Dirac supermultiplet

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The graded extension of the de Sitter space-time algebra so(3,2) is identical to the structure defined by polynomials of order 1 and 2 in the natural coordinates $\xi_1,\ldots,\xi_4$ of four-dimensional phase space. Ordinary Weyl quantization gives a representation that is unique among all the representations of the graded algebra in that the Poisson bracket relations $\{\xi_i,\xi_j\}=-C_{ij}$ (which are not part of the structure of the graded algebra) are preserved. The restriction of this representation to the Lie subalgebra so(3,2) is the direct sum Di $\oplus$ Rac of the two singleton representations. There exists a unique, supersymmetric, interacting field theory of a single Dirac multiplet. The interaction Lagrangian has the form $\frac{1}{2}\int dy(3g\phi^2\overline{\psi}\chi+g^2\phi^6)$, where ϕ is the scalar Rac field, ψ and χ are the spinor Di "field strength" and associated "potential," and g is a real coupling constant. Applications to confinement and to composite massless particles is discussed.

I. INTRODUCTION

The existence of a quantum field theory that includes an interacting, massless field of helicity exceeding 2 has been doubted. No limitation on helicity is encountered in the theory of free fields,[1] and interactions with fixed, external sources can also be introduced,[1] but a genuine theory of interacting fields has not yet been found within the conventional framework. In fact, it was suggested[1] that any field theory of a massless, interacting field with helicity greater than 2 may have to contain an infinite family of massless fields with unbounded helicity.

Some objections have been raised against the existence of higher-spin, massless particles. In particular, it was shown that, under certain conditions, a charged massless particle of spin greater than $\frac{1}{2}$ cannot exist, and that any massless particle with spin greater than 1 cannot have a covariant energy-momentum tensor.[2] However, the conditions under which this result was obtained are not satisfied by Einstein's gravity; evidently one can draw no inference concerning our de Sitter field theory that describes massless particles with all spins.

de Sitter field theory is interesting for at least two reasons. First of all, the conventional view, according to which Minkowski space is the background for quantum gravity, is tenable only if the cosmological constant is rigorously zero. In order to allow for a nonvanishing cosmological constant, however small, one has to take a de Sitter background. Secondly, it turns out that the limit of zero curvature is singular in the following sense. de Sitter covariance allows the existence in de Sitter space of two truly remarkable elementary particles; the spinless Rac and the spin-$\frac{1}{2}$ Di, collectively known as singletons. Emission or absorption of one singleton is absolutely unobservable in terrestrial experiments.[3] This can be viewed as a type of kinematical confinement and described as follows. If energy is measured in units of the square root of the curvature, and J is the angular momentum in the usual units, then the quantum number $c\equiv(-)^{2E-2J}$ is -1 for singletons and $+1$ for all massless particles. Hence singletons are "colored" and massless particles are not. No particles with similar properties exist in flat space.

An even more remarkable property of singletons is the purely kinematical fact that all two-particle states are massless.[3,4] We have proposed to interpret all massless particles as two-singleton states. The idea that massless particles may be composite is now current; it should be strongly emphasized, however, that this idea is implemented much more easily in de Sitter space than in flat space. Free, two-particle states in flat space have all masses and all spins, and detailed dynamical models are required to prove the existence of a small set of massless bound states with all the correct properties. Not the least difficult is to explain why these bound states dominate the subsequent dynamical picture. In contrast, two-singleton states in de Sitter space are massless for purely kinematical reasons, and all one needs to explain is why the states of lower spin have more evident dynamical manifestations than those of higher spin. If the relative strength of electromagnetic and gravitational couplings is a guide, then it is not difficult to imagine that direct observation of massless particles with spin higher than 2 may be difficult. It seems to us, then, that ideas such as quark confinement and gluon compositeness are much more likely to work in de Sitter space, with singletons playing a role similar to that of quarks or preons.

The theory of local Rac fields on de Sitter space is characterized by a gaugelike structure that necessitates the introduction of an indefinite metric and Gupta-Bleuler-type quantization.[5] Internal consistency requires that the interactions be "gauge invariant," as well as invariant under the de Sitter group SO(3,2) of space-time transformations, and this limits the interaction Lagrangian to an expression that contains only two real coupling constants.[4,5] Here it will be shown that supersymmetry imposes an algebraic relation between these two coupling constants and makes the interaction Lagrangian essentially unique (up to a single real coupling constant).

The Dirac supermultiplet Di $\oplus$ Rac is unique and fundamental in the following sense. Let (ξ_i), $i=1,\ldots,4$ denote the odd elements, and $(L_{\alpha\beta})$, $\alpha<\beta=1,2,3,5$ the even elements of a basis for a graded Lie algebra that we shall call osp(2). It is also known as osp(4,1). For on-shell, free Di and Rac fields, the following "metaplectic" structure relations hold:

$$\xi_i\xi_j=(\Sigma^{\alpha\beta})_{ij}L_{\alpha\beta}-\tfrac{1}{2}C_{ij} \ , \tag{1.1}$$

where C is the charge-conjugation matrix and $\Sigma_{\alpha\beta}$ is the representative of $L_{\alpha\beta}$ in the four-dimensional symplectic representation of so(3,2). The entire osp(2) graded-Lie-algebra structure is contained in this equation, including anticommutation relations between the ξ_i, and commutation relations between ξ_i and $L_{\alpha\beta}$, and between the $L_{\alpha\beta}$. This metaplectic structure, defined by Eq. (1.1) and the associativity of operator products, is an extension of osp(2), since not only the anticommutator, but the commutator as well of two odd elements is specified. This suggests the possibility of a deeper interpretation, not only of Dis and Racs and massless particles, but perhaps of space-time itself, possibly along the lines of the twistor program.[6]

Alternatively, Eq. (1.1) may be interpreted as defining an ideal in the enveloping algebra of osp(2). The fact that this ideal uniquely characterizes the free Dirac supermultiplet may be of some utility. For example, it should be possible to give a new and more direct proof of the fact[3,4] that the representation (Di $\oplus$ Rac) $\otimes$ (Di $\oplus$ Rac) of SO(3,2) has an extension to a unitary representation of the conformal group SO(4,2) that is a direct sum of all the "massless representations"; by this means one could then investigate whether or not conformal invariance is preserved by the interaction.

Summary. Di and Rac field theories can be formulated in two ways. As field theories formulated directly on the space-time de Sitter manifold, they exhibit interesting features that are normally encountered only in vector field gauge theories. The case of the scalar Rac field has been worked out in detail.[5] A physically equivalent field theory, that is, one that gives the same S matrix, can be formulated on a three-dimensional manifold that is the boundary of de Sitter space at spatial infinity.[4,5] This avoids all the gauge complications as far as the Rac is concerned, and leaves only an easily manageable "chiral gauge" problem associated with the Di field. Section II contains the details.

The fact that free singleton one-particle states form a de Sitter supermultiplet was pointed out in a recent review on Dis and Racs.[7] In Sec. III we give the details and work out the free-field transformation rules. The algebra closes on-shell only, but this is remedied in Sec. IV with the help of an auxiliary scalar field. Interactions can now be introduced in a unique way, up to a single, real coupling constant. Finally, the auxiliary field can be eliminated; this leads to a new, nonlinear form of the supersymmetry transformation law for the Rac field ϕ and the Di field χ:

$$\xi_i\phi=(\gamma\chi)_i, \quad \xi_i\chi_j=\partial_{ji}\phi-gC_{ji}\phi^3 \ ,$$

where $\xi_1,\ldots,\xi_4$ is a basis for the odd part of the graded supersymmetry algebra, C is the charge-conjugation matrix, and g is the coupling constant.

II. DI AND RAC FIELDS ON THE CONE

Space-time is a four-dimensional de Sitter space with very small constant curvature ρ, or perhaps the flat-space limit $\rho\to 0$. The theory of Rac fields on de Sitter space is an interesting and unconventional type of local field theory.[5] Although the Rac field is a scalar field, features appear that are usually associated with gauge theories, and Gupta-Bleuler techniques are required for quantization. A similar approach can certainly be attempted for the Di field, but that will not be done here.

In fact, it is not necessary, nor useful, to develop Di and Rac field theory on de Sitter space because, as was shown in detail in the case of the Rac,[4,5] there exists an equivalent three-dimensional field theory that is much simpler in that the gauge fields disappear. By calling this three-dimensional field theory equivalent to the four-dimensional field theory, we mean that both give the same S matrix.

Let (y_α), $\alpha=0,1,2,3,5$ be the Cartesian coordinates of a five-dimensional pseudo-Euclidean space with metric δ given by

$$y^2=\delta^{\alpha\beta}y_\alpha y_\beta={y_0}^2-\vec{y}^2+{y_5}^2 .$$

The hyperboloid $y^2=1/\rho$ is a model for a four-dimensional de Sitter space with curvature ρ. Fields defined on de Sitter space can be extended to $y^2>0$ by fixing the degree of homogeneity. In the case of the Rac field ϕ we choose[3]

$$\hat{N}\phi=-\tfrac{1}{2}\phi,\quad \hat{N}\equiv y^\alpha\partial_\alpha . \tag{2.1}$$

Now it turns out[3,5] that all the physical information contained in ϕ is retained when one takes the limit of ϕ as $y^2\to 0$, with all coordinates y_α finite. This limit will henceforth be denoted ϕ, so that the field ϕ is defined only on $y^2=0$ from now on. We have

$$\phi(y)=r^{-1/2}\underline{\phi}(t,\hat{y}),\quad r\equiv|\vec{y}| \tag{2.2}$$

where $\underline{\phi}$ is defined on $S_1\times S_2$ and the angles are defined by $y_5+iy_0=re^{it}$, $\hat{y}=\vec{y}/r$.

A. Rac fields

We shall give a very brief summary of the basic facts relating to classical and quantized Rac fields on the cone.[4,5] The free wave equation is

$$\partial^2\phi=0,\quad \partial^2\equiv\delta^{\alpha\beta}\partial_\alpha\partial_\beta . \tag{2.3}$$

This can be derived from the Lagrangian

$$\tfrac{1}{2}\int dy\,\phi\,\partial^2\phi . \tag{2.4}$$

Integration over the cone is defined by[4,5]

$$\int dy L(y)=\int_{S_1\times S_2} dt\,d\hat{y}\,\underline{L}(t,\hat{y}) , \tag{2.5}$$

for scalar fields L of degree -3 only, where

$$L(y)=r^{-3}\underline{L}(t,\hat{y}) .$$

Such integrals are SO(3,2) invariant. A complete set of positive-energy solutions is given by

$$\phi_{LM}(y)=r^{-1/2}(2L+1)^{-1/2}e^{-it(L+1/2)}Y_{LM}(\hat{y}) , \tag{2.6}$$

with $M=-L,\ldots,L$ and $L=0,1,\ldots$. The free, quantized Rac field is given by

$$\phi(y)=\sum_{LM}[b_{LM}\phi_{LM}(y)+b^*_{LM}\bar{\phi}_{LM}(y)] ,$$

$$[b_{LM},b^*_{L'M'}]_-=\delta_{LL'}\delta_{MM'} ,$$

and the homogeneous propagator is

$$\begin{aligned}D(y,y')&\equiv\langle 0|\phi(y)\phi(y')|0\rangle\\&=\sum_{LM}\phi_{LM}(y)\bar{\phi}_{LM}(y')\\&=\frac{1}{4\pi}(2y\cdot y')^{-1/2} .\end{aligned}$$

Commutators and time-ordered products have also been calculated.[5]

B. Di fields

The action of so(3,2) on spinor fields is given by

$$L_{\alpha\beta}=i(y_\alpha\partial_\beta-y_\beta\partial_\alpha)+\Sigma_{\alpha\beta} . \tag{2.7}$$

A basis for the irreducible representation $\mathrm{Di}=D(1,1/2)$ is given by ($J=1/2$, $3/2,\ldots=$ total angular momentum)

$$\psi_{JM}(y)=\boldsymbol{y}\chi_{JM}(y) , \tag{2.8}$$

where

$$\chi_{JM}(y)=r^{-3/2}-e^{iJt}\mathscr{Y}_{JM}(\hat{y}) . \tag{2.9}$$

The spherical spinors $\mathscr{Y}_{JM}$ are given in the Appendix. The space spanned by the ψ_{JM}'s can be characterized by

$$\psi=\boldsymbol{y}\chi,\quad \hat{N}\psi=-\tfrac{1}{2}\psi , \tag{2.10}$$

$$(\kappa+\tfrac{3}{2})\chi=0,\quad \hat{N}\chi=-\tfrac{3}{2}\chi , \tag{2.11}$$

$$\kappa\equiv 2i\Sigma^{\alpha\beta}y_\alpha\partial_\beta .$$

Since $\boldsymbol{y}^2=-y^2=0$ on the cone, we encounter here, somewhat unexpectedly, a new "gauge" phenomenon.

The relation $\psi=\boldsymbol{y}\chi$ is equivalent to $\boldsymbol{y}\psi=0$ and may be regarded as a subsidiary condition. It is possible to express the wave equation $(\kappa+\frac{3}{2})\chi=0$ in terms of ψ, as will be seen later; to express the theory in terms of ψ alone is nevertheless very inconvenient and probably not compatible with an action-principle formulation. We shall therefore regard the field χ as a type of potential that contains unphysical degrees of freedom, eliminated by multiplication by $\boldsymbol{y}$. We shall see that the wave equation (2.11) is invariant under what we shall call "chiral gauge transformations" (see the Appendix),

$$\chi \to \chi + y\lambda, \quad \hat{N}\lambda = -\tfrac{5}{2}\lambda ; \tag{2.12}$$

that is, $(\kappa+\frac{3}{2})y\lambda$ is identically zero.

For κ we have the expressions

$$\kappa = y\partial + \hat{N} = -\partial y - \hat{N} - 5 , \tag{2.13}$$

from which we deduce that, on the cone,

$$y(\kappa - \hat{N}) = (\kappa + \hat{N} + 3)y = 0 . \tag{2.14}$$

Since the degrees of homogeneity of ψ, χ, and λ are $-\frac{1}{2}$, $-\frac{3}{2}$, and $-\frac{5}{2}$, respectively, the "chiral gauge invariance" of the wave equation (2.11) follows at once.

The wave equation can be derived from the Lagrangian

$$\tfrac{1}{2}\int dy \bar{\chi}(\kappa+\tfrac{3}{2})\chi = \tfrac{1}{2}\int dy \bar{\chi} y \partial \chi , \tag{2.15}$$

where $\bar{\chi}^i = C^{ij}\chi_j$ (see the Appendix). This is real, and invariant under chiral gauge transformations. The integrand is of degree -3, as required by invariance.

A quantized free ψ field may be defined by

$$\psi(y) = \sum_{JM} [d_{JM}\psi_{JM}(y) + d^*_{JM}\bar{\psi}_{JM}(y)C] ,$$

$$[d_{JM}, d^*_{J'M'}]_+ = \delta_{JJ'}\delta_{MM'} .$$

The homogeneous propagator is evaluated in the Appendix, with the following covariant result:

$$\begin{aligned} S(y,y') &\equiv \langle 0 \,|\, \psi(y)\bar{\psi}(y') \,|\, 0 \rangle \\ &= \sum_{JM} \psi_{JM}(y)\bar{\psi}_{JM}(y') \\ &= -\frac{1}{4\pi} y (2y\cdot y')^{-3/2} y' \\ &= (\kappa + \tfrac{3}{2}) D(y,y') . \end{aligned} \tag{2.16}$$

This suggests that the homogeneous χ propagator should be

$$-\frac{1}{4\pi}(2y\cdot y')^{-3/2} ,$$

but to achieve this it would have been necessary to introduce additional ghost states with nonpositive norm, and to include creation and destruction operators associated with these states in the definition of the quantized χ field. Because our Lagrangian and hence our S matrix will be invariant under chiral gauge transformations, we shall not need to carry out this construction. No error will result from substituting

$$\langle 0 \,|\, \chi_i(y)\bar{\chi}^j(y') \,|\, 0 \rangle \to -\frac{1}{4\pi}(2y\cdot y')^{-3/2}$$

during the evaluation of the S matrix.

III. FREE-FIELD SUPERMULTIPLET

The construction of the graded extension of so(3,2) to be carried out here was already done by Keck.[8] It is based on the isomorphism between so(3,2) and sp(2,R). This was already exploited by Dirac[9] in his singleton paper; in fact, Dirac's paper makes it pretty evident that Di ⊕ Rac extends to a very special and fundamental representation of the graded extension of so(3,2), although the extension was not mentioned.

Consider a four-dimensional phase space $W = R^4$ with global, natural coordinates $\xi^1, \ldots, \xi^4$ in terms of which the components of the symplectic two-form are $-C_{ij}$, where C is the charge-conjugation matrix (see the Appendix). In other words, if $\{,\}$ denotes the Poisson brackets, then

$$\{\xi_i, \xi_j\} = -C_{ij} . \tag{3.1}$$

The space of second-order polynomials with real coefficients, with the Poisson brackets, has the structure of the Lie algebra sp(2,R)=so(3,2).

Consider the space of polynomials of order 1 or 2, spanned by the ten even elements $L_{\alpha\beta}$ and the four odd elements ξ_i. A structure of graded Lie algebra is defined on this space as follows: If f,g are both odd, then $[f,g]=fg$; if f,g are both even, or if one is even and the other is odd, then $[f,g]=\{f,g\}$. Keck[8] verified that the graded Jacobi identity holds, but this also follows from the existence of faithful representations of the structure. This graded Lie algebra will be denoted osp(2).

A representation of osp(2) is a mapping $f \to \hat{f}$ of osp(2) into a space of linear operators in a Hilbert space such that $[f,g] \to [\hat{f},\hat{g}]_+ \equiv \hat{f}\hat{g} + \hat{g}\hat{f}$ if f and g are odd and $[f,g] \to [\hat{f},\hat{g}]_- \equiv \hat{f}\hat{g} - \hat{g}\hat{f}$ if f and g are both even or if one is even and the other is odd. Representations of osp(2) were given by Keck[8] and others.[10,11] Oddly enough, however, the simplest and perhaps the most fundamental representation appears to have been neglected.

A. Quantization

A very special representation of osp(2) is given by the Weyl quantization map, modified by replacing the conventional factor $i\hbar$ by unity. This map preserves the Lie algebra structure and it was shown[10,12] that the representation of so(3,2) that one obtains is precisely Di $\oplus$ Rac. The easiest way to obtain a representation of osp(2) is to make the trivial observation that the Weyl map also preserves the graded-Lie-algebra structure; hence the representation Di $\oplus$ Rac of so(3,2) may be extended to osp(2). But that is not all, for the Weyl map also preserves the additional structure relation (3.1); that is, $[\hat{\xi}_i,\hat{\xi}_j]_- = -C_{ij}$ (times the identity operator).

All the structure relations may be summarized by a simple formula

$$\hat{\xi}_i\hat{\xi}_j = (\Sigma_{\alpha\beta})_{ij}\hat{L}^{\alpha\beta} - \tfrac{1}{2}C_{ij} \ . \tag{3.2}$$

The symmetric part (with respect to the interchange of i and j) of this formula is the anticommutator structure of osp(2); the antisymmetric part is the structure of the Heisenberg algebra. The remaining structure; that is, so(3,2) commutation relations for the $\hat{L}^{\alpha\beta}$ and the fact that the $\hat{\xi}_i$ span a spinorial ideal, follows from (3.2) and the associativity of the operator products. According to the von Neuman unicity theorem, the representations of (3.2) are unique up to projective equivalence. We construct an explicit realization in the notation of Bargmann-Segal quantization, or ladder operator formalism.[13] From now on we are dealing exclusively with this operator representation, and the carets on $\hat{\xi}_i$, $\hat{L}_{\alpha\beta}$ will be dropped.

We interpret ξ^1 and ξ^2 as creation operators and ξ^3 and ξ^4 as destruction operators ($\xi^i \equiv C^{ij}\xi_j$),

$$\xi_1 = -\partial_1, \quad \xi_2 = -\partial_2,$$

$$\xi_3 = \xi^2, \quad \xi_4 = -\xi^1 \ ,$$

acting on a Fock space spanned by

$$F_{JM} = f_{JM}(\xi^1)^{J+M}(\xi^2)^{J-M}$$
$$M = -J, -J+1, \ldots, J; \quad J = 0, \tfrac{1}{2}, 1, \ldots \ . \tag{3.3}$$

The normalization constants f_{JM} can be chosen so that the induced matrix representation of the ξ_i satisfy the Hermiticity condition $\xi_i^\dagger = \xi^i$. A unitary representation of SO(3,2) is now generated by the operators $L_{\alpha\beta}$ defined by

$$\xi_i\xi_j = (\Sigma_{\alpha\beta})_{ij}L^{\alpha\beta} - \tfrac{1}{2}C_{ij} \tag{3.4}$$

or

$$L_{\alpha\beta} = -\tfrac{1}{2}(\Sigma_{\alpha\beta})^{ij}\xi_i\xi_j \ . \tag{3.5}$$

This representation is precisely Di $\oplus$ Rac, as was well known.[9]

B. Free-field transformations

To express these results as transformations of free Di and Rac field operators, we introduce the linear map generated by

$$F_{JM} \to \begin{cases} b^*_{JM}, & J,M \text{ integer} \\ d^*_{JM}, & J,M \text{ half-integer} . \end{cases}$$

This allows the Fock space spanned by (3.3) to be identified with the direct sum of the spaces of Di and Rac one-particle states, and to reinterpret the ξ_i accordingly. Finally, this action transfers to the field operators. The result is

$$\xi_i\phi = \psi_i \ , \tag{3.6}$$

$$\xi_i\psi_j = (\kappa_{ij} - \tfrac{1}{2}C_{ij})\phi \ . \tag{3.7}$$

Equation (3.7) follows immediately from (3.6) and the structure relation (3.4).

In principle it is superfluous to verify that (3.6) and (3.7) is compatible with (3.4) and with the free wave equations, since our construction guarantees it. Nevertheless, it will be useful to do so. First, (3.6) and the free Rac wave equation $\partial^2\phi = 0$ imply that

$$\partial^2\psi = 0 \ . \tag{3.8}$$

We shall verify that this holds on-shell; that is, by virtue of the Di wave equations (2.10) and (2.11). First, the identity (2.13) shows that $(\kappa + \frac{3}{2})\chi = 0$ is equivalent to $\boldsymbol{y}\boldsymbol{\partial}\chi = 0$; hence $\boldsymbol{\partial}\chi = \boldsymbol{y}\lambda$ on the cone, and $\boldsymbol{\partial}\boldsymbol{y}\boldsymbol{\partial}\chi = -\boldsymbol{\partial}y^2\lambda = -2\boldsymbol{y}\lambda = -2\boldsymbol{\partial}\chi$ on the cone. Again, (2.13) gives $\boldsymbol{\partial}\boldsymbol{y}\chi = (-2 - \boldsymbol{y}\boldsymbol{\partial})\chi$ identically, so finally $\partial^2\psi = \boldsymbol{\partial}\boldsymbol{\partial}\boldsymbol{y}\chi = (-2\boldsymbol{\partial} - \boldsymbol{\partial}\boldsymbol{y}\boldsymbol{\partial})\chi = 0$. [Warning: It is important to distinguish between statements that are independent of the extrapolation to $y^2 > 0$ and those that are not. Equation (3.8) is independent of the choice of (analytic) extrapolation of ψ, because $\hat{N}\psi = -\frac{1}{2}\psi$.] Next, by (2.13), Eq. (3.7) can be written $\xi_i\psi_j = (\boldsymbol{y}\boldsymbol{\partial})_{ji}\phi$, which is compatible with setting

$$\xi_i\chi_j = \boldsymbol{\partial}_{ji}\phi \ . \tag{3.9}$$

In a theory with chiral gauge invariance this is equivalent to (3.7). Together with (2.11), which can be written $\boldsymbol{y}\boldsymbol{\partial}\chi = 0$, (3.9) implies that $\partial^2\phi = 0$, in agreement with (2.3).

From (3.6) and (3.7) we get one expression for $\xi_i\xi_j\psi_k$, and (3.4) gives another. Equating the two we get

$$M_{ijk} \equiv \kappa_{ij}\psi_k - \kappa_{ki}\psi_j - C_{jk}\psi_i + \tfrac{1}{2}C_{ij}\psi_k - \tfrac{1}{2}C_{ik}\psi_j = 0 \ . \tag{3.10}$$

Since $M_{ijk}+M_{jki}+M_{kij}$ vanishes identically, (3.10) is equivalent to $M_{ijk}-M_{kji}=0$ and thus to the equations

$$0=C^{ki}M_{ijk}\equiv[(\kappa+\tfrac{5}{2})\psi]_j \;, \tag{3.11}$$

$$0=\gamma_\alpha{}^{ki}M_{ijk}\equiv[(\kappa+\tfrac{1}{2})\gamma_\alpha\psi]_j \;. \tag{3.12}$$

By means of the identity

$$[\kappa,\gamma_\alpha]=2\partial\!\!\!/ y_\alpha-2\partial_\alpha y\!\!\!/ \;,$$

one can now easily verify that (3.11) and (3.12) are equivalent to (2.10) and (2.11), so that (3.10) holds on-shell. This is a rather surprising result, which perhaps explains why the simple Di-Rac supermultiplet was not found by Keck[8] and others.[10,11] This miraculous closure would have been even more masked if we would have formulated the problem on de Sitter space (rather than on the cone). Of course, such a formulation is possible.

IV. SUPERSYMMETRIC INTERACTIONS

The operators defined by (3.6) and (3.7) satisfy the structure relations (3.4) on-shell only. To facilitate the study of interactions it is very convenient to improve the formulation so that the structure relations are satisfied identically. More precisely, this will turn out to be possible only so far as the osp(2) structure is concerned, while the additional commutation relation $[\xi_i,\xi_j]=-C_{ij}$ will be abandoned off-shell. In all that follows, as in the preceding section, $L_{\alpha\beta}$ stands for the operator that is defined by

$$L_{\alpha\beta}\phi=i(y_\alpha\partial_\beta-y_\beta\partial_\alpha)\phi \;, \tag{4.1}$$

$$L_{\alpha\beta}\psi_i=i(y_\alpha\partial_\beta-y_\beta\partial_\alpha)\psi_i+(\Sigma_{\alpha\beta})_i{}^j\psi_j \;. \tag{4.2}$$

The so(3,2) commutation relations, and the spinorial nature of the ξ_i, will be maintained without comment, while our interest centers on (3.4), of which only the symmetric part is relevant to osp(2) and to supersymmetry.

A direct approach allows for a generalization of (3.4) by introducing an additional term in (3.7):

$$\xi_i\psi_j=\xi_i\xi_j\phi=(\kappa_{ij}-\tfrac{1}{2}C_{ij})\phi+A_{ij} \;, \tag{4.3}$$

with $A_{ij}=-A_{ji}$. The symmetric part of (4.3), applied to ψ_k, leads after some calculation to the following requirement of consistency:

$$\xi_iA_{jk}=M_{ijk}+B_{ijk} \;, \tag{4.4}$$

where B_{ijk} is totally antisymmetric, and M_{ijk} is the quantity (3.10). Since M_{ijk} vanishes on-shell, it is consistent to require A_{ij} and B_{ijk} to vanish on-shell. This approach can be pushed to completion at the cost of a lot of work.

A simpler method makes use of previous work on de Sitter superfields.[10] Let $\theta^1,\ldots,\theta^4$ be the generators of a Grassmann algebra. Then the operators

$$Q_i=(1+\tfrac{1}{2}\vec{\theta}^{\,2})\partial_i-\theta_i\theta\cdot\partial+\theta^j(\kappa_{ij}-\tfrac{1}{2}C_{ij}) \tag{4.5}$$

with $\vec{\theta}^{\,2}=\theta^i\theta_i$, satisfy the osp(2) anticommutation relations. Applying these operators to the superfield

$$\Phi=\phi+\theta^i\psi_i+\frac{1}{2!}\theta^i\theta^jA_{ij}+\frac{1}{3!}\theta^i\theta^j\theta^kB_{ijk}+\frac{1}{4!}\theta^i\theta^j\theta^k\theta^lC_{ijkl}$$

and putting $Q_i\Phi=\xi_i\phi+\theta^j(\xi_j\psi)+\ldots$, we find

$$\xi_i\phi=\psi_i,\quad \xi_i\psi_j=(\kappa_{ij}-\tfrac{1}{2}C_{ij})\phi+A_{ij} \;, \tag{4.6}$$

$$\xi_iA_{jk}=M_{ijk}+B_{ijk} \;, \tag{4.7}$$

in agreement with (3.6), (4.3), and (4.4), and expressions for ξ_iB_{jkl} and ξ_iC_{jklm} that we shall not write down.

The chirality condition (2.10), $\psi=y\!\!\!/\chi$, is expected to remain valid off-shell, so it is interesting to check whether this condition is compatible with (4.6) and (4.7). Using the identities in (3.11) and (3.12), we find that, when $\psi=y\!\!\!/\chi$, then

$$M_{ijk}=-y\!\!\!/_{jk}[(\kappa+\tfrac{3}{2})\chi]_i \;. \tag{4.8}$$

Hence (4.7) is compatible with restricting A_{ij} to the form

$$A_{ij}=-y\!\!\!/_{ij}A \;, \tag{4.9}$$

which is precisely what is needed in order that (4.6) remain compatible with $\psi=y\!\!\!/\chi$.

It is remarkable that, when A_{ij} is of the form (4.9), then one obtains simply $\xi_iB_{jkl}=C_{ijkl}$, and the internal consistency of the entire scheme may be assured by setting

$$B_{ijk}=C_{ijkl}=0 \;.$$

No auxiliary field besides A_{ij} is required in order to obtain off-shell closure. The transformation law (4.6) and (4.7) can be expressed as

$$\xi_i\phi=\psi_i,\quad \xi_i\chi_j=\partial_{ji}\phi+C_{ji}A\ , \tag{4.10}$$

$$\xi_i A=[(\kappa+\tfrac{3}{2})\chi]_i=(y\partial\chi)_i\ . \tag{4.11}$$

The supersymmetric free Lagrangian is

$$\mathscr{L}_0=\tfrac{1}{2}\int dy[\phi\partial^2\phi+\bar{\chi}(\kappa+\tfrac{3}{2})\chi-A^2]\ . \tag{4.12}$$

Interactions

The interaction Lagrangian density, to give an SO(3,2)-invariant action, must be of degree -3. The only possible terms are $A\phi^3$, $\phi^2\bar{\psi}\chi$, and ϕ^6, since $\bar{\psi}\psi$ vanishes and $\bar{\chi}\chi$ is not chiral gauge invariant. The only real combination that is invariant under the supersymmetry transformations (4.10) and (4.11) is

$$\mathscr{L}_I=g\int dy(\tfrac{3}{2}\bar{\psi}\chi\phi^2-A\phi^3)\ , \tag{4.13}$$

with g real. The wave equation for A becomes

$$A+g\phi^3=0\ . \tag{4.14}$$

Therefore, this auxiliary field can be eliminated to give the following total Lagrangian:

$$\mathscr{L}=\tfrac{1}{2}\int dy[\phi\partial^2\phi+\bar{\chi}(\kappa+\tfrac{3}{2})\chi+3g\bar{\psi}\chi\phi^2+g^2\phi^6] \tag{4.15}$$

and the transformation law

$$\xi_i\phi=\psi_i\ , \tag{4.16}$$

$$\xi_i\chi_j=\partial_{ji}\phi-gC_{ji}\phi^3\ . \tag{4.17}$$

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APPENDIX

That (2.8) and (2.9) defines a basis for $D(1,\frac{1}{2})$ is a special case of results given[13] for $D(E_0,\frac{1}{2})$. The spherical spinors are

$$\mathscr{Y}_{JM}=(2J+2)^{-1/2}\begin{bmatrix}-(J+1-M)^{1/2}Y_{J+1/2,M-1/2}\\ (J+1+M)^{1/2}Y_{J+1/2,M+1/2}\\ 0\\ 0\end{bmatrix}.$$

We have used the following conventions:

$$\gamma_i=\begin{bmatrix}-\sigma_i & \\ & \sigma_i\end{bmatrix},\quad \gamma_0=\begin{bmatrix} & -I\\ I & \end{bmatrix},$$

$$\gamma_5=-\gamma_0\gamma_1\gamma_2\gamma_3=i\begin{bmatrix} & I\\ I & \end{bmatrix},\quad C=-i\begin{bmatrix} & \sigma_2\\ \sigma_2 & \end{bmatrix}.$$

Our justification for referring to y as the chirality operator and to (2.12) as a chiral gauge transformation is as follows. Apart from a fixed power of r, the fields are defined on $S_1\times S_2$, which is a covering space of compactified (2+1)-dimensional Minkowski space. In a notation that is adapted to that interpretation one finds that y is the product of the chiral projection $(1+\gamma_3)/2$ and an invertible matrix.[4,14]

The homogeneous propagator (2.16) was obtained as follows. First,

$$\sum_{JM}\psi_{JM}(y)\bar{\psi}_{JM}(y')=\frac{1}{4\pi}y\sum_J(J+\tfrac{1}{2}-\vec{\sigma}\cdot\vec{\mathrm{M}})e^{-iJ\tau}P_{J+1/2}(\hat{y}\cdot\hat{y}')\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}y'$$

$$=\frac{1}{4\pi}y(i\partial_\tau+\tfrac{1}{2}-\vec{\sigma}\cdot\vec{\mathrm{M}})\sum_{E=0}^{\infty}e^{-i(E-1/2)\tau}P_E\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}y'\ ,$$

with $\tau = t - t'$. The Fourier series defines a distribution that can be defined as the limit of an analytic function, as τ tends to the real axis from below, and the final expression in (2.16) must be interpreted in this sense.[5] The final simple and covariant form is found only after restriction to $y^2 = 0$, and some algebra.

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