Renormalization-prescription ambiguity in perturbative quantum chromodynamics: Has Stevenson found the solution?

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Stevenson has claimed to resolve the renormalization-prescription ambiguity inherent in a truncated perturbative expansion by a principle of minimal sensitivity. This is shown to define an α_s such that its coefficients in the perturbative expansion are in the ratio of the corresponding coefficients of the β function, regardless of the physical quantity calculated. Stevenson's principle is then only relevant when the expansion of the β function for the optimal α_s makes sense and is then negligibly different from the commonly assumed criterion of fastest apparent convergence, with no greater claim to "optimality."

I. INTRODUCTION

One of the ambiguities inherent in perturbative QCD is the question of what is the most sensible expansion parameter to use, an ambiguity that would be of little import if α_s were $O(10^{-2})$. However, at present momenta we are far from this happy situation. Yet we still wish to apply perturbation theory and hopefully obtain meaningful results. This ambiguity arises because we are free to remove the infinities in the theory in any way we choose.[1-3] Though we take care of the freedom to scale the momenta for our renormalization, through a simple subset[2-4] of the general renormalization-group equations,[1] we are still free to prescribe how much of the finite parts are left behind on taking the infinities away. This is the so-called renormalization-prescription ambiguity.[5-8] Clearly any physical quantity does not depend on the renormalization scheme in which we choose to work and so the choice of scheme is merely a matter of convenience, provided we can calculate to all orders in perturbation theory. However, in some schemes we may have to calculate many terms to obtain a good approximation, while in others just the first few terms may be sufficient. It is particularly important in perturbative QCD to find such schemes, because we have little hope of calculating beyond the first few terms in the QCD perturbative expansion.

In perturbative QED, the choice of scheme is of minor consequence, because, for any reasonable choice of expansion parameter, the first few terms in the series are always a good approximation. Rather remarkably, moreover we have a natural choice of renormalization prescription in QED—the physical or on-shell scheme. In this scheme, the low-energy limit of the differential cross section for Compton scattering is proportional to α^2 to all orders.[9] This α is just the fine-structure constant, i.e., $\frac{1}{137}$. We can, of course, use any other (running) coupling $\alpha(Q^2)$ in any scheme as discussed by Coquereaux.[10] In QCD, because the gluon, unlike the photon, is not physical on-shell, we have no such "physical" definition of the coupling.

So how do we attempt to find an optimal scheme in perturbative QCD? Since the expansion is likely neither convergent nor probably even an asymptotic series, this "optimality" may be only a temporary phenomenon applicable to the first few orders of perturbation theory. Nonetheless, it is important for applications of perturbative QCD to have some reasonable consistent criterion for deciding when we feel perturbation theory makes sense. In one school of thought it is believed we should ensure that the higher-order terms do not contribute more than the lowest order[11] by making a judicious choice of our expansion parameter. This idea we call the "fastest apparent convergence criterion" (FACC).

For example, consider the calculation of the ratio

$$\mathscr{R}\equiv\sigma(e^+e^-\to \text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)\ ,$$

then[12]

$$\mathscr{R}=(3\sum_i e_i^2)\left[1+\frac{\alpha(\text{MS})}{\pi}+5\left[\frac{\alpha(\text{MS})}{\pi}\right]^2+\cdots\right], \tag{1}$$

where α(MS) is defined in 't Hooft's minimal-subtraction scheme,[13] and the Q^2 dependence of $\mathscr{R}$

and α is implicit[14] (and the numerics are for five flavors). As seen, the leading-order correction to the parton-model result ($\mathscr{R}=3\sum e_i^2$) is not the dominant one until $\alpha(\mathrm{MS})$ is very small. In fact, the next-to-leading term is only less than a third of the leading-order correction if $\alpha(\mathrm{MS})<0.2$, i.e., $|Q^2/\Lambda_{\mathrm{MS}}{}^2|>400$. However, in contrast, if we expand in terms of α_s defined in the $\overline{\mathrm{MS}}$ scheme,[15] in which the arbitrary $(\ln 4\pi-\gamma_E)$ factors of dimensional regularization have been removed, then

$$\mathscr{R}=(3\sum_i e_i^2)\left[1+\frac{\alpha(\overline{\mathrm{MS}})}{\pi}+1.4\left[\frac{\alpha(\overline{\mathrm{MS}})}{\pi}\right]^2+\cdots\right]. \quad (2)$$

The next-to-leading correction is then smaller than a third of the leading one if $\alpha(\overline{\mathrm{MS}})<0.7$, i.e., when $|Q^2/\Lambda_{\overline{\mathrm{MS}}}{}^2|>6$. Thus, according to FACC, $\alpha(\overline{\mathrm{MS}})$ is a much more satisfactory expansion parameter for this process, as is also α_s defined in the momentum subtraction scheme,[6] in which[12]

$$\mathscr{R}=(3\sum_i e_i^2)\left[1+\frac{\alpha(\mathrm{MOM})}{\pi}-0.9\left[\frac{\alpha(\mathrm{MOM})}{\pi}\right]^2+\cdots\right] \quad (3)$$

with similarly rapid apparent convergence.

Of course, although the FAC criterion tells us the expansion for $\mathscr{R}$ in terms of $\alpha(\overline{\mathrm{MS}})$ or $\alpha(\mathrm{MOM})$ is more sensible than $\alpha(\mathrm{MS})$, at least to this order, we do not know that Eq. (1), with $\alpha(\mathrm{MS})$, is not the whole story with all the still higher-order terms negligible, while in terms of $\alpha(\overline{\mathrm{MS}})$ or $\alpha(\mathrm{MOM})$ these would be very large. We obviously have to calculate these higher terms to be certain. Since this is currently impossible, we must either hope that fastest apparent convergence is meaningful, and this is just not supposed to happen, or think of a surer criterion. Stevenson[16] has proposed such a condition—the principle of minimal sensitivity (PMS), which he claims provides an optimal approximation. The aim of this paper is to test this claim and explore whether PMS really tells us anything different, or more believable, than FACC. In Sec. II, we detail what renormalization-prescription dependence of the perturbative expansion means. In Sec. III, we compare FACC and PMS, and in Sec. IV give our conclusions.

II. RENORMALIZATION-SCHEME DEPENDENCE

To illustrate how we shall treat the dependence of a truncated perturbative expansion on an unphysical parameter, let us first consider the well-known calculation of the running coupling $\alpha(Q^2,R)$ itself. In the leading-logarithm approximation, in any scheme,

$$\alpha(Q^2,R)=\alpha(Q_0{}^2,R)-\frac{\beta_0}{4\pi}\alpha(Q_0{}^2,R)\ln\frac{Q^2}{Q_0{}^2}+\left[\frac{\beta_0}{4\pi}\right]^2\alpha(Q_0{}^2,R)^2\ln^2\frac{Q^2}{Q_0{}^2}+\cdots, \quad (4)$$

where Q_0 is some arbitrary momentum at which we have chosen to fix the coupling to be $\alpha(Q_0{}^2,R)$. The renormalization-group equation resulting from a rescaling of Q^2 gives

$$4\pi\frac{\partial\alpha(Q^2,R)}{\partial\ln Q^2}=-\beta_0\alpha(Q^2,R)^2 \quad (5)$$

to leading order. This can be integrated, summing the series of Eq. (4), to give

$$\alpha(Q^2,R)=\frac{\alpha(Q_0{}^2,R)}{1+(\beta_0/4\pi)\alpha(Q_0{}^2,R)\ln(Q^2/Q_0{}^2)}. \quad (6)$$

It is, of course, essential to use this result, which is exact to all orders in the leading logarithm, if we want to study the behavior of $\alpha(Q^2,R)$ as $Q^2\to\infty$. However, in the neighborhood of some finite momentum Q_0, the approximation given by Eq. (4) is equally good, provided $\alpha(Q_0{}^2,R)\ln(Q^2/Q_0{}^2)\ll 1$. In fact, the approximate form Eq. (4) differs from the exact expression [Eq. (6)] by less than 1% for $\alpha(Q_0{}^2,R)\ln(Q^2/Q_0{}^2)<0.3$. By adding more and more terms to the expansion and including the higher β-function coefficients β_1,β_2, etc., we can obtain an expansion which is correct to yet higher orders. Although the use of a form like Eq. (4) appears to give some special role to the choice of Q_0, this is illusory. In any practical situation, we know the region of interest and Q_0 is just some convenient momentum in this range.

In a closely analogous manner, we study the scheme dependence by expanding $\alpha(Q^2,R)$, for example, about some scheme R_0, labeled by 0, as a

series in $\alpha(Q^2,0)$. Although any physical quantity is invariant to a change of R_0, it is convenient when studying small changes of scheme to expand about some suitable R_0. This will not only be equal to the exact, R_0-independent result to the order of perturbation theory we consider, but will also be negligibly different numerically, provided $\alpha(Q^2,0)\Delta R << 1$ [just as with Eqs. (4) and (6)].

In this spirit, consider some physical quantity Π, which we calculate perturbatively as a series in the parameter $\alpha(0)$ defined in any renormalization scheme R_0:

$$\Pi = \Pi_0+\alpha(0)\Pi_1(0)+\alpha(0)^2\Pi_2(0) +\alpha(0)^3\Pi_3(0)+\cdots . \tag{7}$$

Since it is with the scheme dependence that we are concerned, we ignore the kinematic variables on which Π, $\alpha(0)$, and the coefficients $\Pi_n(0)$ depend, as these will take care of themselves. Now if instead we choose to work in some other renormalization scheme labeled R, then

$$\Pi = \Pi_0+\alpha(R)\Pi_1(R)+\alpha(R)^2\Pi_2(R) +\alpha(R)^3\Pi_3(R)+\cdots . \tag{8}$$

Of course, the physical quantity Π does not depend on the scheme we choose, although the individual terms in the expansion do. Thus, as $\alpha(0)$ is changed to $\alpha(R)$, so the coefficients $\Pi_n(0)$ change appropriately to ensure the renormalization-group equation

$$\partial\Pi/\partial R = 0 \tag{9}$$

is satisfied for all schemes R.

How this happens is determined by how $\alpha(R)$ changes from scheme to scheme. R is the set of parameters $\{\rho_1,\rho_2,\dots\}$, which specify $\alpha(R)$ in terms of $\alpha(0)$:

$$\alpha(R)=\alpha(0)[\,1+\rho_1\alpha(0)+\rho_2\alpha(0)^2+\rho_3\alpha(0)^3+\cdots\,] . \tag{10}$$

The parameters ρ_i are in turn specified by the coefficients $\beta_i(R)$ of the β function, which define the running coupling[17] $\alpha(R)$ by

$$4\pi\frac{\partial\alpha(R)}{\partial \ln Q^2} \equiv \beta(\alpha(R)) = -\alpha(R)^2[\,\beta_0+\beta_1\alpha(R) +\beta_2(R)\alpha(R)^2+\cdots\,] , \tag{11}$$

where, most importantly, the coefficients β_0,β_1 are independent of the scheme and are given by[18,19]

$$\beta_0 = \left[11-\tfrac{2}{3}N_f\right] , \qquad \beta_1 = \left[102-\tfrac{38}{3}N_f\right]\Big/4\pi , \tag{12}$$

where N_f is the number of flavors. β_2 has only been calculated in the MS scheme[20] and is found to be even more strongly varying with N_f:

$$\beta_2(\mathrm{MS}) = \left[\tfrac{2857}{2}-\tfrac{5033}{18}N_f+\tfrac{325}{54}N_f^2\right]\Big/(4\pi)^2 .$$

It is straightforward to show, by substituting Eq. (10) into Eq. (11) and noting that $\alpha(0)$ obviously satisfies Eq. (11) with coefficients $\beta_i(0)$, that

$$\rho_2=\rho_1^2+\frac{\beta_1}{\beta_0}\rho_1+\frac{\beta_2(R)-\beta_2(0)}{\beta_0} , \tag{13}$$

$$\rho_3=\rho_1\rho_2+\frac{3\beta_1}{2\beta_0}\rho_1^2+\frac{2\beta_2(R)-\beta_2(0)}{\beta_0}\rho_1 +\frac{\beta_3(R)-\beta_3(0)}{2\beta_0} , \tag{14}$$

$$\rho_4=\tfrac{2}{3}\rho_1\rho_3-\frac{\beta_1}{3\beta_0}\rho_3+\tfrac{1}{3}\rho_2^2+2\frac{\beta_1}{\beta_0}\rho_1\rho_2 +\frac{4\beta_2(R)-3\beta_2(0)}{3\beta_0}\rho_2+\frac{\beta_1}{3\beta_0}\rho_1^3+2\frac{\beta_2(R)}{\beta_0}\rho_1^2 +\frac{5\beta_3(R)-2\beta_3(0)}{3\beta_0}\rho_1+\frac{\beta_4(R)-\beta_4(0)}{3\beta_0} . \tag{15}$$

Incidentally, the subset of scheme changes in which the coefficients of the β function do not vary can always be absorbed[6] into a scaling of the momentum argument of α [Eq. (11)] $Q^2\to Q^2\exp(-4\pi\rho_1/\beta_0)$ with the scheme fixed.

It is equally simple to show that the scheme independence of the physical quantity Π means

$$\Pi_1(R)=\Pi_1(0)\equiv\Pi_1 , \tag{16}$$

$$\Pi_2(R)=\Pi_2(0)-\rho_1\Pi_1 , \tag{17}$$

$$\Pi_3(R)=\Pi_3(0)-2\rho_1\Pi_2(0)-(\rho_2-2\rho_1^2)\Pi_1 , \tag{18}$$

$$\Pi_4(R)=\Pi_4(0)-3\rho_1\Pi_3(0)-(2\rho_2-5\rho_1^2)\Pi_2(0) -(\rho_3-5\rho_1\rho_2+5\rho_1^3)\Pi_1 , \tag{19}$$

etc.

In practice, when we compare any expression [Eq. (7)] with data, we truncate the expansion at

some order and then compare. In doing this we have explicitly set the higher-order terms to zero and implicitly fixed the renormalization scheme to be the one that minimizes the effect of higher orders.[21] This is not an arbitrary scheme, but is totally fixed by the approximation made. This is most easily described by illustration.

In Fig. 1, we show successive approximations to a quantity Π, given by

$$\Pi^{(N)}(R)=\sum_{n=0}^{N}\Pi_n(R)\alpha^{(N)}(R)^n \tag{20}$$

and labeled by N. The approximation $\alpha^{(N)}(R)$ satisfies Eq. (11) with the β function having coefficients β_N and beyond zero. As an example, we assume Π (Ref. 22) to be given by

$$\Pi=1+\frac{\alpha_0}{\pi}+1.5\left[\frac{\alpha_0}{\pi}\right]^2 -1.5\left[\frac{\alpha_0}{\pi}\right]^3+1.5\left[\frac{\alpha_0}{\pi}\right]^4 \tag{21}$$

with $\alpha_0=\frac{1}{3}$. To allow just a two-dimensional plot to be drawn, we restrict attention to the class of schemes in which the β functions [Eq. (11)] are identical to $O(\alpha^6)$, so the set R depends only on $\rho(\equiv\rho_1)$, a single parameter measuring the change from the reference scheme in which $\alpha_0=\frac{1}{3}$ and Eq. (21) holds.

From Fig. 1, we see that as N increases, the successive approximations $\Pi^{(N)}(\rho)$ become more weakly dependent on ρ, for a bigger and bigger range of ρ, just as they should. Indeed, when $N\to\infty$, $\Pi^{(N)}$ will be independent of ρ, Eq. (9). Of course, we do not actually know the value of $\Pi=\lim_{N\to\infty}\Pi^{(N)}$, which we call $\Pi_{\rm phys}$, but we can guess from Fig. 1 that it must be around 1.12.

When we approximate the data by a truncated series, e.g., $\Pi^{(1)}(R)$, we explicitly fix the scheme to be the one for which $\Pi^{(1)}(R)=\Pi_{\rm phys}$ (see Fig. 1). *A priori,* we have no idea which scheme this is, though we do know it is the scheme that minimizes the effect of higher orders. Moreover, we know such a scheme must exist because provided $\Pi_{\rm phys}$ is not far from the zeroth-order result (of the parton model, for example)—here 1, $\Pi^{(1)}$ takes all values in the neighborhood of 1. We can gain some idea of which scheme this is by asking which prescription has the smallest next-to-leading corrections, i.e., the scheme for which $\Pi^{(1)}(R)\simeq\Pi^{(2)}(R)$. In Fig. 1, this is arrowed as "fastest convergence $(\mathrm{FC})_1$." This is not far from the reference scheme ($\rho\equiv0$) as expected from Eq. (21).

The fastest apparent convergence criterion, of course, always makes the leading-order QCD correction the dominant one, and so it may appear at first sight that higher-order calculations are irrelevant. However, this is not the case, the higher-order contributions must be calculated to know in which scheme, i.e., for which choice of expansion parameter α_s, the leading-order term is dominant. This is essential if we are to make testable predictions for different processes.

While in deep-inelastic scattering, whether on a nucleon[23,15,24] or a photon[25] or in e^+e^- annihilation,[12] α(MOM) does seem most satisfactory for the FACC, this is not always so.[26] For example, in quarkonium decays, the higher-order corrections, though smaller in momentum subtraction, are still large. In fact, in the minimal-subtraction scheme,[27,28]

$$\frac{\Gamma(q\bar{q}\to gg)}{\Gamma(q\bar{q}\to\gamma\gamma)}=\frac{2}{9e_q{}^4}\left[\frac{\alpha(\mathrm{MS})}{\alpha_{\rm QED}}\right]^2\times\left[1+22\frac{\alpha(\mathrm{MS})}{\pi}+\cdots\right] \tag{22}$$

and the next-to-leading-order term changes the leading-order result by a factor of 3. According to

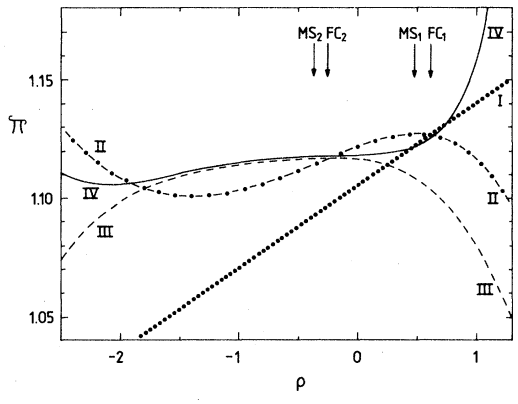

FIG. 1. Successive approximants $\Pi^{(N)}$ [Eq. (20)] labeled by $N=\mathrm{I},\ldots,\mathrm{IV}$, are shown as a function of the scheme parameter ρ. When $\rho=0$, the $\Pi^{(N)}$ are given by Eq. (21). As N increases, note that the $\Pi^{(N)}$ become less dependent on ρ. $(\mathrm{FC})_1$ marks where the $N=\mathrm{I,II}$ approximants intersect and $(\mathrm{FC})_2$ the nearby intersection of the $N=\mathrm{II,III}$ approximants. $(\mathrm{MS})_{1,2}$ are the neighboring turning points of the $N=\mathrm{II,III}$ approximants, respectively, The $(\mathrm{FC})_i$, $(\mathrm{MS})_i$ scheme are picked out by the fastest-convergence and minimal-sensitivity conditions, respectively.

the FACC, this is a quite unsatisfactory application of perturbative QCD with α_s defined in the standard schemes,[6,7,13,29] as much discussed.[7,8]

Of course, this need not be the case. It could well be argued that the expression Eq. (22) with $\alpha(\overline{\text{MS}})$ could give the whole answer with no further higher-order corrections, while in other schemes with smaller next-to-leading terms the still higher orders may be very large. It is clear we have no way of checking that this is incorrect. However, we can ask whether there is some other "reasonable" criterion which may make this possible.

Halliday, Suranyi, and Caswell[30] have studied the problem of perturbative solutions to the anharmonic oscillator and shown that one can obtain a good approximation to a solution Π by expanding perturbatively in terms of a parameter itself dependent on an unphysical parameter (R) and then fixing this unphysical parameter by the requirement that $\partial\Pi/\partial R=0$, to the order to which one works. Stevenson[16] has proposed this same idea in perturbative QCD to fix the renormalization scheme "optimally" for a given process to a given order. This he calls the "principle of minimal sensitivity" (PMS), with its eponymous acronym. The justification that this is sensible, beyond noting that it works in the numerically checkable and explicitly convergent example of Halliday and Suranyi,[30] is the belief that when one has made an optimal choice of scheme, at some order, the result should by only minimally sensitive to changes about this optimal scheme. Thus, there the approximation $\Pi^{(N)}(R)$ [Eq. (20)] should have a turning point, and so

$$\left.\frac{\partial\Pi^{(N)}}{\partial R}\right|_{R=\text{PMS}}=0\ . \tag{23}$$

This result is, of course, true, when $N\to\infty$, for all schemes R as a general consequence of renormalization-group invariance [Eq. (9)].[1]

In contrast, what is meant by fastest apparent convergence is not so precise. The equality of successive approximations $\Pi^{(N)}=\Pi^{(N+1)}$ is really only meant in an approximate way. Rather, we mean by FACC

$$\left|\frac{\alpha_s\Pi_{N+1}}{\Pi_N}\right|\simeq\left|\frac{\Pi^{(N+1)}-\Pi^{(N)}}{\Pi^{(N)}-\Pi^{(N-1)}}\right|<\epsilon<<1 \tag{24}$$

for some range of ϵ. But how different is this from the requirement of minimal sensitivity with its clear mathematical definition?

To gain some feeling, let us consider our example of Eqs. (20) and (21) shown in Fig. 1. At second order, FACC allows $0\lesssim\rho\lesssim 1$, where $\Pi^{(2)}=\Pi^{(1)}$ fixes $\rho=0.6$ (labeled FC_1), while PMS requires $\partial\Pi^{(2)}/\partial\rho=0$, which occurs at $\rho\sim 0.5$ (arrowed MS_1) defining a scheme rather close by with almost the same value of Π. A similar result occurs at the next order too (see FC_2 and MS_2 in Fig. 1), again with very similar values for Π.[31] Let us ask how general this is.

III. PMS AND FACC: IS THERE A DIFFERENCE?

Let us consider the simplest example

$$\Pi^{(2)}(R)=\Pi_0+\alpha(R)\Pi_1+\alpha(R)^2\Pi_2(R)\ , \tag{25}$$

where $\alpha(R)\equiv\alpha^{(2)}(R)$. To this order, where

$$\alpha(R)-\alpha(0)=\rho_1\alpha(0)^2+\rho_2\alpha(0)^3+\cdots\ , \tag{26}$$

α_s has the β function [Eq. (11)]

$$\beta=\beta_0\alpha_s{}^2+\beta_1\alpha_s{}^3$$

and so only depends on ρ_1 [see Eqs. (13)–(15)]. The Stevenson principle then requires ρ_1 to be fixed by $\partial\Pi^{(2)}(\rho_1)/\partial\rho_1=0$, i.e.,

$$\frac{\partial\alpha(\rho_1)}{\partial\rho_1}\left[\Pi_1+2\alpha(\rho_1)\Pi_2(\rho_1)\right]+\alpha(\rho_1)^2\frac{\partial\Pi_2(\rho_1)}{\partial\rho_1}=0\ . \tag{27}$$

For small changes in scheme about R_0, it is convenient to treat this equation perturbatively, in terms of its Taylor series. Though Stevenson[16] insists that this assigns a special role to R_0, out of keeping with renormalization-group invariance,[1] this treatment will, as discussed at the beginning of Sec. II, nevertheless be equivalent to the exact result to the order we consider and furthermore will be negligibly different numerically to all orders, for the small scheme changes we study when $\alpha(0)<<1$. To ameliorate Stevenson's objection we may regard R_0 as his optimal scheme.[32] Equation (17) tells us $\partial\Pi_2/\partial\rho_1=-\Pi_1$, and Eqs. (26) and (13) give

$$\frac{\partial\alpha(\rho_1)}{\partial\rho_1}=\alpha(0)^2\left[1+\left[2\rho_1+\frac{\beta_1}{\beta_0}\right]\alpha(0)+\cdots\right]\ .$$

Substituting in Eq. (27) and working to $O(\alpha^3)$, we easily find

$$\Pi_2(\mathrm{PMS}) = -\frac{1}{2}\frac{\beta_1}{\beta_0}\Pi_1 . \tag{28}$$

Thus the form for Π in the "optimal" scheme is to this order

$$\Pi^{(2)} = \Pi_0 + \alpha(\mathrm{PMS})\Pi_1 - \frac{\beta_1}{2\beta_0}\alpha(\mathrm{PMS})^2\Pi_1 . \tag{29}$$

The coefficient of the next-to-leading-order contribution Π_2 is wholly determined by the leading order. Nevertheless, Π_2 has to be calculated in some (reference) scheme ($\rho_1 \equiv 0$) so that we know in which scheme this optimal form occurs. Thus from Eqs. (13), (17) and (28), we note

$$\alpha(\mathrm{PMS}) = \alpha(0) + \left[\frac{\Pi_2(0)}{\Pi_1} + \frac{\beta_1}{2\beta_0}\right]\alpha(0)^2 + \left[\left[\frac{\Pi_2(0)}{\Pi_1}\right]^2 + \frac{2\beta_1}{\beta_0}\frac{\Pi_2(0)}{\Pi_1} + \frac{3\beta_1^2}{4\beta_0^2}\right]\alpha(0)^3 + \cdots . \tag{30}$$

How does this differ from FACC? FACC would require

$$|\Pi_2(R)/\Pi_1| < 1 .$$

PMS fixes $\Pi_2(R)$, so that

$$\left|\frac{\Pi_2(\mathrm{PMS})}{\Pi_1}\right| = \frac{1}{2}\frac{\beta_1}{\beta_0} = \frac{(306-38N_F)}{8\pi(33-2N_f)} , \tag{31}$$

which equals 0.20 for $N_f = 5$, a value quite in keeping with fastest apparent convergence. This can be illustrated by applying PMS, Eq. (29), to the ratio $\mathscr{R}$ for e^+e^- annihilation, discussed in the Introduction. PMS fixes

$$\mathscr{R} = (3\sum_i e_i^2)\left[1 + \frac{\alpha(\mathrm{PMS})}{\pi} - \frac{\pi\beta_1}{2\beta_0}\left[\frac{\alpha(\mathrm{PMS})}{\pi}\right]^2\right] . \tag{32}$$

With the coefficient of $(\alpha/\pi)^2$ equal to -0.63 for five flavors very close to the value in Eq. (3), we see $\alpha(\mathrm{PMS})$ is almost that of the momentum subtraction scheme as FACC would also want.

We can generalize the result to this order by easily calculating that if

$$\Pi^{(2)}(R) = [\alpha(R)]^d[\Pi_1 + \alpha(R)\Pi_2(R)] ,$$

PMS fixes

$$\Pi_2(\mathrm{PMS}) = -\left[\frac{d}{d+1}\right]\frac{\beta_1}{\beta_0}\Pi_1 ,$$

of which Eq. (28) is the case with $d = 1$. In deep-inelastic scattering, where Π is a moment, each has a different power d and so PMS fixes a different "optimal" scheme for each. For the quarkonium decays we discussed above, for which $d = 2$, PMS does not allow the large terms of the minimal-subtraction scheme [Eq. (22)] anymore than FACC does. Indeed, it is even more restrictive, and, to this order [Eq. (22)] forces an optimal $\alpha(\mathrm{PMS})$ far from $\alpha(\mathrm{MS})$ or even $\alpha(\mathrm{MOM})$ [see the analog of Eq. (30)].

From Eq. (28), we see the Stevenson principle fixes Π_2/Π_1 in terms of the ratio of the corresponding coefficients in the β function, viz. β_1/β_0. This appears to be a quite general result, which can best be illustrated by considering as complicated an example as we are likely to meet in practice.[33] We work to order α_s^4 so that

$$\Pi^{(4)}(R) = \Pi_0 + \alpha(R)\Pi_1 + \alpha(R)^2\Pi_2(R) + \alpha(R)^3\Pi_3(R) + \alpha(R)^4\Pi_4(R) , \tag{33}$$

where

$$\alpha(R) \equiv \alpha^{(4)}(R) = \alpha^{(4)}(0)\left[1 + \rho_1\alpha^{(4)}(0) + \cdots + \rho_4\alpha^{(4)}(0)^4 + \cdots\right] \tag{34}$$

has the Callan-Symanzik function [Eq. (11)] with coefficients $\beta_i(R)$ $(i = 2,3)$ scheme dependent. PMS requires $\partial\Pi^{(4)}(R)/\partial R = 0$, so we have to choose how to specify R—there are several possibilities for independent variables, e.g., $R = \{\rho_1,\rho_2,\rho_3\}$ or $R = \{\rho_1,\beta_2(R),\beta_3(R)\}$. As it matters not which we use, we choose $\{\rho_1,\rho_2,\rho_3\}$ and set $\partial\Pi^{(4)}/\partial\rho_1 = 0$, considering $i = 3,2,1$ in turn:

(a)

$$\frac{\partial\Pi^{(4)}}{\partial\rho_3} = \frac{\partial\alpha(R)}{\partial\rho_3}\left[\Pi_1 + 2\alpha(R)\Pi_2(R) + \cdots\right] + \alpha(R)^4\frac{\partial\Pi_4(R)}{\partial\rho_3} = 0 . \tag{35}$$

Substituting Eqs. (13)–(15) in Eq. (34),

$$\frac{\partial\alpha(R)}{\partial\rho_3} = \alpha(0)^4\left[1 + \left[4\rho_1 - \frac{1}{3}\frac{\beta_1}{\beta_0}\right]\alpha(0) + \cdots\right] . \tag{36}$$

Then, using Eq. (19) and working to $O(\alpha^5)$ in Eq. (35), we easily find

$$\Pi_2(\text{PMS})=\frac{1}{6}\frac{\beta_1}{\beta_0}\Pi_1 \tag{37}$$

rather similar to Eq. (28) for the more restrictive form, Eq. (25).

(b) Similarly,

$$\frac{\partial\Pi^{(4)}}{\partial\rho_2}=\frac{\partial\alpha(R)}{\partial\rho_2}\left[\Pi_1+2\alpha(R)\Pi_2(R)+3\alpha(R)^2\Pi_3(R)+\cdots\right]+\alpha(R)^3\frac{\partial\Pi_3(R)}{\partial\rho_2}+\alpha(R)^4\frac{\partial\Pi_4(R)}{\partial\rho_2}=0 \tag{38}$$

with

$$\frac{\partial\alpha(R)}{\partial\rho_2}=\alpha(0)^3\left[1+\tfrac{2}{3}\alpha(0)^2\left[5\rho_2-14\rho_1{}^2+\frac{\beta_1}{\beta_0}\rho_1+\frac{1}{2}\frac{\beta_2(0)}{\beta_0}\right]+\cdots\right]. \tag{39}$$

Again using Eqs. (13)–(15) in Eq. (34) and keeping terms in Eq. (38) to $O(\alpha^5)$, straightforward algebra yields, on recalling Eq. (37),

$$\Pi_3(\text{PMS})=-\frac{1}{9}\frac{\beta_2(\text{PMS})}{\beta_0}\Pi_1\,. \tag{40}$$

(c) Next

$$\frac{\partial\Pi^{(4)}}{\partial\rho_1}=\frac{\partial\alpha(R)}{\partial\rho_1}\left[\Pi_1+2\alpha(R)\Pi_2(R)+3\alpha(R)^2\Pi_3(R)+4\alpha(R)^3\Pi_4(R)\right]+\alpha(R)^2\frac{\partial\Pi_2(R)}{\partial\rho_1}+\alpha(R)^3\frac{\partial\Pi_3(R)}{\partial\rho_1}+\alpha(R)^4\frac{\partial\Pi_4(R)}{\partial\rho_1}=0 \tag{41}$$

with

$$\frac{\partial\alpha(R)}{\partial\rho_1}=\alpha(0)^2\left[1+\tfrac{2}{3}\alpha(0)^3\left[6\rho_3-28\rho_1\rho_2+\frac{\beta_1}{\beta_0}\rho_2+28\rho_1{}^3-\frac{4\beta_2(0)}{\beta_0}\rho_1+\frac{3\beta_3(0)}{2\beta_0}\right]+\cdots\right] \tag{42}$$

from Eqs. (13)–(15). Once again substituting Eqs. (13)–(15) into Eq. (34), keeping terms to $O(\alpha^5)$ in Eq. (41), tedious algebra yields on using our previous results of Eqs. (37)–(40)

$$\Pi_4(\text{PMS})=-\frac{1}{4}\frac{\beta_3(\text{PMS})}{\beta_0}\Pi_1\,. \tag{43}$$

The optimal form for Π of Eq. (33) is then

$$\Pi^{(4)}=\Pi_0+\alpha(\text{PMS})\Pi_1+\frac{1}{6}\frac{\beta_1}{\beta_0}\alpha(\text{PMS})^2\Pi_1-\frac{1}{9}\frac{\beta_2(\text{PMS})}{\beta_0}\alpha(\text{PMS})^3\Pi_1-\frac{1}{4}\frac{\beta_3(\text{PMS})}{\beta_0}\alpha(\text{PMS})^4\Pi_1\,. \tag{44}$$

We see a general pattern emerging in the Stevenson scheme with

$$\Pi^{(N)}=\Pi_0+\left[\sum_{n=1}^{N}\lambda_n^N\frac{\beta_{n-1}(\text{PMS})}{\beta_0}\alpha(\text{PMS})^n\right]\Pi_1\,, \tag{45}$$

where the λ_i^N depend on the order at which we truncate (except, by definition, $\lambda_1^N\equiv 1$).

[*Note added*: This pattern, shown in Eq. (45), is confirmed by further study[33] and the general form of λ_n^N for any order of truncation N is deduced to be

$$\lambda_n^N=\frac{1}{n}\left[\frac{N-2n+1}{N-1}\right]. \tag{46}$$

This succintly summarizes Eq. (29) for $N=2$ and Eq. (44) for $N=4$.] Most importantly, the form Eq. (45) is quite independent of the particular physical quantity we consider. However, the scheme in which each $\alpha(\text{PMS})$ appropriately truncated is defined does depend on the process, for we must know what the corresponding $\Pi_2(0)$, $\Pi_3(0)$, ... are, as well as $\beta_2(0)$, $\beta_3(0)$, ..., in some (arbitrary reference) scheme. Equations (10), (13)–(19) then fix $\alpha(\text{PMS})$ in terms of $\alpha(0)$ and determine $\beta_2(\text{PMS})$, $\beta_2(\text{PMS})$, ... that appear in Eq. (45).

For example, with $N=3$ in the PMS optimal scheme,

$$\frac{\beta_2(\mathrm{PMS})}{\beta_0}=\frac{3}{2}\left[\frac{\beta_2(0)}{\beta_0}+\frac{\Pi_3(0)}{\Pi_1}-\left[\frac{\Pi_2(0)}{\Pi_1}\right]^2-\frac{\beta_1\Pi_2(0)}{\beta_0\Pi_1}\right],$$

while truncating with $N=4$,

$$\frac{\beta_2(\mathrm{PMS})}{\beta_0}=\frac{9}{8}\left[\frac{\beta_2(0)}{\beta_0}+\frac{\Pi_3(0)}{\Pi_1}-\left[\frac{\Pi_2(0)}{\Pi_1}\right]^2-\frac{\beta_1\Pi_2(0)}{\beta_0\Pi_1}+\frac{7\beta_1^2}{36\beta_0^2}\right]$$

and

$$\frac{\beta_3(\mathrm{PMS})}{\beta_0}=\frac{2\beta_3(0)}{\beta_0}+\frac{4\Pi_4(0)}{\Pi_1}+\frac{\beta_1\Pi_3(0)}{2\beta_0\Pi_1}-12\frac{\Pi_2(0)\Pi_3(0)}{\Pi_1^2}+8\left[\frac{\Pi_2(0)}{\Pi_1}\right]^3+\frac{3\beta_1}{2\beta_0}\left[\frac{\Pi_2(0)}{\Pi_1}\right]^2-\frac{\beta_1^2\Pi_2(0)}{2\beta_0^2\Pi_1}-\frac{4\beta_2(0)\Pi_2(0)}{\beta_0\Pi_1}+\frac{\beta_1\beta_2(0)}{2\beta_0^2}+\frac{\beta_1^3}{216\beta_0^3}$$

with more complicated expressions than Eq. (30) for $\alpha(\mathrm{PMS})$ in terms of $\alpha(0)$.

Note that the optimal PMS form at a given order is not trivally obtained from that at some higher order, i.e., the different truncations of Eq. (45) are not equivalent. The addition of more terms in the series both for Π and α_s, makes the scheme-dependence space more complex and so the turning points of Π that determine Eq. (45) are shifted (as seen in the simple example of Fig. 1). Nonetheless, renormalization-group invariance, Eq. (9), implemented through Eqs. (16)–(19), etc., ensures that as the order of truncation increases, the range of schemes over which there is little scheme dependence is enlarged (as in the simple example of Fig. 1). Any sensible criterion will pick out schemes for the truncated expansion in this (flat) scheme-insensitive region.

What PMS does is to force the coefficients in the expansion of Π, viz. $\Pi_1{:}\Pi_2{:}\Pi_3{:}\Pi_4{:}\cdots$, to be in just the ratio of the corresponding coefficients of the β function, viz. $\beta_0{:}\beta_1{:}\beta_2{:}\beta_3{:}\cdots$ (aside from simple rational fractions). That this happens should come as no surprise, since how the truncated perturbative expansion for Π changes with scheme is dictated by how α does[34] and this is fixed by the β function through the Callan-Symanzik equation, Eq. (11). Thus, not surprisingly, Eq. (45) generalizes to still higher orders in PMS.[33]

FACC requires that we choose an α_s so that the truncated expansions, both for the Callan-Symanzik function and for the physical Π, are apparently convergent. PMS, by forcing the coefficients of one to be given in terms of the other, is thus a particular implementation of FACC where the expansion of Π is sensible if that for β is too.

IV. CONCLUSIONS

As a minimum we can only believe a perturbative expansion in terms of some α_s if its β-function expansion makes sense. For without that, α_s itself is ill defined. Since we can only calculate the first few terms of this expansion, a truncation criterion is essential, even in this definition. The expansion of the Callan-Symanzik function is not, in general, convergent, nor perhaps an asymptotic series. Thus at some order, the perturbative expansion for the β function will break down.

Typically, in QED, expansions in α, e.g., for $(g-2)$, have a hundred or so decreasing terms and then ever increasing ones, which make the series diverge. In perturbative QCD, we would expect the change to increasing terms to occur after perhaps only 3 or 4 and the β_N to show an $N!$ behavior sooner rather than later. At what N this happens will differ from scheme to scheme. Indeed, 't Hooft[35] has pointed out that there do exist schemes in which all the coefficients of the β function, but β_0,β_1, are zero. Though formally defined,[36] unfortunately we do not know what renormalization prescriptions these are[37] and how to calculate in them. Schemes, like MS or MOM have, in general, nonzero β_i, though hopefully[38] some known schemes may have β-function expansions which are Borel summable.

Our hope is that we can obtain a good approximation to physical quantities in suitable kinematic

domains by calculating the terms in the nonincreasing part of the series, as in QED (but see Ref. 35). For this to be meaningful, the truncated expansion of the β function must be sensible. The fastest apparent convergence criterion (FACC), by its very nature, embodies this idea. In this paper, we have deduced that Stevenson's principle of minimal sensitivity (PMS)[16] requires the terms in the analogous expansion for any physical quantity to behave just as for the β function (aside from rational fractions). Thus PMS tells us nothing that FACC does not. The claimed optimality of Stevenson's principle is then no more or less believable, or less transitory, than the naive criterion of fastest apparent convergence.

In any event, these criteria, whether FACC or PMS, force a choice of expansion parameter that makes the first few terms in the perturbative series dominant. Nonetheless, this does not render futile calculation of higher-order terms. For without knowing these, we cannot know in any given situation what expansion parameter is optimal, and without knowing this much of the predictive power of perturbative QCD is lost.[39]

Note added in proof. Following the preprint version of this paper, Wrigley (Ref. 40) has provided a more elegant proof of the general result of Eq. (46).

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