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Quantum-chromodynamic vacuum as a glueball condensate

T. H. Hansson* and K. Johnson†
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106

C. Peterson‡
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
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It is argued semiquantitatively that the magnetic instability of the perturbative QCD vacuum leads to the formation of a Bose-Einstein condensate that locally can be described as a $J^{PC}=0^{++}$ gluon color-singlet bound state or "glueball." We also indicate how this result might be obtained by a more rigorous field-theoretical treatment. A general discussion of the global properties of the "glueball vacuum" is given, and as an illustration of our ideas we construct a phenomenological model in which the bag constant is related to the QCD scale Λ.

I. INTRODUCTION

Quark and gluon confinement is generally attributed to a nonperturbative structure of the QCD vacuum, which prevents the separation of color except over regions of hadronic size.[1] Such a picture is the basis of the MIT bag model where hadrons are "bubbles" of perturbative vacuum immersed in a nonperturbative physical vacuum.[2,3] In the bag model one does not attempt to give a microscopic description of the nonperturbative phase, but characterizes it with only one bulk property, namely the difference in energy density between the "perturbative" (i.e., "empty") and the physical vacuum. This difference is the bag constant B.

Here we shall try to argue that the perturbative vacuum is unstable against the formation of "constituent" gluon bound states hereafter referred to as glueballs. The vacuum state may then be described as a Bose-Einstein condensed liquid of such glueballs, together with the standard short-wavelength fluctuations present in the ordinary perturbative vacuum.[4] The size of the "constituent" gluon wave functions which describe the condensed glueballs will be comparable to the distance between them. There will also be an effective strong short-distance repulsion between the glueballs which inhibits the overlap of their constituent gluon wave functions. Since there is no conservation law to prevent the spontaneous creation of negative-energy glueballs, they will fill up all space, and form a liquidlike state.

We are at present not able to rigorously derive the above picture of the ground state from the equations of QCD. Instead we shall argue semiquantitatively that the instability mentioned above occurs, and that consequently such a glueball "condensate" should be formed. We will construct a simple model for the condensate, which enables us to estimate the bag constant in terms of the QCD coupling strength Λ. It is also indicated how a more rigorous treatment based on field theory might be achieved by explicitly constructing a trial wave functional for the QCD vacuum. The basic idea here is to find a complete set of functions in which the ground-state wave functional has a simple expansion, and we shall argue that the so-called Wannier functions known from solid-state physics may provide such a set.

The paper is organized as follows. In the next section we first discuss the nature of the instability of the perturbative vacuum and then argue that it is connected to the strong color-magnetic forces between localized gluons. Next we give a general discussion of localized gluons in the context of quantum field theory and consider the validity of approximations used later. In Secs. II C and II D rough estimates based on a bag-model approach are used to show that the perturbative vacuum is unstable against formation of 0^{++} glueballs and that these glueballs stabilize themselves at a size $\sim 1/\Lambda$ because of asymptotic freedom. The section ends with a description of the expected global properties of the QCD vacuum and its phenomenological consequences. In Sec. III we construct a simple mean-field model to give a semiquantitative illustration of the previous discussion.

II. THE QCD VACUUM AS A GLUEBALL CONDENSATE

A. The instability of the perturbative vacuum

In QCD the quarks are color triplets and the gluons color octets. Hence, one would expect that the dominant effects should be associated with the gluon-gluon interactions since their color charge is the largest. Since QCD is a theory of "charged" spinning particles, one would also expect that magnetic effects related to their intrinsic magnetic moments to be very important. The spin of the gluons is twice that of the quarks so this should also tend to enhance the importance of the gluon-gluon interaction. We shall therefore omit the quarks and consider the gluons only.

Recently, a simple physical picture of the origin of the asymptotic freedom of QCD has been given.[5] This picture at the same time helps to explain the nature of the instability of the perturbative ground state. For clarity, let us use the example of electrodynamics. Anything we say can be immediately generalized to any non-Abelian gauge theory. We shall investigate how the vacuum responds to the application of an external electromagnetic field. To do this, let us view the perturbative vacuum of any charge field as a medium of charged particles (i.e., the "particles" which occur in the zero motion associated with each mode of the field). We shall for simplicity always assume the particles to be massless. In the case of fermions, we have the Dirac sea of negative-energy particles, and in the Bose case a "one-half" sea of positive-energy particles. The vacuum can now be considered as a polarizable medium with the very special property of looking the same in all Lorentz frames. Thus, the product of the electric permeability ϵ, and the magnetic susceptibility μ, is one; $\epsilon\mu=1$, i.e., we have either $\epsilon>1$, "screening" and $\mu<1$, diamagnetism, or $\epsilon<1$, "antiscreening" and $\mu>1$, paramagnetism.

The particles carry both electric charge and intrinsic magnetic moments. It is, however, simplest to study the polarization properties by considering what happens in an external magnetic field, rather than in an electric one. A calculation of the magnetic susceptibility valid for any medium of free charged particles with spin S and gyromagnetic ratio g yields ($\mu=1+\chi$)

$$\chi=e^2\{\mathrm{Tr}[(2S_z)^2-\tfrac{1}{3}]\}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E^3}\ , \qquad (1)$$

where $(2S_z)^2$ is the contribution from spin paramagnetism and the $-\frac{1}{3}$ is the contribution from Landau diamagnetism, with $E=|\vec{p}|$ for bosons and $E=-|\vec{p}|$ for fermions. The integral in Eq. (1) diverges in both the ultraviolet and infrared. The regularization of the UV divergence leads in the standard way to the introduction of a scale and a running coupling constant, while the IR divergence is regulated by the magnetic field.

For spin-zero particles the medium is diamagnetic due to the Landau diamagnetism associated with the quantized orbits. This corresponds to the standard intuitive picture associated with charge screening in the case of an applied electric field. In contrast, although it would appear that fermions ($g=2$) should be paramagnetic because of their intrinsic magnetic moment, this is not so, since the medium carries negative energy. Thus spin-zero and spin-one-half charged particles both give rise to a diamagnetic vacuum which screens electric charges but for totally different reasons. The first possibility to have a paramagnetic vacuum is with spin-one particles ($g=2$). That is, $\mu>1$ and hence $\epsilon<1$ in the case of pure Yang-Mills theories is a simple consequence of the spin of the vector bosons and the positive energy carried by the vacuum "medium." Here we must mention the existence of the "unstable mode"[6]—that is, because of the sign of the magnetic susceptibility, the effective potential becomes complex in a background magnetic field (this is analogous to the effective potential being complex in a background electric field in ordinary spin-one-half electrodynamics).

Since the perturbative vacuum filled with free vector particles is paramagnetic, it is possible for it to gain energy by spontaneous magnetization in which case it is unstable. To describe this phenomenon by mean-field theory (as in the "Copenhagen vacuum"[7]), one must be careful in handling the "unstable mode." Furthermore if the system can lower its energy by spontaneous magnetization, the question might be raised as to whether enough quanta will be present locally to define semiclassical color-magnetic fields. Certainly QCD could stabilize itself in this way because of the repulsive A^4 term in the Lagrangian. However, we shall argue below that there is a possible QCD ground state which is stabilized at the quantum level, that is, with a few field quanta present in the volume $1/\Lambda^3$ where Λ is the basic scale of QCD defined by the short-distance perturbative region of the vacuum wave functional. In this case we do not expect the fluctuations around the ground state

to include any "long-wavelength gluons" so the system may not be troubled with any unstable modes. We shall not be able to establish even qualitatively that such a quantum vacuum has a lower energy than a possible competing semiclassical state. However, a "shallow" vacuum seems to be indicated by the success of phenomenological models such as the MIT bag model. Here, the bag constant $B \simeq 55$ MeV/fm^3 is the energy density in the condensed phase (the "outside vacuum"). In terms of the conventional QCD scale Λ where Λ is in the range 100 to 500 MeV, $B^{1/4}/\Lambda$ is between 0.3 and 1.5. If we express this in terms of a "number" of gluons with negative energy $\sim\Lambda$, per volume $1/\Lambda^3$, $B \sim n\Lambda^4$ or $n^{1/4} \sim B^{1/4}/\Lambda$. Thus, n is not expected to be a very large number. Hence, we shall in the following sections study the "local" stability of the perturbative vacuum by adding a few localized gluons to it.

B. Properties of localized gluons

Since the gluons are massless, they can not simply be put in a Lorentz rest frame, so we must make precise what is meant by a localized gluon. In Hamiltonian quantum field theory, a one-gluon state is defined in terms of a complete set $\{\vec{A}_\alpha\}$ of orthonormal wave functions, i.e, the field operator is expanded as

$$\vec{A} = \sum_\alpha [\vec{A}_\alpha(\vec{x}) a_\alpha + \text{h.c.}] , \tag{2}$$

where α represents all quantum numbers necessary for identifying the state. A one-particle state

$$|1,\alpha\rangle = a_\alpha^\dagger |0\rangle \tag{3}$$

is localized if the corresponding wave function $\vec{A}_\alpha$ is localized. Note that these states will in general not be eigenstates of the Hamiltonian. However, if any "constituent" model for the QCD ground state (and nearby low-lying states) is approximately correct, then a relatively simple description of the vacuum in terms of such a set of wave functions should be possible. The "constituent" gluons will be the particles which occupy the "modes" characterized by the wave functions $\vec{A}_\alpha$. We shall now propose a systematic way of handling the QCD vacuum along these lines. The fields will be expanded in terms of localized wave functions centered on the sites of a periodic spatial lattice.

To be explicit, let us describe the gauge field theory in the $A^0=0$ or "temporal" gauge where the Hamiltonian is

$$H = \tfrac{1}{2} \int d^3x (E_a^{\,2} + B_a^{\,2}) \tag{4}$$

with

$$[A_a^k(\vec{x}), -E_b^l(\vec{y})] = i\delta^{kl}\delta_{ab}\delta^3(\vec{x}-\vec{y}) \tag{5}$$

and

$$\vec{B}_a = \nabla \times \vec{A}_a + g f_{abc} \vec{A}_b \times \vec{A}_c . \tag{6}$$

The generator of local (time-independent) gauge transformations is

$$G_a(\vec{x}) = \nabla \cdot \vec{E}_a + g f_{abc} \vec{E}_b \cdot \vec{A}_c , \tag{7}$$

so

$$[H, G_a(\vec{x})] = 0 \tag{8}$$

and

$$[G_a(\vec{x}), G_b(\vec{y})] = i\delta^3(\vec{x}-\vec{y}) f_{abc} G_c(\vec{x}) . \tag{9}$$

The "physical" states we are interested in are locally gauge invariant,

$$G_a(\vec{x})\,|S\rangle = 0 . \tag{10}$$

We now introduce a periodic lattice in space, and thereby forfeit at the start any hope of a very simple description of the Lorentz group—or even the translation and rotation groups. Hopefully, the long-range spatial order, associated with the regularity of the lattice, and which destroys these symmetries, is an artifact. The localization of the wave functions over distances of order $1/\Lambda$, which will be defined using the lattice scale, should however be a real effect. That is, we shall expect that the physical cell size is of order $1/\Lambda$. Further, we hope that the energy cost associated with the artificial long-range order is small in comparison with the energy gain which we shall obtain from the localization.

To construct a complete orthogonal set of localized wave functions centered around the lattice points, we use the following procedure due to Kohn.[8] Start with a complete and normalized (but not necessarily orthogonal) set of localized functions $a_n^k(\vec{x})$, $n = 1,2,\ldots,$. From these one can construct a set of Bloch functions $\hat{A}^k_{\vec{q},n}(\vec{x})$ having the translational symmetry of the lattice

$$\hat{A}^k_{\vec{q},n}(\vec{x}) = \frac{1}{\sqrt{N}} \sum_S a_n(\vec{x}-\vec{x}_S) \exp(i\vec{q}\cdot\vec{x}_S) , \tag{11}$$

where N is the number of cells in the lattice and

$\vec{x}_S$ is a lattice vector. These functions are complete but in general not orthogonal. It is not difficult, however, to take linear combinations which form a new complete set of Bloch functions $A^k_{\vec{q},n}$ which are orthogonal. Because of the periodicity, these satisfy the Bloch condition

$$A^k_{\vec{q},n}(\vec{x}+\vec{x}_S)=\exp(i\vec{q}\cdot\vec{x}_S)A^k_{\vec{q},n}(\vec{x}) , \tag{12}$$

and from the completeness of the functions $a_n(\vec{x})$ we get the completeness relation for the Bloch functions

$$\int_{BZ}\frac{d^3q}{V}\sum_n A^{k*}_{\vec{q},n}(\vec{x})A^l_{\vec{q},n}(\vec{y})=\delta^{kl}\delta^3(\vec{x}-\vec{y}) . \tag{13}$$

Here the integral over the reciprocal lattice vector $\vec{q}$ goes over the first Brillouin zone with volume V. We are now ready to define the localized Wannier functions $A^k_{S,n}$ by

$$A^k_{S,n}(\vec{x})=\int_{BZ}\frac{d^3q}{V}\exp(-i\vec{q}\cdot\vec{x}_S)A^k_{\vec{q},n}(\vec{x}) \tag{14}$$

which satisfies the completeness relation

$$\sum_{S,n}A^k_{S,n}(\vec{x})A^l_{S,n}(\vec{y})=\delta^{kl}\delta^3(\vec{x}-\vec{y}) . \tag{15}$$

Here S labels a cell of the lattice, and n the "band" which we may think of as characterized by the number of nodes in $A^k_{S,n}(\vec{x})$ over a distance of the order of the lattice scale. As $n\to\infty$, renormalizability and asymptotic freedom should allow us to use perturbative methods. Over the long-distance scales associated with the momenta $|\vec{q}|<1/\Lambda$ we hope to use variational methods.

The Bloch and Wannier functions are unitarily equivalent. However, it is the localization associated with the Wannier functions which we believe will be most useful, rather than the lattice periodicity of the Bloch functions.

The functions $A^k_{S,n}(\vec{x})$ is a set of the type referred to in the beginning of this section and in the $A^0=0$ gauge the expansion corresponding to Eq. (2) is

$$A^k_a(\vec{x})=\sum_{n,S}q^a_{n,S}A^k_{n,S}(\vec{x}) , \tag{16a}$$

$$E^k_a(\vec{x})=-\sum_{n,S}p^a_{n,S}A^k_{n,S}(\vec{x}) . \tag{16b}$$

From the above it is clear that there is an enormous freedom in our choice of $A^k_{n,S}(\vec{x})$. We now show, that by imposing an additional condition on these functions, we get a systematic way of constructing "physical" trial states, i.e., states fulfilling the condition Eq. (10). To do this, separate the local gauge invariance into a "coarse-grained" gauge invariance associated with the lattice scale, and a "fine-grained" one related to local transformation on distance scales short in comparison with the lattice spacing. First consider the coarse-grained transformations defined by

$$G^a_S=\int_S d^3xG^a(\vec{x}) , \tag{17}$$

where the integration takes place over one cell of the lattice. Let us use the freedom to define the set $A^k_{S,n}(\vec{x})$ of Wannier functions to make

$$\int_S d^3x\partial_kA^k_{S,n}(\vec{x})=\int_{\delta S}d^2xn_kA^k_{S,n}(\vec{x})=0 . \tag{18}$$

(For the construction of such a set in $2+1$ dimensions, see Appendix A.) It will then be true that

$$\int_{S'}d^3x\partial_kA^k_{S,n}(\vec{x})=0 , \tag{19}$$

where S' is any other cell. (This is most easily proved by using the unitarily related Bloch functions.) In this case there will never be any *net* electric flux from any cell S':

$$\int_{S'}d^3x\,\vec{\nabla}\cdot\vec{E}_a(x)=0 . \tag{20}$$

One might worry about the compatability of the completeness Eq. (15) and the boundary conditions Eq. (18). If we expand functions $A_n(\vec{x})$ which do not obey the boundary condition in terms of the set $A^k_{n,q}$ the expansion will not converge uniformly on the boundary surfaces, that is, many modes will contribute to represent such a field. However, if the true vacuum is such that these configurations are unprobable, then our expansion will be reasonable.

Because of Eq. (20) the operator G_S takes the form

$$G^a_S=\int_S d^3x[gf^{abc}\vec{E}_b(x)\cdot\vec{A}_c(x)] \tag{21}$$

so the condition for "coarse-grained" local gauge invariance is that there be no net "global" color carried by the fields in any cell of the lattice. If we describe our states in terms of operators corresponding to a set $A^k_{S,n}$ fulfilling Eq. (18), this requirement is simply that no net color is carried in these modes. There is of course still the requirement of fine-grained gauge invariance, namely that Eq. (10) should be satisfied over short-distance scales; this can be achieved by using perturbation

theory. The ordinary IR divergences will now be cut off by the lattice scale.

The degree of localization, and scale size associated with the lattice are some of the remaining parameters in the complete set. It is our hope that these will be fixed by variational principles applied to specific proposals for the ground state, and that we can then fix the lattice scale in terms of the QCD scale Λ. In the following sections we shall argue semiquantitatively that when expressed in such a local basis, the ground state of QCD is a state in which the cells will contain occupied modes; that is, we shall argue that there can be a condensed phase of gluon "quasiparticles" present in the vacuum.

The program just outlined is obviously very ambitious. In the rest of the paper we shall use a much simpler and cruder way to construct the localized functions $A^k_{S,n}(\vec{x})$. We shall also not attempt to construct an explicit field-theoretical trial state for the vacuum, but instead study the quantum mechanics of the "constituent" gluons.

A very simple way to construct a complete, orthogonal, localized set $A^k_{S,n}(\vec{x})$ is to divide space into cells and then solve the Maxwell equations with suitable boundary conditions in each cell. The resulting set of "cavity modes" is complete and orthonormal (ON) in each cell. A complete set for all space is then trivially the set of all such sets in all cells. The price one pays for this simple recipe is the presence of discontinuities in the wave functions and their derivatives (and corresponding singularities in, e.g., the energy) at the cell walls. The obvious way to bypass this difficulty is to smoothen the walls and thus also the wave functions. This causes the functions defined in different cells to overlap and we are led to the Wannier functions discussed above. If, however, as suggested in the previous section, the vacuum stabilizes locally by excitation of a few localized gluon modes, the effect of this overlap should not be too important. Hence, we shall for an approximation use cavity modes to describe the localized gluons, and simply ignore the discontinuities introduced by the sharp cell walls. From the previous discussion it is clear that we want to impose a local boundary condition on the surface of the cells, which ensures that the global condition Eq. (20) is satisfied. Still there is a large freedom in the choice of boundary conditions and cell shape which defines the cavity modes. Since our prejudice is that the important phenomenon of vacuum instability is a local one we shall try to locally keep as much as possible of the symmetry of the full theory. Thus we take a spherical cavity to keep rotational invariance and the gauge-invariant boundary condition

$$n_\mu G_a^{\mu\nu}=0\ , \tag{22}$$

where $n_\mu=(0,\vec{n})$ is the spacelike normal to the cell surface. Written in terms of the $\vec{E}$ and $\vec{B}$ fields, Eq. (22) reads

$$\vec{n}\cdot\vec{E}_a=0\ , \tag{23a}$$

$$\vec{n}\cdot\vec{B}_a=0 \tag{23b}$$

so Eq. (20) is trivially fulfilled and we of course recognize Eqs. (23) as the usual bag boundary conditions.

C. The 0^{++} glueball instability

We have been led to study color-singlet states built from spherical-cavity gluon modes defined by the boundary condition Eq. (23). These states are of course the "glueballs" of the bag model, which have been extensively studied in the literature.[9–11] Let us briefly review the properties of these states.

The different modes of a gluon in a spherical cavity satisfying Eq. (23) can be classified as transverse electric (TE) or transverse magnetic (TM). The lowest TE and TM modes have energies $2.74/R$ and $4.49/R$ and parities $P=-(-1)^l$ and $P=(-1)^l$, respectively. The lowest-lying two-gluon glueballs are thus formed by two TE modes and has $J^{PC}=0^{++},2^{++}$.

It is well known from hadron spectroscopy, that the chromomagnetic spin-spin interactions are crucial for understanding the mass differences in the low-lying hadron multiplets as, e.g., K-K^* and p-Δ. One would naively expect these chromomagnetic forces to be even more important for gluons than for quarks since the gluons both have larger spin (1) and higher color charge (8) than the quarks. The chromomagnetic energy shift is however only one part of the lowest-order interaction shown in Fig. 1. The full interaction Hamiltonian between two gluons 1 and 2 in color spin space to order α_s is given by

$$H_{12}=-\frac{\alpha_s}{R}[\,a\,\vec{\Lambda}_1\cdot\vec{\Lambda}_2\vec{S}_1\cdot\vec{S}_2 +b\,\vec{\Lambda}_1\cdot\vec{\Lambda}_2T_{12}+2c_1\vec{\Lambda}_1\cdot\vec{\Lambda}_2 +c_2(\Lambda_1^2+\Lambda_2^2)]\ , \tag{24}$$

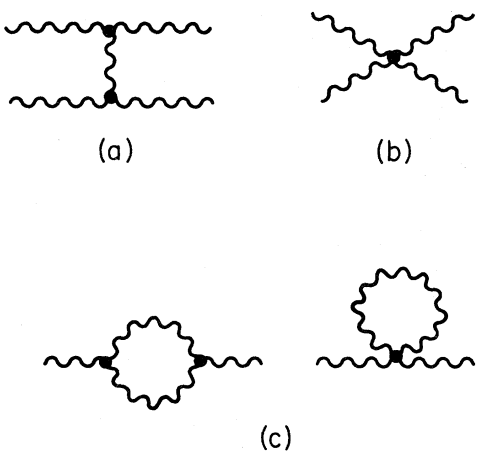

FIG. 1. Lowest-order two-gluon interaction diagrams (a),(b) and gluon self-energy diagrams (c).

where $\vec{\Lambda}$ and $\vec{S}$ are the usual color and spin generators and the tensor T_{12} is

$$T_{12}=2[(\vec{S}_1\cdot\vec{S}_2)^2-1]+\vec{S}_1\cdot\vec{S}_2 . \tag{25}$$

The first three terms in Eq. (24) come entirely from the interaction diagrams 1(a) and 1(b) while the last one is the self-energies of the gluons [Fig. 1(c)]. Both the "magnetic" term a and the "electric" term b are separately independent of gauge, while for the last two terms only their sum is gauge independent. The coefficients a and b have been calculated to be[12,13]

$$\begin{aligned} a&=0.26 , \\ b&=-0.04 . \end{aligned} \tag{26}$$

An alternative way of expressing this result is

$$\begin{aligned} H_{12}=-\frac{\alpha_s}{R}\{&(a+b)\vec{\Lambda}_1\cdot\vec{\Lambda}_2\vec{S}_1\cdot\vec{S}_2 \\ &+2b[(\vec{S}_1\cdot\vec{S}_2)^2-\tfrac{4}{3}] \\ &+2(c_1+\tfrac{1}{3}b)\vec{\Lambda}_1\cdot\vec{\Lambda}_2 \\ &+c_2(\Lambda_1^2+\Lambda_2^2)\} , \end{aligned} \tag{27}$$

where the second term is an "induced" quadrupole-quadrupole interaction present in the electric energy shift.

We see from Eq. (26) that the electric energy is small compared to the magnetic one. The coefficients c_1 and c_2 have not yet been calculated because of the computational difficulties related to the extraction of the ultraviolet divergent parts. There is, however, a possibility that the effect of the two last terms is small compared with the magnetic energy. Namely, if $c_1\simeq c_2$, then we get for color singlets

$$2c_1\vec{\Lambda}_1\cdot\vec{\Lambda}_2+c_2(\Lambda_1^2+\Lambda_2^2)\simeq c_1(\vec{\Lambda}_1+\vec{\Lambda}_2)^2=0 . \tag{28}$$

Since b is small compared to a it does not matter much whether we use Eq. (24) or Eq. (27) in the calculations.

In the following we shall mainly concern ourselves with the effect of the magnetic interaction energy only. This is reasonable if the self-energy terms really are small. Everything we will say will hold true even for large self-energies as long as they are negative, and in Appendix B we argue that this might be the case. If on the other side the self-energies turn out to be large and positive, our whole picture will be seriously questioned. Needless to say, an explicit calculation of the diagrams 1(c) is of great importance.

Let us now consider the magnitude of the magnetic term in the various possible color and spin states for a single gluon pair. The relative values of the strength of this interaction energy as given by the eigenvalues of the operator

$$\Omega=-\tfrac{1}{2}\sum_{i\neq j}\vec{\Lambda}_i\cdot\vec{\Lambda}_j\vec{S}_i\cdot\vec{S}_j \tag{29}$$

are (for $i,j=1,2$) shown in Table I. The states are listed in order of decreasing strength of attraction. We see that there are four states with negative color-magnetic energy. Since the coupling becomes large for large distances, all these configurations are potentially unstable. We now discuss the several possibilities.

The paramagnetic instability of the perturbative vacuum discussed earlier occurs in the state in

TABLE I. Eigenvalues of the operator Ω in two-gluon states of definite color and spin.

Spin	Color	$\Omega=-\Lambda_1\cdot\Lambda_2 S_1\cdot S_2$
0	1	-6
0	8_S	-3
1	8_A	$-\frac{3}{2}$
2	27	-1
1	$10,\overline{10}$	0
0	27	$+2$
2	8_S	$+\frac{3}{2}$
2	1	$+3$

which the pair forms a color-octet vector state, where $\Omega=-\frac{3}{2}$. If we were to use mean-field theory to describe the global state in this case, we would be at the starting point of the Copenhagen group. We see a stronger attraction in the color-octect, spin-singlet channel. In this case, a mean-field treatment of the pair bound state would lead to a Higgs-type spontaneous breakdown of color symmetry. However, the attraction is strongest in the color-singlet, spin-singlet state, and hence we shall focus here on the possibility that the global state will correspond to a Bose condensed system of spinless and colorless gluon pairs.[14] In the next section, we will argue that such a condensate can be locally stable due to a balance between kinetic and interaction energy.

Up to now we have only considered spherically symmetric glueballs. One might worry that there are other shapes with even stronger attraction. To study this problem carefully one really should have to describe the shape with a set of parameters, and then minimize the vacuum energy density with respect to these. In the following sections we shall assume a spherical shape, and keep only R as a variational parameter. The reason for this is that for color singlets the interaction in the spin-zero state is strongly attractive while the five spin-two states are strongly repulsive. Thus we expect that any change of the spherical shape will mix in spin-two components in the wave function and hence rapidly increase the energy. So we conclude that the spherical shape is locally stable, but can not exclude other stable shapes corresponding to, e.g., long tubes or thin "pancakes." The presence of such configurations however would introduce a dimensionless number in the description of the vacuum, and that seems to us to be somewhat unnatural.

D. Local stabilization by gluon pair condensation

We shall now qualitatively show how, as a consequence of asymptotic freedom, the color- and spin-singlet two-gluon state becomes locally stable and acquires a negative total energy.

The local two-gluon state just mentioned should, of course, except for the "valence" gluons, also contain the "medium" of the perturbative vacuum (or at least the high-momentum parts of it). The "medium" modifies the interaction between the "valence" gluons and causes the coupling constant to "run" as a function of ΛR where R is the radius of the bound state and Λ the basic QCD scale parameter. The renormalization group tells us how α_s behaves for small R, i.e., for $\Lambda R \ll 1$,

$$\lim_{R\to 0}\alpha_S(R)=\frac{2\pi}{9}\frac{-1}{\ln\Lambda R}\ . \tag{30}$$

In order to connect the Λ parameter of this formula to, e.g., $\Lambda_{\overline{\rm MS}}$ measured in deep-inelastic scattering, one has to evaluate the one-loop corrections to the interaction diagrams 1(a) and 1(b). Such a calculation is on the same level of complication as that of the self-energies mentioned earlier, and has not yet been carried out. It is to be stressed, however, that nothing in principle prevents this calculation from being done, and in that way eliminates the need for a free scale parameter.

As a consequence of Eq. (30), the total energy of the gluon pair becomes

$$E(R)\simeq E_k(R)-6(0.26)\left[\frac{\alpha_s(\Lambda R)}{R}\right]\ . \tag{31}$$

To be realistic, the estimate of the kinetic energy caused by localizing the pair, $E_k=2(2.74/R)$, is considerably too high. To see this, consider the general case of localizing N gluons (in the lowest energy mode) in a sphere of radius R. The naive value for the kinetic energy is $E_k=N(2.74/R)$. This estimate, however, includes not only the cost of the relative localization of the particles, which is all that is necessary for them to benefit from the strong attractive coupling, but also a contribution for localizing all the particles in a fixed region of space (center-of-"mass" energy). This effect may be simply estimated by using the relation

$$E_{\rm true}{}^2+\langle(\vec{p}_1+\vec{p}_2+\cdots+\vec{p}_N)^2\rangle=E_{\rm naive}{}^2\ , \tag{32}$$

where $\vec{p}_1+\vec{p}_2+\cdots+\vec{p}_N=\vec{p}$ is the total momentum of the N-gluon state. Since they are all in the same spatial state,

$$\langle p^2\rangle=N\langle p_1{}^2\rangle\simeq N\left[\frac{2.74}{R}\right]^2 \tag{33}$$

and therefore

$$E_k^{\rm true}\simeq[N(N-1)]^{1/2}\left[\frac{2.74}{R}\right]\ . \tag{34}$$

Thus, we find for a localized gluon pair,

$$E_{\rm pair}\simeq\sqrt{2}\left[\frac{2.74}{R}\right]-\left[\frac{1.6\alpha_s(\Lambda R)}{R}\right]\ . \tag{35}$$

Because α_s runs, we see that E reaches a minimum for $E<0$, and becomes negative for a value of R where $\alpha_s>2.4$. Note that the stabilization can take place because of asymptotic freedom as reflected in the running coupling constant $\alpha_s(\Lambda R)$, and that the size of the stabilized state will be of order $1/\Lambda$.

A perhaps slightly more convincing way to handle the localization energy would be to apply Eq. (32) directly to the total rather than to only the kinetic energy. In that case, since the energy connected with localization is subtracted, E^2 is simply m^2 for the bound state. For $N=2$ we get

$$m^2=E_{\text{naive}}{}^2-\langle p^2\rangle = \left[2\left[\frac{2.74}{R}\right]-\left[\frac{1.6\alpha_s(R)}{R}\right]\right]^2-2\left[\frac{2.74}{R}\right]^2 \tag{36}$$

which becomes negative for $\alpha_s>1.0$. The presence of such a tachyonic state is the signal of an instability of the perturbative ground state. The difference between this and the above estimate of the "critical" α_s^{crit} (2.4 vs 1.0) needed for the instability to occur gives an idea about the uncertainties involved in these calculations.

One might be worried that the rather high value of α_s in these estimates indicates that a perturbative treatment is not to be trusted. Although nothing can be said with certainty, one should remember that in the bag model, good fits to the low-lying hadron spectrum are obtained using first-order perturbation theory with $\alpha_s=2.2$ (see Refs. 3, and 16).[17] Also, if we include the self-energies of the gluons as estimated in Appendix B, α_s^{crit} drops to $\sim 0.5-1.0$.

E. Stability against addition of extra gluons

Before we focus on the global form of the ground state, we must study the local state to see whether it is energetically profitable to add more than two gluons locally to the perturbative vacuum. Since gluons are bosons, it is by no means obvious that two gluons will be the most likely configuration locally present in the ground state.

In analogy to Eqs. (35) to (36) we have for N gluons

$$E_N\simeq\sqrt{N(N-1)}\left[\frac{2.74}{R}\right]+\frac{\Omega^{(N)}0.26\alpha_s(\Lambda R)}{R}, \tag{37}$$

$$m_N{}^2\simeq\left[N\left[\frac{2.74}{R}\right]+\left[\frac{\Omega^{(N)}0.26\alpha_s(R)}{R}\right]\right]^2 -N\left[\frac{2.74}{R}\right]^2, \tag{38}$$

where $\Omega^{(N)}$ is the expectation value of the operator Ω in the N-gluon color- and spin-singlet state. A very rough criterion for the relative importance of different N would be the value of the "critical" coupling constant α_s^{crit} at which E_N or $m_N{}^2$ becomes negative. The strongest instability should be related to the shortest distance and hence to the lowest α_s^{crit}. We can clearly not investigate all values of N but shall consider the two limiting cases of N small ($N=3$ and 4), and N large.

When N becomes large, it is easily shown that for any even number of gluons in the same spatial state

$$\lim_{N\to\infty}\Omega_{\min}^{(N)}=-3N \tag{39}$$

which corresponds to a critical coupling $\alpha_s^{\text{crit}}\simeq 3.5$ as compared to $\alpha_s^{\text{crit}}\simeq 1.0-2.4$ in the case of two gluons [the two numbers shown here and below come from using Eqs. (38) and (37), respectively].

For $N=3$ each pair of gluons forms a color-octet spin-one state, so trivially,

$$\Omega^{(3)}=-3(-\tfrac{3}{2})(-1)=-\tfrac{9}{2} \tag{40}$$

which gives $\alpha_s^{\text{crit}}\simeq 3.0-5.7$. In the case of four gluons, it is shown in Appendix C that there are four possible spin- and color-singlet configurations having $\Omega^{(4)}\simeq -16$, -4, -3, and $+7.8$, respectively. (The state with maximum attraction may be compared to the large-N estimate $-3N=-12$ given earlier.) For the most attractive state ($\Omega^{(4)}=-16$) we get $\alpha_s^{\text{crit}}\simeq 1.3-2.3$.

From these estimates[18] one could be tempted to exclude $N=3$ and $N=\infty$, and say that $N=2$ and $N=4$ are equally favored. Such a conclusion would, however, be very dangerous for two reasons. First, we omitted the self-energies which could be important. Second, to find the most stable global configuration we should minimize the energy density E/V, and not the energy of the one glueball state. We postpone the problem of self-energies, and try to estimate the energy density.

For this we shall have to anticipate some results from Sec. III, and use the following relation for

the energy density [Eq. (51)]:

$$E/V \sim -\frac{(-m_N{}^2)^{1/2}}{R^3} . \tag{41}$$

From Eqs. (38) and (39) it follows that in the large-N limit, $m_N{}^2$ becomes negative for

$$\alpha_s(R) > 3.5\frac{3.5}{N^{1/2}} . \tag{42}$$

This means that large N implies large α_s^{crit} and consequently large R. Since $E/V \sim R^{-4}$ a moderate change in R gives a rather large change in the energy density. Thus, it seems likely that the few-gluon configurations are favored over the many-gluon states, although it should be remembered that this whole discussion is based on a very simple model for the condensate.

For the few-gluon states the situation is more complicated. The $N=3$ case is certainly unfavorable because of the rather large α_s^{crit}, but for $N=4$ the values of α_s^{crit} are comparable to those in the two-gluon case. In Sec. III we shall show that, using identical parameters, the two-gluon condensate is slightly favored over the four-gluon one, but the difference is clearly within the error of the calculation.

We shall now comment on the possible effects of including self-energies. As before, a large positive self-energy could destroy our whole picture, so we will consider a contribution of type $-N\alpha_s c/R$ where $c>0$. The effect of such a term in the large-N limit will simply be to change Eq. (42) to

$$\alpha_s(R) > \left[\frac{2.74}{0.78+c}\right]\left[1-\frac{1}{N^{1/2}}\right] \tag{43}$$

so the conclusions following Eq. (42) are still valid. If the self-energies are large, it could very well happen that the glueballs in the condensate are almost pure two-gluon states.

It is clear that from these considerations nothing can be said for sure about the number of gluons. A reasonable guess would be a state dominated by $N=2$ but with a (perhaps sizable) admixture of $N=4$. Our main point, however, is that the number of gluons should be fairly small, and that seems to be indicated by all of the above estimates.

F. Global structure of the glueball vacuum

In what follows we shall assume that the energetically most favorable state in QCD is one which locally is a two-gluon 0^{++} color-singlet glueball. Given this, we are naturally led to the picture of the QCD vacuum as globally being a zero-momentum Bose condensate of 0^{++} glueballs. We now proceed to a qualitative discussion of the properties of such a condensate.

First, if a glueball condensate is formed at all, it must be formed everywhere, i.e., the vacuum is densely filled with the glueballs. The reason for this is that the instability is a local phenomenon, so no region of perturbative vacuum of size > (size of a glueball) can be stable against the creation of a glueball. Second, one could argue that even if the most attractive state of four gluons is close in energy to that of two gluon pairs, there would not be much admixture of this four-gluon component in the two-gluon states. This is because if we form a wave function of two pairs, and let them interact in an S wave at small separation, they will most of the time find themselves in a state which is much higher in energy than the one where they are separated. The reason for this is that the probability of finding a spin-color-singlet pair in the four-gluon state with the maximum attraction $\Omega=-16$ has been calculated to be only ~ 0.21. This is the overlap of the two-gluon-pair state with the most attractive of the color-spin-singlet four-gluon states mentioned in the previous section. In all the other states, the effective interaction between the two pairs is very repulsive at short distances. Of course, we cannot from this kind of naive static consideration conclude too much about the obviously very complicated dynamics of the glueball "liquid." Consequently it is an assumption that in a global state densely filled with glueballs, there will be an effective strong repulsion between them to prevent any sizable overlap. Another way to express this is to say that the Bose condensate of glueballs is of liquid-helium type. The above discussion has, it is hoped, made this assumption somewhat plausible.

A condensate of the type just described will not break any symmetry since it carries vacuum quantum numbers.[19] This is consistent with the belief that color is not spontaneously broken in QCD. (Conversely, any scheme based on condensation in color-nonsinglet channels must find a mechanism to effectively restore the color symmetry. One mechanism of this kind proposed in the context of the "Copenhagen vacuum" is the formation of random colored domains.[7])

Presumably, the presence of the glueball condensate will strongly modify the effective long-distance interaction between colored particles. The hope of course is that it will give rise to confine-

ment; that is, all excited states are to be color singlets. In the absence of quarks this means that the only excitations are physical glueballs. Furthermore, since there experimentally are no light glueballs, the spectrum must show a mass gap.

At present we cannot prove that colored excitations are absent in our vacuum, although if we believe that the two-gluon 0^{++} state is the energetically favored one, it is natural to think that addition of a free (i.e., plane-wave) gluon will tend to "destroy" this 0^{++} color-singlet configuration over a large region, and thus be pushed high up in energy. The way to prove this would be to construct an explicit trial wave functional and calculate the expectation value of a large Wilson loop. An area law (i.e., confinement) is expected if the vacuum is disordered, i.e., if correlation functions fall exponentially at large distances. This could very well happen since, because of the glueballs, our model has a domain structure, which is usually a signal of disorder.

Now we shall argue that there are no low-lying excitations connected to the presence of the condensate. The similarity between the glueball condensate and liquid helium II has already been pointed out. We can then by analogy use an argument due to Feynman to show that the only possible low-lying excitations of the condensate are "phonons" or density waves.[20] The presence of phonons in liquid helium is, however, related to the incompressibility of the liquid, or equivalently, to the conservation of the number of particles. The glueball vacuum has no such conservation law. The glueball density is set by the scale of the instability and remains constant. Any attempt to "push" the condensate, e.g., with an electric field, will only modify the state locally. Hence we do not expect any low-lying excitations of the condensate at all.

So far we have completely ignored the quarks, assuming that they can be added perturbatively. If this is done, what will happen? Because of the quark-gluon coupling the glueballs in the condensate will acquire a $\bar{q}q$ component,[21] and furthermore the whole Dirac sea will be changed due to the presence of the condensate.[22] Both these effects are calculable and will give a value for the chiral-breaking parameter $\langle \bar{q}q \rangle$ which is related to the pion decay constant f_π.

G. Phenomenological consequences

Without working out any details, we shall now mention several consequences of having a QCD vacuum of the type proposed here. At this stage, we cannot directly compute, e.g., the excitation spectrum but on a less ambitious level we can ask what our model for the QCD vacuum can tell us about existing hadron phenomenology.

First we discuss the relevance for bag-model calculations. Except for the obviously appealing possibility of calculating the bag constant in terms of say $\Lambda_{\overline{MS}}$, one can get a better qualitative understanding of several features of previous calculations. If we picture a hadron as an excited state of one of the 0^{++} glueballs making up the vacuum condensate, it is natural that a description of hadrons purely in terms of valence quarks (or gluons) works very well. The reason is that there is a large overlap between the vacuum and a "hadron" state where the valence particles have been removed (by e.g., a weak or electromagnetic decay). Another set of predictions concerns the (physical) glueball spectrum. In the original bag model, spherical glueballs were not stable, since a vector field can never give rise to a spherically symmetric pressure. In our picture, however, the approximately spherical shape of low-lying hadrons is "built" into the vacuum so spherical (or almost spherical) glueballs might very well be allowed.[10]

Another type of possible phenomenological application of our model would be in the estimate of different condensate densities associated with the vacuum itself. One such quantity has already been mentioned, namely, $\langle \bar{q}(\vec{x})q(\vec{x}) \rangle$ which is directly related to observables. Another is $\langle G_{\mu\nu}(\vec{x})G^{\mu\nu}(\vec{x}) \rangle$ which will be discussed in Sec. III. This and other expectation values of more complicated local operators characterizing the QCD vacuum, have been used by Shifman *et al.* to calculate masses and widths of hadrons using dispersion relations and moment sum rules.[23] If we manage to construct a reasonable trial wave functional along the lines of Sec. II B we could clearly use that to calculate the various expectation values that go into the approach of Shifman *et al.* (A very rough estimate of $\langle G_{\mu\nu}G^{\mu\nu} \rangle$ is given at the end of Sec. III.)

III. A MEAN-FIELD MODEL FOR THE QCD VACUUM

We shall now use a simple model to illustrate the ideas about the QCD vacuum described in the earlier sections. Since the glueballs have a size R, set by the coupling strength, their self-interaction should be softened by form-factor effects. This

will suppress loop diagrams and suggests a classical (or tree-diagram) treatment of an effective theory of interacting glueballs.

Instead of attempting a microscopic derivation of such a theory we shall simply postulate that the QCD ground state is a dense gas or liquid of interacting glueballs, and try to describe this "condensate" with conventional mean-field methods.[24] The analogy with superconductivity or liquid helium is obvious and we shall use a relativistic version of the usual Ginzburg-Landau Lagrangian, i.e., a scalar field with self-coupling,

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-\frac{m^2}{2}\phi^2-\frac{\lambda}{4}\phi^4 , \tag{44}$$

where ϕ represents the scalar glueball condensate. The form of the interaction term is of course quite arbitrary, but according to the discussion in the previous sections it has to be short range and repulsive at small distances. The form suggested above is the simplest one having these properties. For small values of ϕ Eq. (44) describes a noninteracting dilute gas of particles of mass m, so we identify the parameter m^2 with the bag-model value for 0^{++} glueballs given by Eq. (36). Since m^2 becomes negative for large enough R, we get a vacuum expectation value ϕ_0 by minimizing the classical potential with respect to ϕ. This gives

$$\phi_0=\left[\frac{-m^2(R)}{\lambda}\right]^{1/2} . \tag{45}$$

Connected with this "condensate" is a negative vacuum energy density

$$E/V=\tfrac{1}{4}m^2(R)\phi_0^2 , \tag{46}$$

or, expressed in terms of the bag constant B,

$$B=-\tfrac{1}{4}m^2(R)\phi_0^2>0 . \tag{47}$$

To actually compute B we need to know $\alpha_s(R)$ and λ. For α_s we shall use the parametrization

$$\alpha_s(R)=\frac{2\pi n}{9}\left[\ln\left[1+\frac{1}{(\Lambda R)^n}\right]\right]^{-1} . \tag{48}$$

This form is consistent with asymptotic freedom for small R and becomes large for large R. The parameter n reflects our lack of knowledge about the large-R behavior of α_s, and clearly other parametrizations consistent with asymptotic freedom could be given. Qualitatively, however, our results should not be too dependent on the parametrization since we believe the model only if the vacuum stabilizes at an R where $\Lambda R\simeq 1$. Below we show that a change of n does change the parameters but not the nature of our results.

Rather than try to estimate the strength of the interaction λ and use Eq. (45) to calculate ϕ_0, we shall use as input that the vacuum is densely filled by the glueballs to get ϕ_0 directly. To do this we take the classical field to be the expectation value of the field operator in a coherent state of zero momentum. In such a state, the number density ρ is[25]

$$\rho=\frac{m}{2}\phi_0^2 . \tag{49}$$

This is strictly true for a free field, but we shall use this relation also in the interacting case. The requirement that the vacuum is dense gives

$$\rho=\frac{1}{V} , \tag{50}$$

TABLE II. Results for the glueball radius R in GeV^{-1}, the bag constant $B^{1/4}$ in MeV, the effective coupling $\alpha_s(\Lambda R)$, and the quantity $\tilde{G}$ in GeV^4 for different values of the parameters Λ and n.

$\Lambda=150$ MeV	$\Lambda=200$ MeV	$\Lambda=300$ MeV
	$n=1$	
$R=7.7$	$R=5.8$	$R=3.9$
$B^{1/4}=80$	$B^{1/4}=105$	$B^{1/4}=160$
$\alpha_s=1.1$	$\alpha_s=1.1$	$\alpha_s=1.1$
$\tilde{G}=0.0005$	$\tilde{G}=0.002$	$\tilde{G}=0.007$
	$n=2$	
$R=4.5$	$R=3.3$	$R=2.2$
$B^{1/4}=145$	$B^{1/4}=190$	$B^{1/4}=290$
$\alpha_s=1.2$	$\alpha_s=1.2$	$\alpha_s=1.2$
$\tilde{G}=0.004$	$\tilde{G}=0.01$	$\tilde{G}=0.08$

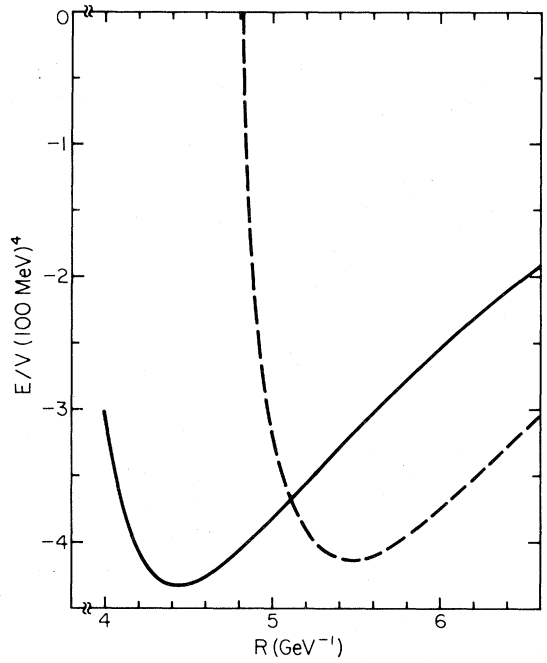

FIG. 2. Energy density as a function of the glueball radius R, for $\Lambda=150$ MeV and $n=2$. The solid and dashed curves correspond to two- and four-gluon glueballs, respectively.

where $V=4\pi R^3/3$ is the volume of one glueball. Combining Eqs. (46), (49), and (50) gives the vacuum energy density as a function of R and $m^2(R)$ only,

$$\frac{E}{V}=-\frac{3(-m^2)^{1/2}}{8\pi R^3} . \tag{51}$$

We can now determine both the bag constant and the radius of the glueballs in the condensate by minimizing this expression with respect to the radius R. Figure 2 shows the typical dependence of E/V on R, and in Table II we have compiled the results for R, B, and α_s for various values of Λ and the parameter n in Eq. (48).

From the table we see that although the values of R, B, and α_s are quite dependent on the details of the parametrization, we indeed have solutions with small $\Lambda\simeq$ few 100 MeV, $R\simeq 3-4$ GeV, and reasonably small $\alpha_s(R)$.[26] Since we do not know the relation between our Λ and, e.g., $\Lambda_{\overline{\rm MS}}$, nothing conclusive can be said about the numerical results. It is, however, satisfying that the glueballs get small (i.e., $<$ size of hadron) using a relatively small scale Λ and small coupling $\alpha_s(R)$.

The dashed curve in Fig. 2 shows E/V as a function of R for the case of the lowest-energy four-gluon 0^{++} glueball condensate. As mentioned in the last section, the minimum is slightly above the one found in the two-gluon case using the same parameters.

In this model we can also make a crude estimate of the renormalization-group-invariant quantity $\tilde{G}$ defined by

$$\tilde{G}=\alpha_s\langle 0\,|\,G_{\mu\nu}G^{\mu\nu}\,|\,0\rangle=2\alpha_s\langle 0\,|\,\vec{\mathrm{B}}^2-\vec{\mathrm{E}}^2\,|\,0\rangle . \tag{52}$$

It is natural to estimate this density by first computing the expectation value of the operator $\alpha_s G^2$ in one glueball, and then average over its volume, i.e.,

$$\tilde{G}\simeq\frac{3}{4\pi R^3}\alpha_s(\Lambda R)\langle 0^{++}\,|\,G^2\,|\,0^{++}\rangle , \tag{53}$$

where R is the radius of the glueballs in the condensate. To zeroth order in α_s, $\langle 0^{++}\,|\,G^2\,|\,0^{++}\rangle=0$ due to the $\vec{\mathrm{E}}$-$\vec{\mathrm{B}}$ symmetry. To $O(\alpha_s)$ this symmetry is broken by the non-Abelian interactions, and we can relate $\tilde{G}$ to the energy shift (see Appendix B),

$$\tilde{G}=-\frac{3\alpha_s}{4\pi R^3}4\Delta E\simeq\frac{1.5\alpha_s(\Lambda R)^2}{R^4} , \tag{54}$$

where we neglected self-energy terms. Note that since in our model $\alpha_s(\Lambda R)$ is almost independent of Λ and R, the quantity $\tilde{G}$ is a very sensitive measure of the size of the glueballs. The values of G listed in Table II range from 0.0005 to 0.09 and are to be compared to the phenomenological value $\tilde{G}\simeq 0.04$ extracted from QCD sum rules.[23] Although, as above, nothing conclusive can be said about the numerics, it is pleasing to note that the phenomenological value is within the range of our estimates, and corresponds to a small glueball radius.

In this context, we shall also comment on a proposed mean-field-theory (or phenomenological Lagrangian) model based on treating $G^2(\vec{x})$ as a local composite scalar glueball operator in some approximation.[27] A local description may be justified if the "size" of the glueball is small in comparison to the density in the condensed phase. We imagine that the size is comparable to the spacing in which case using a local mean-field theory may not be possible. Our treatment is based upon a dynamical picture in which locally we can discuss pure QCD in terms of a well-defined number of "constituent" gluons in the same way that the conventional hadrons are characterized as being composed of a

well-defined number of "constituent" quarks. In this picture the "structure" of the composite particles which condense is of crucial importance in obtaining a value for the energy density in the condensed phase.

In conclusion, we have shown that a simple model with reasonable values for the coupling strength shows many of the features we expect from the QCD vacuum. One might even hope that eventually there will be a microscopic derivation from QCD of such an effective theory, much in the same way as the Ginzburg-Landau model of superconductivity is derived from the BCS theory.

IV. SUMMARY AND OUTLOOK

In this paper we have argued that the magnetic instability of the perturbative QCD vacuum makes it unstable against creation of 0^{++} color-singlet glueballs. The glueballs will then form a Bose condensate with negative-energy density. In a qualitative discussion we have tried to show that such a condensate might have many of the properties expected from the QCD ground state, and using a mean-field model we obtained reasonable numerical relations between the energy density, the size of the glueballs and the QCD scale Λ.

Obviously much remains to be done. The most promising way to proceed is presumably to construct trial wave functionals along the lines described in Sec. II B. Concerning the cruder approximations discussed in this work, it is of importance to calculate the self-energies of confined gluons and quarks and to understand how $\alpha_s(\Lambda R)$ runs at the one-loop level.

Once we get a good model for the vacuum a number of interesting possibilities arise, such as understanding the low-lying excitations (the hadron spectrum) and the finite-T behavior.

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APPENDIX A

Here we shall give an explicit recipe for the construction of a complete set of potentials $A_i(x)$ which satisfy the flux constraint Eq. (18). For simplicity, we shall do this for a square lattice in two dimensions. The potentials A_i can be separated into mutually orthogonal and longitudinal parts. The flux constraint only applies to the longitudinal potentials which we write as

$$A^{\alpha}_{i,\vec{q}} = -\partial_i[\phi^{\alpha}_{\vec{q}}(\vec{x})] , \tag{A1}$$

where the scalar functions $\phi^{\alpha}_{\vec{q}}$ obey the Bloch condition $\phi^{\alpha}_{\vec{q}}(\vec{x}+\vec{x}_S)=\exp(i\vec{q}\cdot\vec{x}_S)\phi^{\alpha}_{\vec{q}}(\vec{x})$, and the orthonormality requirement

$$\int_{\text{cell}} d^2x \phi^{\alpha}_{\vec{q}}(\vec{x})^*[-\nabla^2\phi^{\alpha'}_{\vec{q}}(\vec{x})]=\delta^{\alpha\alpha'} . \tag{A2}$$

It is convenient to use the Bloch functions for the present construction. Since, however, the flux constraint is linear, the corresponding Wannier functions can be obtained immediately using Eq. (14).

The conditions Eqs. (A1) and (A2) are most generally met by the functions

$$\phi^{\alpha}_{\vec{q}}(\vec{x})=e^{i\vec{q}\cdot\vec{x}}\sum_{n,m}\frac{(\alpha\mid U\mid nm)\exp\left[i2\pi\left[n\frac{x}{a}+m\frac{Y}{a}\right]\right]}{\left[\left[q_x+\frac{2n\pi}{a}\right]^2+\left[q_y+\frac{2m\pi}{a}\right]^2\right]^{1/2}} , \tag{A3}$$

where $n,m=0,\pm1,\pm2,\ldots,$ a is the lattice spacing, and $(\alpha \mid U \mid nm)$ an aribtrary unitary matrix,

$$\sum_{nm}(\alpha \mid U \mid nm)(\alpha' \mid U \mid nm)^*=\delta_{\alpha\alpha'} . \tag{A4}$$

The potentials $A^{\alpha}_{i,\vec{q}}$ constructed from these $\phi^{\alpha}_{\vec{q}}$ will be continuous if $(\alpha \mid U \mid nm)$ falls at least like $O(1/n^2m^2)$ for $n,m,\to\infty$. The additional constraint imposed by the flux condition

$$\int_{\text{cell}} d^2x\,\partial^i\vec{A}^{\alpha}_{i,\vec{q}}=0 \tag{A5}$$

then takes the form

$$\sum_{nm}(\alpha \mid U \mid nm)\frac{\left[\left[q_x+\frac{2n\pi}{a}\right]^2+\left[q_y+\frac{2m\pi}{a}\right]^2\right]^{1/2}}{\left[q_x+\frac{2n\pi}{a}\right]\left[q_y+\frac{2m\pi}{a}\right]}=0 . \tag{A6}$$

Because the "vector"

$$\frac{\left[\left[q_x+\frac{2n\pi}{a}\right]^2+\left[q_y+\frac{2m\pi}{a}\right]^2\right]^{1/2}}{\left[q_x+\frac{2n\pi}{a}\right]\left[q_y+\frac{2m\pi}{a}\right]}$$

is non-normalizable, the condition Eq. (A6) is not necessarily in conflict with the unitarity (or completeness) requirement Eq. (A4). However, $(\alpha \mid U \mid nm)$ certainly must be such that the sum in Eq. (A6) does not converge uniformly in the row labeled α. [Otherwise we could multiply Eq. (A6) on the left by $(\alpha \mid U \mid n'm')^*$ sum on α and then interchange the α and n,m sums. This would produce a contradiction.] It is not difficult to give a simple example of $(\alpha \mid U \mid nm)$ which satisfies the requirements Eqs. (A4) and (A6). The corresponding potentials are however not differentiable across the cell walls, and they give Wannier functions which only fall off like a power. We do not have an explicit example which meets all requirements.

APPENDIX B

In this appendix we shall make an estimate of the self-energy diagrams [Fig. 1(c)] which is rather different from the one in the text. We note that for the true vacuum state Lorentz invariance alone requires

$$\langle 0 \mid G_{\mu\nu}G_{\sigma\lambda} \mid 0\rangle=\text{const}(g_{\mu\sigma}g_{\nu\lambda}-g_{\mu\lambda}g_{\nu\sigma}) , \tag{B1}$$

which immediately gives

$$\langle 0 \mid B_i^2 \mid 0\rangle=-\langle 0 \mid E_i^2 \mid 0\rangle=\text{const} . \tag{B2}$$

Now remember that the electric and magnetic energy shifts in any state are[3]

$$\Delta E_{\text{el}}=\tfrac{1}{2}\langle E^2\rangle , \tag{B3a}$$

$$\Delta E_{\text{mag}}=-\tfrac{1}{2}\langle B^2\rangle . \tag{B3b}$$

So for the vacuum we expect

$$\Delta E^{\text{vac}}=\Delta E^{\text{vac}}_{\text{el}}+\Delta E^{\text{vac}}_{\text{mag}}=2\Delta E^{\text{vac}}_{\text{mag}} . \tag{B4}$$

Since the interaction diagrams in Figs. 1(a) and 1(b) give a large magnetic and small electric contribution, one might conclude from Eq. (B4) that there is an electric contribution of the same size and sign in the self-energy. Thus one would estimate the total energy shift for the 0^{++} color-singlet state to be twice the one in the text, i.e.,

$$\Delta E_{0^{++}}=-\frac{3.2\alpha_s}{R} . \tag{B5}$$

Using this estimate will change nothing of the discussion in the text. The only difference is that the simple mean-field model in Sec. III allows for smaller values of $\Lambda\sim100-150$ MeV, corresponding to a change in the effective coupling $\alpha(R)$. For the spectrum of physical glueballs, no relation like (B4) holds and the spectrum depends crucially on the estimate of the self-energy terms.

One obvious objection against applying Eq. (B4) to our vacuum state is that it is in fact not Lorentz invariant. This criticism is certainly valid, but it is interesting to note that in the case of the zero point energy for a sphere, an explicit calculation shows that for a large region $\langle E^2-B^2\rangle\simeq0$ as expected from Lorentz invariance.[28] Since we have the same geometry a similar "effective restoration" of Lorentz invariance might take place.

APPENDIX C

There are four possible color-spin configurations for four gluons which are symmetric in spin and color. To use a compact notation we shall represent the spin wave function of a gluon by a traceless Hermitian 2×2 matrix, S. The color wave function will be a traceless Hermitian $n\times n$ matrix [in $\text{SU}(n)^C$], we may then form the symmetric configurations

$$\alpha=[\mathrm{Tr}(S_1S_2)\mathrm{Tr}(S_3S_4)+\mathrm{Tr}(S_1S_3)\mathrm{Tr}(S_2S_4)+\mathrm{Tr}(S_1S_4)\mathrm{Tr}(S_2S_3)]$$
$$\times[\mathrm{Tr}(C_1C_2)\mathrm{Tr}(C_3C_4)+\mathrm{perms}]\ , \tag{C1a}$$
$$\beta=[\mathrm{Tr}(S_1S_2)\mathrm{Tr}(S_3S_4)+\mathrm{perms}][\mathrm{Tr}(C_1C_2C_3C_4)+\mathrm{perms}]\ , \tag{C1b}$$
$$\gamma=[2\,\mathrm{Tr}(S_1S_2)\mathrm{Tr}(S_3S_4)-\mathrm{Tr}(S_1S_3)\mathrm{Tr}(S_2S_4)-\mathrm{Tr}(S_1S_4)\mathrm{Tr}(S_2S_3)]$$
$$\times[2\,\mathrm{Tr}(C_1C_2)\mathrm{Tr}(C_3C_4)-\mathrm{Tr}(C_1C_2)\mathrm{Tr}(C_2C_4)-\mathrm{Tr}(C_1C_4)\mathrm{Tr}(C_2C_3)]+\mathrm{perms}\ , \tag{C1c}$$
$$\delta=[2\,\mathrm{Tr}(S_1S_2)\mathrm{Tr}(S_3S_4)-\mathrm{Tr}(S_1S_3)\mathrm{Tr}(S_2S_4)-\mathrm{Tr}(S_1S_4)\mathrm{Tr}(S_2S_3)]$$
$$\times[2\,\mathrm{Tr}(C_1C_2C_3C_4)+2\,\mathrm{Tr}(C_1C_2C_4C_3)-\mathrm{Tr}(C_1C_3C_2C_4)-\mathrm{Tr}(C_1C_3C_4C_2)$$
$$-\mathrm{Tr}(C_1C_4C_2C_3)-\mathrm{Tr}(C_4C_3C_2C_1)]+\mathrm{perms}\ . \tag{C1d}$$

For $n=2$ the set is over complete and the relatively trivial computation must be done separately. If $n\geq 3$ this set is complete *but not orthonormal.* In this space of $(\alpha,\ldots)$ the color-spin matrix $\Omega=(\vec{\Lambda}_1\cdot\vec{\Lambda}_2\vec{S}_1\cdot\vec{S}_2+\cdots)$ is a 4×4 matrix which by elementary means can be calculated:

$$\Omega=\begin{pmatrix} \frac{4n}{3} & 0 & \frac{10n}{9} & \frac{-5}{3} \\ 0 & \frac{4n}{3} & \frac{10}{3} & \frac{-5n}{9} \\ 8n & 0 & \frac{2n}{3} & -1 \\ -24 & -4n & 0 & \frac{5n}{3} \end{pmatrix}\ . \tag{C2}$$

The characteristic polynomial for Ω is then found to be

$$1600(n/3)^2[(n/3)^2-1]+(4n/3-\lambda)$$
$$\times\{(4n/3-\lambda)(2n/3-\lambda)(5n/3-\lambda)+\lambda[100(n/3)^2+40]-(40n/3)[11(n/3)^2-1]\}=0\ . \tag{C3}$$

For the case of interest, SU(3), where $n=3$, the roots of Eq. (C3) are $\lambda=15.97$, 4.000, 2.872, and -7.847. In the $n\to\infty$ limit the roots take the forms, $\lambda=4n$, $3n$, 0, and $-2n$.

If we write the (not normalized) eigenvectors in the form

$$V=\alpha_1\alpha+\alpha_2\beta+\alpha_3\gamma+\alpha_4\delta\ , \tag{C4}$$

then

$$\alpha_1=\frac{3}{20n}(\lambda-2n/3)(\lambda-4n/3)(\lambda-5n/3)$$
$$-n\lambda/3+2[(n/3)^2-1]\ , \tag{C5a}$$
$$\alpha_2=\lambda\ , \tag{C5b}$$
$$\alpha_3=\tfrac{1}{6}(\lambda-5n/3)(\lambda-4n/3)$$
$$-\tfrac{10}{3}[(n/3)^2-1]\ , \tag{C5c}$$
$$\alpha_4=-(\lambda/4n)(\lambda-4n/3)\ , \tag{C5d}$$

where λ is the appropriate eigenvalue of Ω.

Using the above expressions it is algebraically tedious but straightforward to calculate the probability p that a pair in the states V is a color singlet, spin singlet.

*Address after 1 February 1982: Center for Theoretical Physics, M.I.T., Cambridge, MA 02139.
†On leave from Center for Theoretical Physics, M.I.T., Cambridge, MA 02139.
‡On leave from NORDITA, Copenhagen, Denmark.

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[13]These values will be compared to $a=0.70$ and $b=0$ in the $\bar{q}q$ and qq cases.

[14]It is of interest to compare the value $\Omega=6$ in this case to $\Omega=-1$, and $-\frac{1}{24}$ obtained for a $\bar{q}q$ system in the configurations (0,1) and $(1,8_A)$, respectively. This may indicate that the vector bosons always condense before the fermions (this would indeed happen according to our bag-type estimates since the coefficient a is only $\sim$ a factor 2 larger in the $\bar{q}q$ case) which would have consequences for weak-interaction models (Ref. 15).

[15]L. F. Abbot and E. Farhi, Phys. Lett. 101B, 69 (1980).

[16]To get $\alpha_s=2.2$ one needs a phenomenological term $\sim -1.8/R$ in the Hamiltonian. Part of this term is due to center-of-mass motion, and the rest is attributed to self-energies and perhaps also to Casimir energy. Any inclusion of such a term will decrease our estimates of the critical α_s.

[17]The bag model predicts a light (but not tachyonic) 0^{-+} glueball [see C. Carlson, T. H. Hansson, and C. Peterson, Report No. MIT CTP 1020, 1982 (unpublished)]. We believe that this problem is intimately linked to the corresponding problem in the quark mass spectrum; that is the large mass of the (mostly flavor-singlet) η' meson. The η' is a serious problem for all realistic quark models and we have nothing very intelligent to say about it. Although we cannot provide any solution, it seems that it would be even more puzzling if the solution to the η' problem did not also help with the 0^{-+} glueball since these particles have the same quantum numbers.

[18]For simplicity, we have here omitted the s-channel annihilation diagram, which together with an extra contact term, contributes an additional energy shift $+0.53(\alpha_s/R)P$, where P is the projection operator on the color-octet spin-one channel. Since this makes the total shift positive in the channel, the estimates given in this section are too low. The 0^{++} two-gluon state will be even more favored when we include the extra repulsion in the octet spin-one channel. For example, for the four-gluon state, we can estimate the maximum effect of including this repulsion by computing the expectation values of Ω and P in the state formed by combining two gluon pairs (12) and (34) both in color- and spin-singlet channels, and then symmetrizing. In this state where $\langle\Omega\rangle=-12$ the probability of finding any pair in the octet, triplet channel is very small ($\langle P\rangle\simeq 0.051$) and we find $0.26\langle\Omega\rangle+0.53\times 6\langle P\rangle\simeq -3.12+0.16\simeq -2.96$.

[19]As always, when using a Hamiltonian formalism, care must be taken to maintain Lorentz invariance. In this context it is important to remember that both the boundary condition Eq. (22), and having $E/V=\mathrm{Tr}T_{\mu\nu}\neq 0$, is consistent with Lorentz invariance. Of course, any trial state based on glueballs at fixed positions in space, explicitly breaks the spatial symmetries. As mentioned in Sec. II B however, this might not introduce too large errors in estimate of, e.g., the vacuum energy.

[20]R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, Mass., 1972), and references therein. The essence of Feynman's argument is the observation that any change of state which does not involve a change in density can always be achieved by only short-distance (size of atoms) displacements of the atoms. This is possible since all the atoms are identical which means that long-distance exchanges unrelated to density changes are unobservable, and do not alter the state.

[21]The recipe for determining this component is given in C. Carlson and T. H. Hansson, Nucl. Phys. B199, 441 (1982), where the 0^{-+} $\bar{q}q$-glueball overlap is calculated.

[22]K. Johnson, in Proceedings of the International Conference on High Energy Physics and Nuclear Structure, IXth Versailles, 1980 (unpublished).

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Phys. B147, 385 (1979); B147, 448 (1979); Phys. Rev. Lett. 42, 297 (1979); Nucl. Phys. B191, 301 (1981); L. J. Reinders, H. R. Rubinstein, and S. Yazaki, *ibid.* B186, 109 (1981).

[24]Other models for the QCD vacuum based on effective Lagrangians can be found in the following references. J. Kogut and L. Susskind, Phys. Rev. D 9, 3501 (1974); W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, *ibid.* 11, 1094 (1975); R. Friedberg and T. D. Lee, *ibid.* 18, 2623 (1978).

[25]To understand this, note that the energy density of such a coherent state is $m\rho=m^2\phi^2/2=V(\phi)$ as expected for a free scalar theory.

[26]If we use the alternative way of estimating the self-energies of the gluons described in Appendix B, we typically decrease α_s by a factor 2 for given values of R and B. In this case again α_s stays constant but with a value $\alpha(\Lambda R)=0.55$, which is certainly more appealing from the point of view of perturbation theory (cf. the discussion at the end of Sec. II D).

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