xi_1 - k)} - (\vec{k} \leftrightarrow \vec{l})\ ,\end{aligned} \tag{C4}$$

where in the last step we have used the SC form of $\alpha(z)$. We see from (C1) and (C4) that the identification of the form factors once again leads to the result

$$\psi_\lambda(k,l) \overset{s}{=} \Xi_\lambda(k,l)\ .$$

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[7]As in the bosonic case, the repulsive fermionic separable potential admits a discrete state beyond the cutoff but not below the cutoff, i.e., in any case there is only one discrete case in the $N\theta$ sector.

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[9]The corresponding situation in the case of the bosonic Lee model is different in that in the fermionic case the equations of motion for $\lambda = M_1 + M_2$ do not require any relationship between $\chi_\lambda{}^1$ and $\chi_\lambda{}^2$. Thus, these can be chosen to satisfy the constraint conditions. In the bosonic case, $\lambda = M_1 + M_2$ does not belong to the spectrum since the relationship between $\chi_\lambda{}^1$ and $\chi_\lambda{}^2$ that is required by the equations of motion [see Eq. (3.25b), Ref. 5(b)] is incompatible with the constraint conditions.

[10]It is interesting to compare this situation with that in the case of the bosonic Lee model. In the present case there is just one discrete state, irrespective of the coupling g_0. In the case of the bosonic Lee model there were two discrete states at Λ_1 and Λ_2 with $2M_1 < \Lambda_1 < M_1$ and $M_2 + L < \Lambda_2 < 2M_2$. In the SC limit, $\Lambda_1 \to 2M_1$ whereas Λ_2 remains finitely removed from $M_2 + L$.

[11]See, for example, T. Eguchi, Ref. 3.