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Equations of motion of a non-Abelian charged spin particle in a Yang-Mills field

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The equations of motion of a non-Abelian charged spin particle are derived from energy-momentum conservation and covariant charge conservation following the moment method of Papapetrou.

I. INTRODUCTION

In a recent paper[1] the laws of motion for a pointlike non-Abelian charge were derived from energy-momentum conservation and covariant charge conservation following a procedure first introduced by Mathisson.[2] In this paper we shall extend these results for a particle with spin and SU(2) electromagnetic moments. We shall follow the moment method of Papapetrou,[3] which has been used recently[4] in the analysis of pole-dipole charged particles with electromagnetic moments in an electromagnetic-gravitational field. The method will also give the volume integral of the energy-momentum tensor and of the current, and of their first moments.

II. FIELD EQUATIONS AND LAWS OF MOTION

The field equations for SU(2) gauge theory are given by

$$D^{\mu}\vec{F}_{\mu\nu}=\partial^{\mu}\vec{F}_{\mu\nu}-\vec{A}^{\mu}\times\vec{F}_{\mu\nu}=\vec{J}_{\nu}\ , \tag{1}$$

with

$$\vec{F}_{\mu\nu}=\partial_{\mu}\vec{A}_{\nu}-\partial_{\nu}\vec{A}_{\mu}-\vec{A}_{\mu}\times\vec{A}_{\nu}\ . \tag{2}$$

Three-dimensional vector notation is used for the degrees of freedom referring to the local isospin and $\times$ denotes the cross product.

As a consequence of the field equations (1) the current $\vec{J}_{\nu}$ is covariantly conserved, i.e.,

$$D^{\nu}\vec{J}_{\nu}=\partial^{\nu}\vec{J}_{\nu}-\vec{A}^{\nu}\times\vec{J}_{\nu}=0\ . \tag{3}$$

We introduce now the energy-momentum tensor of the gauge fields $\vec{F}^{\mu\nu}$,

$$T^{\mu\nu}_{(f)}=\tfrac{1}{4}\eta^{\mu\nu}\vec{F}^{\alpha\beta}\cdot\vec{F}_{\alpha\beta}-\vec{F}^{\mu\alpha}\cdot\vec{F}^{\nu}{}_{\alpha}\ , \tag{4}$$

with the metric $\eta^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ in Minkowski space. From (1) it is easily shown that $T^{\mu\nu}_{(f)}$ satisfies the equation

$$\partial_{\mu}T^{\mu\nu}_{(f)}=-\vec{F}^{\nu}{}_{\alpha}\cdot\vec{J}^{\alpha}\ . \tag{5}$$

We introduce now the matter-symmetric energy-momentum tensor $T^{\mu\nu}$ by the requirement

$$\partial_{\mu}T^{\mu\nu}=\vec{F}^{\nu}{}_{\alpha}\cdot\vec{J}^{\alpha}\ . \tag{6}$$

From (5), this is equivalent to the statement that $T^{\mu\nu}$ is such that the overall energy-momentum for the system matter-gauge fields is conserved,

$$\partial_{\mu}(T^{\mu\nu}+T^{\mu\nu}_{(f)})=0\ . \tag{7}$$

From (3) and (6) we shall derive the equations of motion for a particle with charge and spin following the moment procedure of Papapetrou.[3] We consider an extended system with reference point $X^{\alpha}(s)$ with velocity $u^{i}=dX^{i}/ds$ and we shall take moments of $T^{\mu\nu}$ and $\vec{J}^{\nu}$ about this point, up to first order. By demanding that the dimensions of the system tend to zero around $X^{\alpha}(s)$ at the very end of the calculation, this point will give the world line of our pointlike charged spin particle.

From (6) we have the equations

$$\partial_{\alpha}(x^{\mu}T^{\alpha\nu})=T^{\mu\nu}+x^{\mu}\vec{F}^{\nu}{}_{\alpha}\cdot\vec{J}^{\alpha}\ , \tag{8}$$

$$\partial_{\alpha}(x^{\mu}x^{\beta}T^{\alpha\nu})=x^{\beta}T^{\mu\nu}+x^{\mu}T^{\beta\nu}+x^{\mu}x^{\beta}\vec{F}^{\nu}{}_{\alpha}\cdot\vec{J}^{\alpha}\ . \tag{9}$$

Integrating Eqs. (6), (8), and (9) over the three-dimensional space volume for $t=$const of our system, then

$$\frac{d}{dt}\int T^{0\nu}dV=\int\vec{F}^{\nu\alpha}\cdot\vec{J}_{\alpha}dV\ , \tag{10}$$

$$\frac{d}{dt}\int x^{\mu}T^{0\nu}dV=\int T^{\mu\nu}dV+\int x^{\mu}\vec{F}^{\nu\alpha}\cdot\vec{J}_{\alpha}dV\ , \tag{11}$$

and

$$\frac{d}{dt}\int x^\mu x^\beta T^{0\nu}dV=\int x^\beta T^{\mu\nu}dV+\int x^\mu T^{\beta\nu}dV + \int x^\mu x^\beta \vec{F}^{\nu\alpha}\cdot\vec{J}_\alpha dV\,. \tag{12}$$

Next we write

$$x^\mu = X^\mu + \delta x^\mu \tag{13}$$

with $X^4=t$, that is, $\delta x^0=0$. Substituting Eq. (13) in (11) and using (10) we obtain

$$\frac{dX^\mu}{dt}\int T^{0\nu}dV+\frac{d}{dt}\int \delta x^\mu T^{0\nu}dV = \int T^{\mu\nu}dV+\int \delta x^\mu \vec{F}^{\nu\alpha}\cdot\vec{J}_\alpha dV\,. \tag{14}$$

Next we substitute Eq. (13) into (12) and use (11), and afterwards use (13) again for x^β and use Eq. (14). Neglecting second-order moments of $T^{\mu\nu}$ one gets

$$\frac{dX^\mu}{dt}\int \delta x^\beta T^{0\nu}dV+\frac{dX^\beta}{dt}\int \delta x^\mu T^{0\nu}dV = \int \delta x^\beta T^{\mu\nu}dV+\int \delta x^\mu T^{\beta\nu}dV\,. \tag{15}$$

Now we expand $\vec{F}^{\nu\alpha}(x)$ around X^α,

$$\vec{F}^{\nu\alpha}(x)=\vec{F}^{\nu\alpha}(X)+\delta x^\beta\partial_\beta\vec{F}^{\nu\alpha}(X)+\cdots\,, \tag{16}$$

where $\partial_\beta=\partial/\partial X^\beta$. Introducing Eq. (16) in (10) and (14) one gets to first order in δx^α

$$\frac{d}{dt}\int T^{0\nu}dV=\vec{F}^\nu{}_\alpha\cdot\int\vec{J}^\alpha dV+\partial_\beta\vec{F}^\nu{}_\alpha\cdot\int\delta x^\beta\vec{J}^\alpha dV\,, \tag{17}$$

$$\frac{dX^\mu}{dt}\int T^{0\nu}dV+\frac{d}{dt}\int\delta x^\mu T^{0\nu}dV = \int T^{\mu\nu}dV+\vec{F}^\nu{}_\alpha\cdot\int\delta x^\mu\vec{J}^\alpha dV\,. \tag{18}$$

We introduce now the notation ($u^0=dt/ds$)

$$M^{\alpha\beta}=u^0\int T^{\alpha\beta}dV,\quad M^{\lambda\alpha\beta}=-u^0\int\delta x^\alpha T^{\alpha\beta}dV \tag{19}$$

and

$$\vec{N}^\alpha=u^0\int\vec{J}^\alpha dV,\quad \vec{N}^{\lambda\alpha}=-u^0\int\delta x^\lambda\vec{J}^\alpha dV\,. \tag{20}$$

Note that

$$M^{0\alpha\beta}=0,\quad \vec{N}^{0\beta}=0\,. \tag{21}$$

Also,

$$\vec{q}=\frac{\vec{N}^0}{u^0}=\int\vec{J}^0dV,\quad \vec{d}^j=-\frac{\vec{N}^{j0}}{u^0}=\int\delta x^j\vec{J}^0dV \tag{22}$$

are the SU(2) charge and electric dipole moment, respectively.

With $u^\alpha=dX^\alpha/ds$, Eqs. (17), (18), and (15) become

$$\frac{d}{ds}\frac{M^{0\nu}}{u^0}=\vec{F}^\nu{}_\alpha\cdot\vec{N}^\alpha-\partial_\beta\vec{F}^\nu{}_\alpha\cdot\vec{N}^{\beta\alpha}\,, \tag{23}$$

$$u^\mu\frac{M^{0\nu}}{u^0}-\frac{d}{ds}\frac{M^{\mu0\nu}}{u^0}=M^{\mu\nu}-\vec{F}^\nu{}_\alpha\cdot\vec{N}^{\mu\alpha}\,, \tag{24}$$

$$u^\mu\frac{M^{\beta0\nu}}{u^0}+u^\beta\frac{M^{\mu0\nu}}{u^0}=M^{\beta\mu\nu}+M^{\mu\beta\nu}\,. \tag{25}$$

Before we go on with these equations we derive those that follow from (3). From this equation we have

$$\partial_\nu(x^\alpha\vec{J}^\nu)=\vec{J}^\alpha+x^\alpha\vec{A}^\nu\times\vec{J}_\nu \tag{26}$$

and

$$\partial_\nu(x^\alpha x^\beta\vec{J}^\nu)=x^\beta\vec{J}^\alpha+x^\alpha\vec{J}^\beta+x^\alpha x^\beta\vec{A}^\nu\times\vec{J}_\nu\,. \tag{27}$$

Space integration of Eq. (3) and of (26) and (27) gives

$$\frac{d}{dt}\int\vec{J}^0dV=\int\vec{A}_\nu\times\vec{J}^\nu dV\,, \tag{28}$$

$$\frac{d}{dt}\int x^\alpha\vec{J}^0dV=\int\vec{J}^\alpha dV + \int x^\alpha\vec{A}_\nu\times\vec{J}^\nu dV\,, \tag{29}$$

$$\frac{d}{dt}\int x^\alpha x^\beta\vec{J}^0=\int x^\beta\vec{J}^\alpha dV+\int x^\alpha\vec{J}^\beta dV + \int x^\alpha x^\beta\vec{A}_\nu\times\vec{J}^\nu dV\,. \tag{30}$$

Now we proceed as before. We use (13) in (29) and make use of (28), then in Eq. (30) first for x^α and afterwards for x^β and make use of (29) each time. Next we expand $\vec{A}^\nu(x)$ around X^α:

$$\vec{A}^\nu(x)=\vec{A}^\nu(X)+\delta x^\beta\partial_\beta\vec{A}^\nu(X)+\cdots\,. \tag{31}$$

Keeping only terms to first order in δx^α we obtain, in the notation (20), the following set of equations analogous to Eqs. (23)–(25):

$$\frac{d}{ds}\frac{\vec{N}^0}{u^0}=\vec{A}_\nu\times\vec{N}^\nu-\partial_\beta\vec{A}_\nu\times\vec{N}^{\beta\nu}\,, \tag{32}$$

$$\frac{u^\alpha}{u^0}\vec{N}^0-\frac{d}{ds}\frac{\vec{N}^{\alpha 0}}{u^0}=\vec{N}^\alpha-\vec{A}_\nu\times\vec{N}^{\alpha\nu}\,, \tag{33}$$

$$\frac{u^\alpha}{u^0}\vec{N}^{\beta 0}+\frac{u^\beta}{u^0}\vec{N}^{\alpha 0}=\vec{N}^{\beta\alpha}+\vec{N}^{\alpha\beta}\,. \tag{34}$$

Substituting the value of $\vec{N}^\alpha$ obtained from Eq. (33) in Eq. (23) one gets

$$\frac{d}{ds}\left[\frac{M^{0\nu}}{u^0}+\vec{F}^\nu{}_\alpha\cdot\frac{\vec{N}^{\alpha 0}}{u^0}\right]=\vec{F}^\nu{}_\alpha\cdot\vec{N}^0\frac{u^\alpha}{u^0}-\partial_\beta\vec{F}^\nu{}_\alpha\cdot\left[\vec{N}^{\beta\alpha}-\frac{u^\beta}{u^0}\vec{N}^{\alpha 0}\right]-\vec{A}_\beta\times\vec{F}^\nu{}_\alpha\cdot\vec{N}^{\alpha\beta}\,. \tag{35}$$

Aiming to obtain a covariant derivative for $\vec{F}^\nu{}_\alpha$ we express $\vec{N}^{\alpha\beta}$ in the last term of (35) in terms of $\vec{N}^{\beta\alpha}$ with the help of Eq. (34). The last term of (35) becomes

$$\vec{A}_\beta\times\vec{F}^\nu{}_\alpha\cdot\vec{N}^{\alpha\beta}=-(\vec{A}_\beta\times\vec{F}^\nu{}_\alpha)\cdot\left[\vec{N}^{\beta\alpha}-\frac{u^\beta}{u^0}\vec{N}^{\alpha 0}\right]+\left[\frac{\vec{N}^{\beta 0}}{u^0}\times\vec{A}_\beta\right]\cdot\vec{F}^\nu{}_\alpha u^\alpha\,. \tag{36}$$

Equation (35) can now be written as

$$\frac{dp^\nu}{ds}=\vec{Q}\cdot\vec{F}^\nu{}_\alpha u^\alpha-D_\beta\vec{F}^\nu{}_\alpha\cdot\vec{j}^{\alpha\beta}\,, \tag{37}$$

where

$$p^\nu=\frac{M^{0\nu}}{u^0}+\vec{F}^\nu{}_\alpha\cdot\frac{\vec{N}^{\alpha 0}}{u^0} \tag{38}$$

is the generalized momentum,

$$\vec{Q}=\frac{\vec{N}^0}{u^0}+\frac{1}{u^0}\vec{A}_\beta\times\vec{N}^{\beta 0}=\vec{q}-\vec{A}_j\times\vec{d}^j \tag{39}$$

is the generalized SU(2) charge, and

$$\vec{j}^{\alpha\beta}=\vec{N}^{\beta\alpha}-\frac{u^\beta}{u^0}\vec{N}^{\alpha 0}\,. \tag{40}$$

Using Eq. (34) we can see that $\vec{j}^{\alpha\beta}$ is an antisymmetric quantity,

$$\vec{j}^{\alpha\beta}=\tfrac{1}{2}(\vec{N}^{\beta\alpha}-\vec{N}^{\alpha\beta})+\frac{1}{2u^0}(u^\alpha\vec{N}^{\beta 0}-u^\beta\vec{N}^{\alpha 0})\,. \tag{41}$$

From Eqs. (40), (41), and (20) we have

$$j^{i0}=-\vec{N}^{i0}=u^0\int\delta x^i\vec{J}^0dV=u^0\vec{d}^i\,, \tag{42}$$

which is the electric dipole moment of the system in its rest frame and

$$\vec{j}^{ij}=\frac{1}{2u^0}\int(\delta x^i\vec{J}^j-\delta x^j\vec{J}^i)dV+\tfrac{1}{2}(u^i\vec{d}^j-u^j\vec{d}^i)\,, \tag{43}$$

which reduces to the usual nonrelativistic magnetic dipole moment in the particle rest frame. Note that from (40) and (42) we have

$$\vec{N}^{\beta\alpha}=\vec{j}^{\alpha\beta}-\frac{u^\beta}{u^0}\vec{j}^{\alpha 0}\,. \tag{44}$$

To obtain the spin equation we interchange μ and ν in Eq. (24) and subtract the resulting equation from Eq. (24) itself. Keeping in mind that $M^{\mu\nu}=M^{\nu\mu}$ we obtain the equation

$$\frac{u^\mu}{u^0}M^{0\nu}-\frac{u^\nu}{u^0}M^{0\mu}+\frac{d}{ds}S^{\mu\nu}=\vec{F}^\nu{}_\alpha\cdot\vec{N}^{\mu\alpha}-\vec{F}^\mu{}_\alpha\cdot\vec{N}^{\nu\alpha}\,, \tag{45}$$

where

$$S^{\mu\nu}=\frac{M^{\nu 0\mu}-M^{\mu 0\nu}}{u^0}=\int(\delta x^\mu T^{0\nu}-\delta x^\nu T^{0\mu})dV \tag{46}$$

is the spin of the system. Note that

$$S^{\mu 0}=-\frac{M^{\mu 00}}{u^0}=\int\delta x^\mu T^{00}dV\,. \tag{47}$$

As shown in Ref. 4 $S^{\mu\nu}$ is a tensor and by the same argument there used for the usual $\vec{J}^{\alpha\beta}$ current moment, $\vec{j}^{\alpha\beta}$ is also a tensor.

Using Eq. (35) and recalling (38), Eq. (45) can be written as

$$\frac{dS^{\mu\nu}}{ds}=p^\mu u^\nu-p^\nu u^\mu+\vec{F}^\mu{}_\alpha\cdot\vec{j}^{\alpha\nu}-\vec{F}^\nu{}_\alpha\cdot\vec{j}^{\mu\alpha}\,. \tag{48}$$

This is the final covariant form of the spin equation. By the tensor character of $S^{\mu\nu}$, $\vec{F}^{\mu\alpha}$, and $\vec{J}^{\alpha\nu}$ we conclude that p^μ is a four-vector. Setting $\nu=0$ in Eq. (24) we can express $M^{0\nu}=M^{\nu 0}$ and consequently p^ν in terms of M^{00}. We have from (24), (47), and (44)

$$M^{\mu 0}=\frac{u^\mu}{u^0}(M^{00}-\vec{F}^0{}_\alpha\cdot\vec{j}^{\alpha 0})+\frac{dS^{\mu 0}}{ds}+\vec{F}^0{}_\alpha\cdot\vec{j}^{\alpha\mu}\,, \tag{49}$$

and introducing this in Eq. (38) we have

$$p^{\mu}=mu^{\mu}+\frac{dS^{\mu 0}}{ds}+\vec{F}^{0}{}_{\alpha}\cdot\vec{J}^{\alpha\mu}-\vec{F}^{\mu}{}_{\alpha}\cdot\vec{J}^{\alpha 0}\ , \qquad (50)$$

where

$$m=\frac{M^{00}}{u^{0}}-\frac{1}{u^{0}}\vec{F}^{0}{}_{\alpha}\cdot\vec{j}^{\alpha 0}$$
$$=\frac{1}{u^{0}}\int T^{00}dV-\vec{F}^{0}{}_{i}\cdot\vec{d}^{i} \qquad (51)$$

is the total mass of our system. Note that

$$p^{0}=mu^{0}\ . \qquad (52)$$

As p^{μ} is a four-vector m is a scalar.

Next $M^{\mu\nu}$ and $M^{\lambda\mu\nu}$ are expressed in terms of the dynamical (tensor) variables. Introducing Eq. (49) in (24) we obtain

$$M^{\mu\nu}=u^{\mu}p^{\nu}-\frac{d}{ds}\frac{M^{\mu\nu 0}}{u^{0}}+\vec{F}^{\nu}{}_{\alpha}\cdot\vec{j}^{\alpha\mu}\ , \qquad (53)$$

where use has been made of Eqs. (44), (50), and (51).

Next we concentrate on Eq. (25). This equation gives $M^{\beta\mu\nu}$ in terms of the spin by the following procedure.[3] Add to Eq. (25) the equation obtained by exchanging β and ν and subtract the equation obtained by exchanging μ and ν. A sum, $M^{\beta 0\nu}+M^{\nu 0\beta}$, will be present in the final equation that can be expressed in terms of $M^{\beta 00}=-u^{0}S^{\beta 0}$. The final result is

$$M^{\beta\mu\nu}=-\tfrac{1}{2}(S^{\beta\mu}u^{\nu}+S^{\beta\nu}u^{\mu})$$
$$+\frac{u^{\beta}}{2u^{0}}(u^{\mu}S^{0\nu}+u^{\nu}S^{0\mu})\ . \qquad (54)$$

From here we have

$$M^{\beta\mu 0}=-\tfrac{1}{2}(S^{\beta\mu}u^{0}+S^{\beta 0}u^{\mu}+u^{\beta}S^{\mu 0})\ . \qquad (55)$$

Taking this result in Eq. (53) we obtain

$$M^{\mu\nu}=u^{\mu}p^{\nu}+\frac{1}{2}\frac{dS^{\mu\nu}}{ds}$$
$$+\frac{1}{2}\left[\frac{u^{\nu}}{u^{0}}S^{\mu 0}-\frac{u^{\mu}}{u^{0}}S^{0\nu}\right]+\vec{F}^{\nu}{}_{\alpha}\cdot\vec{j}^{\alpha\mu}\ . \qquad (56)$$

Equations (56) and (55) give, respectively, the volume integral of $T^{\mu\nu}$ and of its first moment in terms of the dynamical variables of our problem.

Consider now Eq. (33) and concentrate on its last term. Using Eqs. (40) and (2) this term can be written

$$\partial_{\beta}\vec{A}_{\nu}\times\vec{N}^{\beta\nu}=\tfrac{1}{2}(\vec{F}_{\beta\nu}+\vec{A}_{\beta}\times\vec{A}_{\nu})\cdot\vec{j}^{\nu\beta}$$
$$+\frac{d\vec{A}_{\nu}}{ds}\times\frac{\vec{N}^{\nu 0}}{u^{0}}\ . \qquad (57)$$

From Eqs. (33), (40), and (39) we also have

$$\vec{N}^{\alpha}=u^{\alpha}\vec{Q}+\frac{d}{ds}\frac{\vec{j}^{\alpha 0}}{u^{0}}+\vec{A}_{\nu}\times\vec{j}^{\nu\alpha}\ . \qquad (58)$$

Substituting Eqs. (57) and (58) in (32) and using the Jacobi identity for $\vec{A}_{\beta}$, $\vec{A}_{\nu}$, and $\vec{j}^{\nu\beta}$ together with the antisymmetric character of this last quantity one obtains the equation

$$\frac{D\vec{Q}}{Ds}=\frac{d\vec{Q}}{ds}-u^{\alpha}\vec{A}_{\alpha}\times\vec{Q}=\tfrac{1}{2}\vec{F}_{\beta\nu}\times\vec{j}^{\beta\nu}\ , \qquad (59)$$

where $\vec{Q}$ is the SU(2) generalized charge defined in Eq. (39).

Equations (37), (48), and (59) generalize the equations for a point spinless particle obtained in Ref. 1.

Using (39) and (42), Eq. (58) can be written as

$$\vec{N}^{\alpha}=u^{\alpha}\vec{q}\frac{D}{Ds}\frac{\vec{j}^{\alpha 0}}{u^{0}}+\vec{A}_{\nu}\times\vec{j}^{\nu\alpha}\ . \qquad (60)$$

Equations (60) and (44) give, respectively, the space integral of the SU(2) current $\vec{J}^{\alpha}$ and of its first moment in terms of the electromagnetic dipole moment tensor $\vec{j}^{\alpha\beta}$.

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