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# Multiple representations of extended objects

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The two methods of quantizing scalar field theories in the soliton sector currently in use in the literature, one developed by Christ and Lee, the other by Matsumoto and Umezawa, are examined simultaneously. It is shown that both may be derived by the same general technique, but correspond to different adiabatic-switching prescriptions. The Christ-Lee switching leads to an interaction picture which includes bound-state modes and the collective coordinate, while the Matsumoto-Umezawa interaction picture uses the standard massive free field. The asymptotic ground states in both cases are realized as coherent states, in the Christ-Lee method constructed from the exact classical solution, in the Matsumoto-Umezawa method built around a function which satisfies a simple inhomogeneous equation. It is shown that both representations yield identical results when used to calculate unrenormalized Green's functions. The Lehmann-Symanzik-Zimmermann reduction formulas are developed for both approaches and used to show the two methods predict different results for particle-soliton scattering. Advantages and drawbacks of both approaches are discussed, as well as extensions to multisoliton problems.

## I. INTRODUCTION

Considerable effort has been spent in investigating the classical solutions to the equations of motion for various field theories.[1] Generically known as solitons or extended objects, these configurations may be interpreted as nonlocal manifestations of the local dynamics and the structure of the ground state. It is believed that the soliton represents a sector of the theory inaccessible to standard perturbative techniques.

While the attendant problem of quantizing the parent theory in the presence of the soliton has been studied exhaustively,[2] and considerable progress has been made in the multisoliton problem,[3] an interesting and unusual question has arisen. Examination of the literature shows that there currently exist two approaches to canonical quantization in the soliton sector for simple scalar theories. Each gives a unique method for determining the Feynman rules, propagators, and vertices, in the soliton sector, and a cursory examination reveals little similarity between the two. The first method, developed by Christ, Lee, *et al.*[4] (hereafter referred to as CL), consists of a prescription for translating the fields in the action by the exact classical solution and self-consistently decomposing the interaction-picture fields around the principal modes which remove the linear and quadratic terms. The second method, developed by Matsumoto, Umezawa, *et al.*[5] (hereafter referred to to as MU), consists of representing the Heisenberg field of the theory in terms of a functional power series in the free field. The Heisenberg field in the soliton sector is then found by translating the free field by a function which satisfies the same equation of motion as the free field and resumming the series. This method can be shown to yield a classical solution of the original equation of motion in the tree approximation for the vacuum expectation value of the Heisenberg field. In addition, both methods have been formulated in terms of a path integral for the time-ordered products of fields in the soliton sector.[6]

It is the intent of this paper to examine both methods to determine whether these two formulations are equivalent and, if they are not, to find their respective domains of validity. For the purposes of this paper the standard techniques of perturbation theory will be employed.[7] In so doing it is necessary to select an asymptotic particle spectrum with which to populate the scattering states. This is determined by adiabatically switching off the interaction, either completely or partially, and using the remaining part of the Hamiltonian $H_0$ to define a complete set of scattering eigenstates. In order to reflect the presence of the soliton in the ground state, the asymptotic vacuums will be represented by a coherent state[8] constructed using the particle operators which create eigenstates of $H_0$ and a function determined self-consistently. Once these steps are completed it is straightforward to find the evolution operator, which interpolates between these asymptotic vacuums, in terms of a set of time-ordered products of interaction-picture fields *which lie between Fock vacuums, not the coherent states.* The interaction-picture fields must satisfy an equation of motion uniquely determined from $H_0$. It is then possible to examine the vacuum graphs of the theory to determine if the conditions of the Gell-Mann—Low theorem[9] can be met so that the adiabatic-switching parameter may be allowed to vanish.

It will be shown that this method allows both the CL and MU perturbation series to be derived, and it shows that their difference arises from selection of a different adiabatic-switching condition. The CL method corresponds to retaining a piece of the interaction in the asymptotic region which allows the coherent-state function to be an exact classical solution. On the other hand, the MU series is developed by switching to a standard free Hamiltonian and using a coherent state built around a function which satisfies a simple *inhomogeneous* equation of motion. The form of this equation of motion is determined by examining the Yang-Feldman equation[10] for the interpolating Heisenberg field derived from the evolution operator. Demanding that the Heisenberg field satisfy the original equation of motion, and that the coherent-state function interpolate between different constant solutions of the original equation of motion and have a Fourier transform,[11] uniquely determines this function's equation of motion. These restrictions alleviate certain difficulties in the MU method discussed elsewhere.[12]

When the vacuum graphs of the switched theory are examined it is seen that both methods satisfy the criteria of the Gell-Mann—Low theorem, and thus both methods can be used to generate an eigenstate of the full Hamiltonian even when the switching parameter is extinguished. However, the vacuum graphs in the MU theory are given by a phase with an essential singularity in the coupling constant, whereas the CL theory does not have this property. This is symptomatic of the fact that the asymptotic particles of the MU approach scatter off the soliton, while those of the CL method do not. It will be shown that the particle operators of the CL theory can be written as functionals of the particle operators in the MU theory as long as the normal modes of both theories' interaction pictures span the same space of functions, a condition usually satisfied.

It will be seen that, modulo the difference in vacuum graphs and with certain convergence conditions met, the MU series representation of the unrenormalized Green's functions can be resummed to the CL series for the same unrenormalized Green's function once the adiabatic-switching parameter is allowed to vanish. In addition, it will be shown that there exists a denumerable infinity of representations "intermediate" to the MU and CL representations. The question of the equivalence or lack thereof for the renormalized Green's functions will not be resolved in this paper; however, the way is clear to investigate this aspect of the problem using the techniques developed for resumming the bare series.

The remainder of this paper can now be outlined. In Sec. II, the general class of theories to be analyzed is defined and the respective adiabatic-switching conditions are fixed. The interaction-picture fields are constructed, paying special attention to the translation mode and collective coordinate,[13] and the asymptotic ground states are defined in terms of coherent states. The evolution operators for both approaches are developed, and from there the Yang-Feldman equations are derived, so that the equation of motion for the coherent-state function in the MU approach can be found. In Sec. III, the unrenormalized MU series is resummed using functional techniques to find the conditions under which it is equivalent to the unrenormalized CL series. Certain aspects of renormalization are discussed, and the extension to a determination of equivalence for the renormalized Green's functions is sketched. In Sec. IV, the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas for scattering in the presence of the soliton are derived for both approaches and discussed. In Sec. V the results are discussed for their utility to computation in the soliton sector. The advantages and drawbacks of both approaches are evaluated, and extensions of the work presented in this paper are suggested.

## II. THE PERTURBATION SERIES

In this section the two forms for the perturbation series in the soliton sector discussed in the Introduction will be derived using standard operator methods. There is no intention of claiming that the results presented in this section are new; rather, the intent is to show that both results can arise quite naturally from the same general technique. This is a necessary step in order to open the way to their comparison. In order to begin this process the class of theories to be considered will be specified.

### A. Preliminaries and notation

Attention will be restricted in this paper to simple scalar field theories where the action has the general form

$$A=\int d^nx\,dt[-\tfrac{1}{2}(\partial_\mu\psi)^2+\mathscr{L}_I(\psi)]\ . \tag{2.1}$$

It is assumed that $\mathscr{L}_I$ contains no derivatives of the field $\psi$, and that there are $n$ spatial dimensions. From (2.1) the equation of motion

$$\Box\psi=\frac{\partial}{\partial\psi}\mathscr{H}_I(\psi) \tag{2.2}$$

and the Hamiltonian

$$H=\int d^nx[\tfrac{1}{2}\dot{\psi}^2+\tfrac{1}{2}(\vec{\nabla}\psi)^2+\mathscr{H}_I(\psi)] \tag{2.3}$$

are derived, where $\mathscr{H}_I=-\mathscr{L}_I$. The theory is quantized by the relation

$$[\psi(\vec{x},t),\dot{\psi}(\vec{y},t)]_-=i\delta^n(\vec{x}-\vec{y})\ . \tag{2.4}$$

It will be further assumed that the equation

$$\frac{\partial}{\partial v}\mathscr{H}_I(v)=0 \tag{2.5}$$

admits a possibly denumerably infinite set of distinct constant solutions such that

$$\left.\frac{\partial^2}{\partial v^2}\mathscr{H}_I(v)\right|_{v=v_i}=m^2,\quad\forall i\ , \tag{2.6}$$

where $m^2$ is some positive constant, which is the same for each value $v_i$ which satisfies (2.5). Furthermore, it will be assumed that there exists a set of classical static functions $\phi_0(\vec{x})$ to (2.2) which interpolate between two or more different $v_i$. The standard examples of such theories are the $\phi^4$ model

$$\mathscr{H}_I(\phi)=-\tfrac{1}{2}\alpha^2\phi^2+\tfrac{1}{4}\lambda\phi^4 \tag{2.7a}$$

and the sine-Gordon model

$$\mathscr{H}_I(\phi)=\lambda\cos(\alpha\phi)\ . \tag{2.7b}$$

### B. The interaction picture

In order to select the asymptotic particle spectrum, the Hamiltonian (2.3) is replaced by a time-dependent Hamiltonian through an adiabatic-switching procedure. In the CL approach the Hamiltonian density is rewritten as

$$\mathscr{H}^\epsilon_{\rm CL}=\tfrac{1}{2}\left[\dot{\psi}^2+(\vec{\nabla}\psi)^2+\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\psi^2\right]+e^{-\epsilon|t|}\left[\mathscr{H}_I(\psi)-\tfrac{1}{2}\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\psi^2\right]\ , \tag{2.8a}$$

where the second derivative of the interaction Hamiltonian is evaluated at the classical solution of (2.2) of interest. In the MU method the Hamiltonian is switched as

$$\mathscr{H}^\epsilon_{\rm MU}=\tfrac{1}{2}[\dot{\psi}^2+(\vec{\nabla}\psi)^2+m^2\psi^2]+e^{-\epsilon|t|}[\mathscr{H}_I(\psi)-\tfrac{1}{2}m^2\psi^2]\ , \tag{2.8b}$$

where $m^2$ is given by (2.6). Note that at $t=0$ or when $\epsilon=0$, both (2.8a) and (2.8b) coincide with the original expression (2.3).

By selecting different adiabatic-switching prescriptions the two methods cause the Heisenberg field to approach different limits at asymptotic times. The limit of $\psi$ in the CL method is denoted

$$\lim_{t\to\pm\infty}\psi=\tilde{\phi}\ , \tag{2.9a}$$

while for the MU prescription

$$\lim_{t\to\pm\infty}\psi=\phi\ . \tag{2.9b}$$

From the fact that the right-hand side of (2.4) is time independent it follows that both (2.9a) and (2.9b) must satisfy the commutation relation

$$[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)]_-=[\tilde{\phi}(\vec{x},t),\dot{\tilde{\phi}}(\vec{y},t)]_-=i\delta^n(\vec{x}-\vec{y})\ . \tag{2.10}$$

It is then possible to find the respective equations of motion for (2.9a) by taking the large-time limit of the relation

$$\ddot{\psi}=-[H^\epsilon,[H^\epsilon,\psi]] \tag{2.11}$$

and using (2.9) and (2.10) along with the respective Hamiltonians from (2.8). This gives

$$(\Box - m^2)\phi = 0 \tag{2.12a}$$

and

$$\left[\Box - \frac{\partial^2}{\partial \phi_0{}^2}\mathscr{H}_I(\phi_0)\right]\tilde{\phi} = 0 . \tag{2.12b}$$

Combining (2.10) and (2.12) shows that the fields must be expanded around the complete set of eigenfunctions which satisfy (2.12), and that creation and annihilation operators must be introduced for every eigenfunction in the complete set. The time development of these fields will be given by the free Hamiltonians, which can be written in terms of these interaction-picture fields. For the MU approach this gives

$$H_0 = \tfrac{1}{2}\int d^n x[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] , \tag{2.13a}$$

while the CL method gives

$$\tilde{H}_0 = \tfrac{1}{2}\int d^n x\left[\dot{\tilde{\phi}}^2 + (\vec{\nabla}\tilde{\phi})^2 + \frac{\partial^2}{\partial \phi_0{}^2}\mathscr{H}_I(\phi_0)\tilde{\phi}^2\right] . \tag{2.13b}$$

This procedure is straightforward for the case (2.12a), so that

$$\phi(x) = \int \frac{d^n k}{(2\pi)^{n/2}}(2\omega_{\vec{k}})^{-1/2}(a_{\vec{k}}e^{ikx} + a_{\vec{k}}^{\dagger}e^{-ikx}) , \tag{2.14a}$$

where

$$\omega_{\vec{k}}{}^2 = \vec{k}^2 + m^2 \tag{2.14b}$$

and

$$[a_{\vec{k}}, a_{\vec{p}}^{\dagger}] = \delta^n(\vec{k} - \vec{p}) . \tag{2.14c}$$

The case (2.12b) is more complicated. In the event that a static solution $\phi_0$ is being considered, the eigenfunctions which satisfy (2.12b) will be separable and have the form

$$u(\vec{x},t) = u(\vec{x})e^{i\tilde{\omega}t} . \tag{2.15}$$

Under these conditions Eq. (2.12b) reduces to the time-independent eigenvalue equation

$$\left[-\vec{\nabla}^2 + \frac{\partial^2}{\partial \phi_0{}^2}\mathscr{H}_I(\phi_0)\right]u(\vec{x}) = \tilde{\omega}^2 u(\vec{x}) . \tag{2.16}$$

It will be assumed that the potential appearing in (2.16) is sufficiently well behaved that there is a complete set of orthonormal eigenfunctions corresponding to real values of $\tilde{\omega}$. In general there will be a continuum of "scattering" eigenfunctions $\{u_{\vec{k}}(\vec{x})\}$ and a discrete set of "bound-state" eigenfunctions. Among the latter there will be a group of eigenfunctions, whose number will match the spatial dimension of the theory, corresponding to $\tilde{\omega} = 0$. This may be seen by applying the gradient to the static equation of motion for $\phi_0$ to obtain

$$\begin{aligned} 0 &= \vec{\nabla}\left[\nabla^2\phi_0 - \frac{\partial}{\partial \phi_0}\mathscr{H}_I(\phi_0)\right] \\ &= \left[\nabla^2 - \frac{\partial^2}{\partial \phi_0{}^2}\mathscr{H}_I(\phi_0)\right]\vec{\nabla}\phi_0 , \end{aligned} \tag{2.17}$$

matching Eq. (2.16). These are the well-known translation modes of the theory and, after proper normalization, will be denoted $\vec{u}_T(\vec{x})$. The set of discrete eigenfunctions will be denoted $\{\vec{u}_T(\vec{x}) : u_l(\vec{x})\}$, where the $u_l(\vec{x})$ are those discrete eigenfunctions occurring in addition to the $\vec{u}_T(\vec{x})$. Their completeness implies

$$\vec{u}_T(\vec{x})\cdot\vec{u}_T(\vec{y}) + \sum_l u_l(\vec{x})u_l(\vec{y}) + \int \frac{d^n k}{(2\pi)^n}u_{\vec{k}}(\vec{x})u_{\vec{k}}^*(\vec{y}) = \delta^n(\vec{x} - \vec{y}) , \tag{2.18}$$

while orthonormality requires that

$$\int d^n x\, u_j(\vec{x})u_l(\vec{x}) = \delta_{jl} \tag{2.19a}$$

and

$$\int d^n x\, u_{\vec{k}}(\vec{x})u_{\vec{p}}^*(\vec{x}) = \delta^n(\vec{k} - \vec{p}) , \tag{2.19b}$$

where the translation modes are included in (2.19a) and all other inner products vanish.

In order to satisfy (2.10) and (2.12b), $\tilde{\phi}$ is expanded as

$$\tilde{\phi}(x) = \vec{Q}(t)\cdot\vec{u}_T(\vec{x}) + \left[\sum_l \frac{\alpha_l}{(2\tilde{\omega}_l)^{1/2}}u_l(\vec{x})e^{-i\tilde{\omega}_l t} + \int \frac{d^n k}{(2\pi)^{n/2}}\frac{\alpha_{\vec{k}}}{(2\tilde{\omega}_{\vec{k}})^{1/2}}u_{\vec{k}}(\vec{x})e^{-i\tilde{\omega}_{\vec{k}}t} + \text{H.c.}\right] . \tag{2.20}$$

The $\alpha$ satisfies creation and annihilation operator commutation relations

$$[\alpha_l,\alpha_j^\dagger]=\delta_{lj}\ ,\qquad [\alpha_{\vec{k}},\alpha_{\vec{p}}^\dagger]=\delta^n(\vec{k}-\vec{p})\ , \tag{2.21}$$

with all other commutators vanishing. Because $\tilde{\omega}=0$ for the translation modes and because their presence is necessary for (2.10) to hold, the time-dependent Hermitian operator $\vec{Q}(t)$ must be introduced. This operator is referred to as the collective coordinate,[13] and has the property that

$$\frac{\partial}{\partial t}\vec{Q}(t)=\vec{P}(t)\ , \tag{2.22a}$$

where

$$[Q_l(t),P_j(t)]=i\delta_{lj} \tag{2.22b}$$

with $\vec{Q}$ and $\vec{P}$ commuting with all $\alpha$ operators. $\vec{P}$ is not the canonical momentum, denoted $\vec{P}^c$ to avoid confusion, which will be discussed later.

The expansions (2.14a) and (2.20) may be inserted into the respective free Hamiltonians (2.13) to obtain

$$H_0=\int d^nk\,\omega_{\vec{k}}a_{\vec{k}}^\dagger a_{\vec{k}} \tag{2.23a}$$

and

$$\tilde{H}_0=\tfrac{1}{2}\vec{P}^2+\sum_l\tilde{\omega}_l\alpha_l^\dagger\alpha_l+\int d^nk\,\tilde{\omega}_{\vec{k}}\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}\ . \tag{2.23b}$$

It follows that the time independence of $\tilde{H}_0$ requires that

$$\frac{\partial}{\partial t}\vec{P}(t)=0\ , \tag{2.24}$$

and thus an explicit calculation gives[14]

$$\vec{Q}(t)=\vec{Q}(0)+t\vec{P}\ , \tag{2.25}$$

a form which will be useful later.

The asymptotic ground states for the respective methods are written as a coherent state. In order to do this it is necessary to introduce a state which is cyclic with respect to the algebra of the respective operators. For the MU representation this state, denoted $|0\rangle$, assumed to have unit norm, has the property that

$$a_{\vec{k}}\,|0\rangle=0,\quad \forall\vec{k} \tag{2.26a}$$

and therefore,

$$H_0\,|0\rangle=0\ . \tag{2.26b}$$

For the CL method this state, denoted $|\tilde{0}\rangle$, has the properties that

$$\vec{Q}(0)\,|\tilde{0}\rangle=\alpha_l\,|\tilde{0}\rangle=\alpha_{\vec{k}}\,|\tilde{0}\rangle=0,\quad \forall l,\vec{k}\ . \tag{2.27}$$

Having made this selection it is to be noted that $|\tilde{0}\rangle$ is not an eigenstate of $\vec{P}$, due to relation (2.22b), and therefore it is also not an eigenstate of $\tilde{H}_0$. In the conclusion, the technique for using eigenstates of the Hamiltonian will be discussed. Because $\vec{Q}$ is not time independent, it is necessary to define a cyclic state at every time $t$ by the relation

$$|\tilde{0},t\rangle=e^{i\tilde{H}_0t}\,|\tilde{0}\rangle\ , \tag{2.28}$$

which has the obvious properties

$$\vec{Q}(t)\,|\tilde{0},t\rangle=\alpha_l\,|\tilde{0},t\rangle=\alpha_{\vec{k}}\,|\tilde{0},t\rangle=0,\quad \forall l,\vec{k}\ . \tag{2.29}$$

The necessity of introducing $|0\rangle$ and $|\tilde{0}\rangle$ will become clear when deriving the perturbation series.

The ground states of the two approaches are defined as coherent states.[8] In the MU formulation the ground state at time $t$ is defined as

$$V^{-1}(t)\,|0\rangle=\exp\left[-i\int d^nx\,\dot{\phi}(\vec{x},t)f(\vec{x})\right]|0\rangle\equiv|f,t\rangle\ . \tag{2.30}$$

The form of the static function $f(\vec{x})$ will be determined later. The state (2.30) has the property that

$$\langle f,t\,|\,\phi(\vec{x},t)\,|\,f,t\rangle=f(\vec{x})\ . \tag{2.31}$$

The CL approach differs only in its use of a static classical solution $\phi_0$. The ground state is defined as

$$\tilde{V}^{-1}(t)\,|\tilde{0},t\rangle=\exp\left[-i\int d^nx\,\dot{\tilde{\phi}}(\vec{x},t)\phi_0(\vec{x})\right]|\tilde{0},t\rangle\equiv|\phi_0,t\rangle\ , \tag{2.32}$$

and has the property that

$$\langle \phi_0,t \,|\, \tilde{\phi}(\vec{x},t) \,|\, \phi_0,t \rangle = \phi_0(\vec{x}) \; . \tag{2.33}$$

In order for these relations to hold, the existence of the integrals in (2.30) and (2.32) must be verified. It is clear that $f(\vec{x})$ must possess a well-defined Fourier transform because of the decomposition (2.14a), while the product of $\phi_0$ and the principal modes of $\tilde{\phi}$ must yield integrable functions. Under these conditions neither (2.30) nor (2.32) will be time independent.

### C. Time-ordered products and the evolution operator

In Sec. IV, the LSZ reduction formulas will be derived. For both approaches the Green's functions will be given in terms of time-ordered products of Heisenberg fields between the respective ground states (2.30) and (2.32) at large times. The MU Green's functions, derived from the asymptotic condition (2.9b), are of the form

$$G_{\rm MU}(x_1,x_2),\ldots = \langle f,t_+ \,|\, T\{\psi(x_1)\psi(x_2) \,|\, \cdots \} \,|\, f,t_- \rangle \; , \tag{2.34a}$$

while the CL counterparts, derived from (2.9a), are given by

$$G_{\rm CL}(x_1,x_2,\ldots) = \langle \phi_0,t_+ \,|\, T\{\psi(x_1)\psi(x_2)\cdots \} \,|\, \phi_0,t_- \rangle \; , \tag{2.34b}$$

where $t_+$ and $t_-$ are times far in the future and past, respectively.

The standard assumption[15] made to derive the perturbative representation of these amplitudes is that the interpolating Heisenberg fields are related to the interaction-picture fields by a unitary transformation, which thus preserves the canonical commutation relations. For the MU method

$$U(t)\psi(\vec{x},t)U^{-1}(t) = \phi(\vec{x},t) \; , \tag{2.35a}$$

while for the CL method

$$\tilde{U}(t)\psi(\vec{x},t)\tilde{U}^{-1}(t) = \tilde{\phi}(\vec{x},t) \; . \tag{2.35b}$$

It follows from the time development of the respective fields that

$$i\dot{U}(t)U^{-1}(t) = e^{-\epsilon|t|} \int d^n x \, [\mathscr{H}_I(\phi(\vec{x},t)) - \tfrac{1}{2}m^2\phi^2(\vec{x},t)] \equiv H_I(t) \tag{2.36a}$$

and

$$i\dot{\tilde{U}}(t)\tilde{U}^{-1}(t) = e^{-\epsilon|t|} \int d^n x \left[ \mathscr{H}_I(\tilde{\phi}(\vec{x},t)) - \frac{1}{2}\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\tilde{\phi}^2(\vec{x},t) \right] \equiv \tilde{H}_I(t) \; , \tag{2.36b}$$

where an arbitrary $c$-number dependence has been dropped.[16] From the adiabatic-switching conditions it is evident that

$$\lim_{|t|\to\infty} [U(t),\phi(\vec{x},t)] = \lim_{|t|\to\infty} [\tilde{U}(t),\tilde{\phi}(\vec{x},t)] = 0 \; . \tag{2.37}$$

From the definition of time ordering it follows that the amplitudes (2.34) can be rewritten as

$$\begin{aligned} G_{\rm MU}(x_1,x_2,\ldots) = \langle 0 \,|\, & V(t_+)U^{-1}(t_+)V^{-1}(t_+) \\ & \times T\{[\phi(x_1)+f(\vec{x}_1)][\phi(x_2)+f(\vec{x}_2)]\cdots V(t_+)U(t_+)U^{-1}(t_-)V^{-1}(t_-)\} \\ & \times V(t_-)U(t_-)V^{-1}(t_-)\,|\,0\rangle \end{aligned} \tag{2.38a}$$

and

$$\begin{aligned} G_{\rm CL}(x_1,x_2,\ldots) = \langle \tilde{0},t_+ \,|\, & \tilde{V}(t_+)\tilde{U}^{-1}(t_+)\tilde{V}^{-1}(t_+) \\ & \times T\{[\tilde{\phi}(x_1)+\phi_0(\vec{x}_1)][\tilde{\phi}(x_2)+\phi_0(\vec{x}_2)]\cdots \tilde{V}(t_+)\tilde{U}(t_+)\tilde{U}^{-1}(t_-)V^{-1}(t_-)\} \\ & \times \tilde{V}(t_-)\tilde{U}(t_-)V^{-1}(t_-)\,|\,\tilde{0},t_- \rangle \; . \end{aligned} \tag{2.38b}$$

This form is dictated by the fact that the operator selected to act on the asymptotic cyclic vacuum must give back a multiple of that state. Because the algebras generated by the fields $\phi$ and $\tilde{\phi}$ are irreducible in the respective free sectors and $U(t_\pm)$ and $\tilde{U}(t_\pm)$ commute with all operator products by (2.37), it follows that $U$ and $\tilde{U}$ must be multiples of the identity operator in the asymptotic region. It is then written

$$V(t_-)U(t_-)V^{-1}(t_-)\,|\,0\rangle = U(t_-)\,|\,0\rangle = \lambda_-\,|\,0\rangle \tag{2.39a}$$

and

$$\tilde{V}(t_-)\tilde{U}(t_-)\tilde{V}^{-1}(t_-)\,|\,\tilde{0},t_-\rangle = \tilde{U}(t_-)\,|\,\tilde{0},t_-\rangle = \tilde{\lambda}_-\,|\,\tilde{0},t_-\rangle \tag{2.39b}$$

with similar expressions holding at $t_+$ for the constants $\lambda_+$ and $\tilde{\lambda}_+$. It is now clear that the form (2.38) is necessary because there are many functions for which

$$\langle 0\,|\,V(t)\,|\,0\rangle = \langle \tilde{0},t\,|\,\tilde{V}(t)\,|\,\tilde{0},t\rangle = 0, \quad \forall t\ . \tag{2.40}$$

In particular the functions used in both approaches discussed here exhibit this property. Thus, unless the operator $V$ is removed from the asymptotic ground states, such a state could not be used for performing contractions simply. The constants $\lambda_\pm$ and $\tilde{\lambda}_\pm$ are removed when the vacuum transition amplitude is normalized. This will be exhibited explicitly.

Inspection of (2.38a) and (2.38b) shows that the evolution operators for the respective formulations are given by

$$Z(t,t') = V(t)U(t)U^{-1}(t')V^{-1}(t') \tag{2.41a}$$

and

$$\tilde{Z}(t,t') = \tilde{V}(t)\tilde{U}(t)\tilde{U}^{-1}(t')\tilde{V}^{-1}(t')\ . \tag{2.41b}$$

It follows for the MU formulation that

$$\frac{\partial}{\partial t}Z(t,t') = I(t)Z(t,t')\ , \tag{2.42}$$

where

$$I(t) = \dot{V}(t)V^{-1}(t) + V(t)\dot{U}(t)U^{-1}(t)V^{-1}(t)\ . \tag{2.43}$$

Expression (2.42) may be integrated and iterated using the definition of time ordering to obtain

$$Z(t,t') = T\left\{\exp\left[-i\int_{t'}^{t} dt''\, iI(t'')\right]\right\}. \tag{2.44}$$

Identical expressions for the CL method are obtained with the respective operators receiving a tilde.

The operators $I(t)$ and $\tilde{I}(t)$ can be evaluated using (2.30), (2.32), and (2.36). The CL method will be treated first. Clearly, it is easy to see that

$$i\tilde{V}(t)\dot{\tilde{U}}(t)\tilde{U}^{-1}(t)\tilde{V}^{-1}(t) = e^{-\epsilon|t|}\int d^n x\left[\mathscr{H}_I(\tilde{\phi}+\phi_0) - \frac{1}{2}\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)(\tilde{\phi}+\phi_0)^2\right] \tag{2.45a}$$

and

$$i\dot{\tilde{V}}(t)\tilde{V}^{-1}(t) = -\int d^n x\left[\ddot{\tilde{\phi}}\phi_0 - \frac{1}{2}\left[\phi_0{}^2\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0) - \phi_0\nabla^2\phi_0\right]\right]. \tag{2.45b}$$

Expression (2.45a) may be expanded in a power series in $\tilde{\phi}$ to obtain

$$i\tilde{I}(t) = \int d^n x\left\{e^{-\epsilon|t|}\left[\tilde{I}(\tilde{\phi},\phi_0) + \left[\frac{\partial}{\partial\phi_0}\mathscr{H}_I(\phi_0) - \phi_0\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\right]\tilde{\phi}\right] - \ddot{\tilde{\phi}}\phi_0 + C\right\}, \tag{2.46}$$

where $\tilde{I}(\tilde{\phi},\phi_0)$ represents the terms cubic and higher in $\tilde{\phi}$, while $C$ represents the collected $c$-number functions. Using the equation of motion (2.12b) and integrating by parts, assuming good spatial behavior, gives

$$i\tilde{I}(t)=\int d^nx\left[e^{-\epsilon|t|}\tilde{I}(\tilde{\phi},\phi_0)+(e^{-\epsilon|t|}-1)\left[\frac{\partial}{\partial\phi_0}\mathscr{H}_I(\phi_0)-\phi_0\frac{\partial^2}{\partial{\phi_0}^2}\mathscr{H}_I(\phi_0)\right]\tilde{\phi}+C\right]. \tag{2.47}$$

The last term in (2.47) is clearly proportional to $\epsilon$ for most values of $t$ and vanishes in the event that $\epsilon$ is zero. For these reasons it will be discarded. The diligent reader may ascertain that the presence of this term which is not completely switched in no way invalidates the Gell-Mann–Low theorem. Thus, the final form of (2.38b) is

$$G_{\rm CL}(x_1,x_2,\dots)=(\tilde{\lambda}_+\tilde{\lambda}_-)\left\langle\tilde{0},t_+\left|T\left\{[\tilde{\phi}(x_1)+\phi_0(\vec{x}_1)][\tilde{\phi}(x_2)+\phi_0(\vec{x}_2)]\cdots\right.\right.\right.$$
$$\left.\left.\left.\times\exp\left[-i\int_{t_-}^{t_+}dt\,d^nx\,e^{-\epsilon|t|}\tilde{I}(\tilde{\phi},\phi_0)+C\right]\right\}\right|\tilde{0},t_-\right\rangle. \tag{2.48}$$

This amplitude may be normalized by the factor which normalizes the vacuum transition element. This is done by demanding that

$$\tilde{N}(t_+,t_-)\langle\phi_0,t_+\,|\,\phi_0,t_-\rangle=1\ , \tag{2.49}$$

so that

$$[\tilde{N}(t_+,t_-)]^{-1}=(\tilde{\lambda}_+\tilde{\lambda}_-)\left\langle\tilde{0},t_+\left|T\left\{\exp\left[-i\int_{t_-}^{t_+}dt\,d^nx\,e^{-\epsilon|t|}\tilde{I}(\tilde{\phi},\phi_0)+C\right]\right\}\right|\tilde{0},t_-\right\rangle. \tag{2.50}$$

It is clear the $\tilde{N}(t_+,t_-)$ removes all $c$-number dependence as well as canceling the vacuum graph phase present in (2.48).

The way is then clear to apply the Dyson-Wick contraction scheme to (2.48). It is left as an exercise to show, using (2.20) and (2.25), that

$$\begin{aligned}\tilde{\Delta}(x-x')&=\langle\tilde{0},t_+\,|T\{\tilde{\phi}(x)\tilde{\phi}(x')\}\,|\tilde{0},t_-\rangle\\&=\langle\tilde{0},t_+\,|\,\tilde{0},t_-\rangle\left\{\theta(t-t')\left[i\frac{(t_+-t)(t_--t')}{(t_+-t_-)}\vec{u}_T(\vec{x})\cdot\vec{u}_T(\vec{x}')\right.\right.\\&\qquad+\sum_l(2\tilde{\omega}_l)^{-1}u_l(\vec{x})u_l(\vec{x}')e^{-i\tilde{\omega}_l(t-t')}\\&\qquad\left.+\int\frac{d^nk}{(2\pi)^n}(2\tilde{\omega}_{\vec{k}})^{-1}u_{\vec{k}}(\vec{x})u^*_{\vec{k}}(\vec{x}')e^{-i\tilde{\omega}_{\vec{k}}(t-t')}\right]\\&\quad+\theta(t'-t)\left[i\frac{(t_+-t')(t_--t)}{(t_+-t_-)}\vec{u}_T(\vec{x})\cdot\vec{u}_T(\vec{x}')\right.\\&\qquad+\sum_l(2\tilde{\omega}_l)^{-1}u_l(\vec{x})u_l(\vec{x}')e^{i\tilde{\omega}_l(t-t')}\\&\qquad\left.\left.+\int\frac{d^nk}{(2\pi)^n}(2\tilde{\omega}_{\vec{k}})^{-1}u^*_{\vec{k}}(\vec{x})u_{\vec{k}}(\vec{x}')e^{i\tilde{\omega}_{\vec{k}}(t-t')}\right]\right\},\end{aligned} \tag{2.51}$$

where the vacuum element

$$\langle\tilde{0},t_+\,|\,\tilde{0},t_-\rangle=[2\pi i(t_+-t_-)]^{-n/2} \tag{2.52}$$

may be ignored since it will be canceled by the normalizing factor $\tilde{N}(t_+,t_-)$, given by (2.50), which contains the same element. Because odd powers of $\vec{Q}$ vanish when their vacuum expectation value is taken, the Feynman rules consist of the vertices given by $\tilde{I}(\tilde{\phi},\phi_0)$ and the propagator of (2.51). It is shown in the Appendix, using some results of the next section, that these are equivalent to the CL rules.

At this point the Gell-Mann—Low theorem establishes that the state

$$|\tilde{p}\rangle_{\tilde{H}}=\tilde{Z}(0,t_-)\,|\tilde{p},t_-\rangle[\langle\tilde{p},t=0\,|\,\tilde{Z}(0,t_-)\,|\,\tilde{p},t_-\rangle]^{-1}\,, \tag{2.53}$$

where $|\tilde{p},t\rangle$ is an eigenstate of the operator $\vec{P}$, conjugate to the operator $\vec{Q}$, is an eigenstate of the operator

$$\tilde{H}=\tilde{H}_0+\int d^n x\,\tilde{I}(\tilde{\phi},\phi_0) \tag{2.54}$$

in the limit that $\epsilon$ vanishes. The application is straightforward and left to the reader. The operator $\tilde{H}$ is, modulo $c$-number differences, the full Hamiltonian (2.3) shifted by $\phi_0$, and written in terms of the interaction-picture field (2.20).

The evolution operator for the MU approach is developed similarly. It follows from (2.43) that

$$iI(t)=\int d^n x\{e^{-\epsilon|t|}[\mathscr{H}_I(\phi+f)-\tfrac{1}{2}m^2(\phi+f)^2]+\phi(\nabla^2-m^2)f\} \tag{2.55}$$

after an integration by parts *which precludes f from satisfying the static equation of motion* (2.12a).

At this point a second demand is placed upon $f$, and this is that it must interpolate between two or more different constant solutions to (2.5). In particular, to be topologically equivalent to the classical solution $\phi_0$ which it seeks to mimic, $f$ must interpolate between those same values which $\phi_0$ does. To reflect this, $f(\vec{x})$ is written

$$f(\vec{x})=v(\vec{x})+g(\vec{x})\,, \tag{2.56}$$

where $g(\vec{x})$ is assumed to vanish at spatial infinity, a property which will later be verified, and $v(\vec{x})$ consists of constant solutions to (2.5) prefaced by step functions. An example serves to clarify this statement. The two-value case for one-space dimension is written

$$v\equiv v(\vec{x})=v_i\theta(x-a)+v_j\theta(a-x)\,, \tag{2.57}$$

where $a$ is an arbitrary constant. For higher spatial dimensions the spherically symmetric version of the two-value case is written

$$v=v_i\theta(r-a)+v_j\theta(a-r)\,. \tag{2.58}$$

Generalization to many-value cases is obvious, e.g., the three-value case in one dimension,

$$v=v_i\theta(x-a)+v_j[\theta(a-x)-\theta(b-x)]+v_k\theta(b-x)\,, \tag{2.59}$$

where it is assumed that $a>b$. By inspection these forms are spatially dependent solutions to (2.5) which satisfy (2.6) over all space.

Expanding (2.55) about $v$ gives

$$iI(t)=\int d^n x\{C+e^{-\epsilon|t|}I(\phi+g,v)+\phi[(\nabla^2-m^2)f-e^{-\epsilon|t|}m^2 v]\}\,, \tag{2.60}$$

where $C$ represents the collected $c$-number functions and $I(\phi+g,v)$ represents the terms cubic and higher in $\phi+g$. These terms include parts linear and quadratic in $\phi$. Amplitude (2.38a) is then normalized by dividing by the factor

$$N(t_+,t_-)=\langle 0,t_+\,|\,0,t_-\rangle(\lambda_+\lambda_-)\left\langle 0\left|T\left\{\exp\left[-i\int_{t_-}^{t_+}dt\,iI(t)\right]\right\}\right|0\right\rangle\,. \tag{2.61}$$

As in the CL case the Gell-Mann—Low theorem may be applied to show that the state

$$|0\rangle_H\equiv Z(0,t_-)|0\rangle[\langle 0\,|\,Z(0,t_-)\,|\,0\rangle]^{-1} \tag{2.62}$$

is an eigenstate of the operator $H$ given by

$$H=H_0+\int d^nx\{I(\phi+g,v)+\phi[(\nabla^2-m^2)f-m^2v]\} \tag{2.63}$$

in the limit that $\epsilon$ vanishes. Again, $H$ is the Hamiltonian (2.3) shifted by $f$ and written in terms of the interaction-picture field (2.14a).

In order to find the equation of motion for $f$ it is necessary to examine the Yang-Feldman equation for $\psi$ in the soliton sector. In the MU approach this is given by the evolution operator $Z(t_+,t_-)$. It is straightforward to show[17]

$$\begin{aligned}\psi(x)-v(\vec{x})&\equiv\tilde{\psi}(x)\\&=\phi(x)+g(\vec{x})+\int_{t_-}^{t_+}dt'\,d^nx'\Delta_R(x-x')[I'(\tilde{\psi}(x'),v(\vec{x}'))+(\nabla^2-m^2)g(\vec{x}')+\nabla^2v(\vec{x}')]\ ,\end{aligned} \tag{2.64}$$

where $I'$ is the derivative of $I$, and $\Delta_R$ is the retarded Green's function satisfying

$$(\Box-m^2)\Delta_R(x-x')=\delta(t-t')\delta^n(\vec{x}-\vec{x}') \tag{2.65a}$$

and given by

$$\Delta_R(x-x')=i\theta(t-t')[\phi(x),\phi(x')]\ . \tag{2.65b}$$

Expression (2.64) shows that $\tilde{\psi}$, as opposed to $\psi$, is the operator generated by the perturbation series.

For the purposes of this paper the function $g$ can be given the general form

$$g(\vec{x})=g_0(\vec{x})+\bar{g}(\vec{x})\ , \tag{2.66}$$

where $g_0$ satisfies

$$(\nabla^2-m^2)g_0(\vec{x})=-\nabla^2v(\vec{x})\ , \tag{2.67}$$

while $\bar{g}$ is an arbitrary function which does not satisfy (2.67) but does possess a Fourier transform. It is easy to show that

$$\int_{t_-}^{t_+}dt'\,d^nx'\,\Delta_R(x-x')[(\nabla^2-m^2)g(\vec{x}')+\nabla^2v(\vec{x}')]=-\bar{g}(\vec{x})\ , \tag{2.68}$$

so that (2.64) becomes

$$\tilde{\psi}(x)=\phi(x)+g_0(\vec{x})+\int_{t_-}^{t_+}dt'\,d^nx\,\Delta_R(x-x')I'(\tilde{\psi}(x'),v(\vec{x}'))\ . \tag{2.69}$$

Result (2.69) shows that only $g_0$ contributes to the iteration of (2.64) for $\tilde{\psi}$. Thus $f(\vec{x})$ is fixed to be

$$f(\vec{x})=v(\vec{x})+g_0(\vec{x}) \tag{2.70}$$

and satisfies the *inhomogeneous* equation

$$(\nabla^2-m^2)f(\vec{x})=-m^2v(\vec{x})\ . \tag{2.71}$$

Result (2.69) is very similar to the MU formulation of the Yang-Feldman equation in the soliton sector. However, in the MU approach $f(\vec{x})$ is assumed to satisfy the homogeneous form of (2.71) and thus does not have a Fourier transform, creating difficulty in iterating (2.69). This problem is completely removed by demanding that $f(\vec{x})$ satisfy (2.71). Selecting (2.71) removes that equation from (2.60) and (2.63), so that the final form for (2.38a) is

$$G_{\rm MU}(x_1,x_2,\ldots)=\lambda_+\lambda_-\left\langle 0\left|T\left\{[\phi(x_1)+f(\vec{x}_1)]\cdots\exp\left[-i\int_{t_-}^{t_+}dt\,d^nx\,I(\phi+g_0,v)\right]\right\}\right|0\right\rangle\ . \tag{2.72}$$

The Yang-Feldman equation for $\psi$ in the CL approach can also be derived. It is given by

$$\psi(x)=\tilde{\phi}(x)+\phi_0(\vec{x})+\int_{t_-}^{t_+}dt'\,d^nx'\,\tilde{\Delta}_R(x-x')\tilde{I}'(\psi(x')-\phi_0(\vec{x}'),\phi_0(\vec{x}'))\ , \tag{2.73}$$

where $\widetilde{I}'$ is the derivative of $\widetilde{I}$ and the retarded Green's function has the representation

$$\widetilde{\Delta}_R(x-x')=i\theta(t-t')[\widetilde{\phi}(x),\widetilde{\phi}(x')] \ , \qquad (2.74a)$$

and satisfies

$$\left[\Box-\frac{\partial^2}{\partial^2\phi_0{}^2}\mathscr{H}_I(\phi_0)\right]\widetilde{\Delta}_R(x-x') =\delta(t-t')\delta^n(\vec{x}-\vec{x}') \ . \qquad (2.74b)$$

It is straightforward to show that the representation of $\psi$ obtained by either (2.69) or (2.73) satisfies the equation of motion (2.2).

An example of the MU method is useful. The $\phi^4$ theory, given by (2.7a) has been evaluated in one spatial dimension for the two-value form (2.57).[12] Thus $v(\vec{x})$ is given by

$$v(\vec{x})=v\theta(x-a)-v\theta(a-x) \ , \qquad (2.75a)$$

where

$$v=\left[\frac{\alpha^2}{\lambda}\right]^{1/2}, \ m^2=2\alpha^2 \ . \qquad (2.75b)$$

It is straightforward to show that

$$f(\vec{x})=v(\vec{x})-v\theta(x-a)e^{-m(x-a)} +v\theta(a-x)e^{-m(a-x)} \qquad (2.76)$$

satisfies the one-dimensional form of (2.71). The form for $I$ is

$$I(\phi+g_0,v)=\tfrac{1}{4}\lambda(\phi+g_0)^4+\lambda v(\vec{x})(\phi+g_0)^3 \ . \qquad (2.77)$$

Iteration of (2.69) using (2.76) and (2.77) gives, in the limit $\hbar$ vanishes ,

$$\langle 0\,|\,\widetilde{\psi}(x)+v(\vec{x})\,|\,0\rangle=v\tanh\frac{m}{\sqrt{2}}(x-a) \ , \qquad (2.78)$$

the standard kink solution of the classical theory.

### D. The canonical momentum operators

Selection of a spatially dependent ground state has broken translational invariance. This is easy to see from the relation

$$[\vec{P}^c,\psi(x)]=-i\vec{\nabla}\psi(x) \ , \qquad (2.79)$$

so that, if the expectation value of the right-hand side does not vanish, the states of the theory are not eigenstates of the canonical momentum operator. In the interaction picture or in the asymptotic region the canonical momentum operators have the representations

$$\vec{P}^c=\int d^nx\,\phi\overleftrightarrow{\partial}_t(\vec{\nabla}\phi), \ \overleftrightarrow{\partial}_t\equiv\frac{\vec{\partial}}{\partial t}-\frac{\overleftarrow{\partial}}{\partial t} \qquad (2.80a)$$

and

$$\widetilde{\vec{P}}^c=\int d^nx\,\widetilde{\phi}\overleftrightarrow{\partial}_t(\vec{\nabla}\widetilde{\phi}) \ , \qquad (2.80b)$$

in the MU and CL methods, respectively. A straightforward calculation shows that the ground states (2.30) and (2.32) are not eigenstates of these operators, in agreement with (2.79). A significant difference occurs in the algebra of the asymptotic operators. It is easy to show that

$$[H_0,\vec{P}^c]=0 \qquad (2.81a)$$

and

$$[\widetilde{H}_0,\widetilde{\vec{P}}^c]=\int d^nx\,\widetilde{\phi}^2\vec{\nabla}\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0) \ , \qquad (2.81b)$$

so that $\widetilde{\vec{P}}_c$ in the CL approach has an explicit time dependence. The fact that (2.81a) vanishes is a direct consequence of the existence of the state $|\,0\rangle$ which both operators annihilate. Relation (2.81b) is a manifestation of the fact that not even $|\,\widetilde{0},t\,\rangle$ is an eigenstate of these operators.

It follows, however, that

$$\langle f,t\,|\,\vec{P}^c\,|\,f,t\,\rangle=0 \qquad (2.82a)$$

and

$$\langle \phi_0,t\,|\,\widetilde{\vec{P}}^c(t)\,|\,\phi_0,t\,\rangle=0 \ , \qquad (2.82b)$$

because a static solution has been chosen. A time-dependent solution may be found by simply performing a Lorentz boost on the static solution to find

$$\phi_0(\vec{x},t)=\phi_0\left[\frac{\vec{x}-\vec{\beta}t}{(1-\beta^2)^{1/2}}\right] , \qquad (2.83a)$$

or, for the MU case, boosting the function $f$ to

$$f(\vec{x},t)=f\left[\frac{\vec{x}-\vec{\beta}t}{(1-\beta^2)^{1/2}}\right] . \qquad (2.83b)$$

If these solutions are used the coherent states are defined as

$$\tilde{V}^{-1}(t)\,|\,\tilde{0},t\rangle=\exp\left[-i\int d^n x(\dot{\tilde{\phi}}\phi_0-\tilde{\phi}\dot{\phi}_0)\right]|\,\tilde{0},t\rangle\equiv|\,\phi_0,t\rangle \tag{2.84a}$$

and

$$V^{-1}(t)\,|\,0\rangle=\exp\left[-i\int d^n x(\dot{\phi}f-\phi\dot{f})\right]|\,0\rangle\equiv|\,f,t\rangle\ . \tag{2.84b}$$

The analysis of the previous subsections may be repeated to find that these coherent states lead to boosted forms for $\tilde{\phi}$.

## III. RESUMMING THE MU SERIES

In this section the MU representation of the bare Green's functions will be resummed to find the conditions under which it is equivalent to the CL series for the same functions. This is done by removing all linear and quadratic vertices in the Feynman rules given by (2.72). The method which will be used to effect this procedure is a formal manipulation of the functional representation of the bare Green's functions. Furthermore, to simplify and clarify this process it will be done for a specific model; however, the generalization will be obvious. It must be stressed that this resummation could also be accomplished by manipulation of the operator representation, but such an approach would be much more tedious.

The model to be examined is the $\phi^4$ interaction given by (2.7a), (2.75b), and (2.77), although $\nu(\vec{x})$ will be allowed to be more general than (2.75a). In the MU approach the Green's functions are given by

$$\begin{aligned}G_{\rm MU}(x_1,\ldots)&=\langle f,t_+\,|\,T\{\psi(x_1)\cdots\}\,|\,f,t_-\rangle\\&=\left\langle 0\left|T\left\{[\phi(x_1)+f(\vec{x}_1)]\cdots\exp\left[-i\int_{t_-}^{t_+}dt\,d^n x\,\tfrac{1}{4}\lambda(\phi+g_0)^4+\lambda\nu(\phi+g_0)^3\right]\right\}\right|0\right\rangle\ .\end{aligned} \tag{3.1}$$

It is verified by inspection that (3.1) has the functional representation

$$\begin{aligned}G_{\rm MU}(x_1,\ldots)=&\left[\frac{\delta}{i\delta J(x_1)}+f(\vec{x}_1)\right]\cdots\\&\times\exp\left\{-i\int_{t_-}^{t_+}dt\,d^n x\left[\frac{1}{4}\lambda\left[\frac{\delta}{i\delta J(x)}+g_0(\vec{x})\right]^4+\lambda\nu(\vec{x})\left[\frac{\delta}{i\delta J(x)}+g_0(\vec{x})\right]^3\right]\right\}\\&\times\exp\left[-\tfrac{1}{2}i\int d^n x_1\,dt_1\,d^n x_2\,dt_2 J(x_1)\Delta^{(0)}(x_1-x_2)J(x_2)\right]\Big|_{J=0}\ ,\end{aligned} \tag{3.2}$$

where $\Delta^{(0)}$ is given by

$$\Delta^{(0)}(x_1-x_2)=\langle 0\,|\,T\{\phi(x_1)\phi(x_2)\}\,|\,0\rangle\ , \tag{3.3}$$

and it is understood that the series (3.2) is evaluated at $J=0$.

The second exponential appearing in (3.2) can be rewritten

$$\exp\left[-\tfrac{1}{2}i\int d^n x_1 dt_1 d^n x_2 dt_2 J(x_1)\Delta^{(0)}(x_1-x_2)J(x_2)\right]$$

$$=D_0^{-1}\int[d\phi]\exp\left\{i\int d^n x\,dt[\tfrac{1}{2}\phi(\Box-m^2)\phi+J\phi]\right\}\ . \tag{3.4}$$

The right-hand side of (3.4) requires clarification. It is the standard quadratic path integral with the configuration-space measure $[d\phi]$. Such a measure has been discussed elsewhere,[18] and is to be understood as the limit of a lattice measure as the lattice spacing vanishes. It is assumed that $m^2$ is given a small imaginary part which selects time-ordered Green's function (3.3). $D_0$ is a product of Gaussian integrals, and is given explicitly by

$$D_0=[\det\Delta^{(0)}(x-x')]^{1/2}\ . \tag{3.5}$$

Inserting (3.4) into (3.2) yields

$$G_{\rm MU}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+f(\vec{x}_1)\right]\cdots D_0^{-1}\int[d\phi]$$
$$\times\exp\left\{i\int d^nx\,dt[\tfrac{1}{2}\phi(\Box-m^2)\phi-\tfrac{1}{4}\lambda(\phi+g_0)^4-\lambda v(\phi+g_0)^3+J\phi]\right\}\Bigg|_{J=0}\ , \tag{3.6}$$

a form derived previously by other means.[6] Expanding the quartic and cubic terms and collecting those terms with no dependence upon $\phi$ into a phase factor allows (3.6) to be written

$$G_{\rm MU}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+f(\vec{x}_1)\right]\cdots\exp\left\{-i\int d^nx\,dt\left[\frac{1}{4}\lambda\left[\frac{\delta}{i\delta K(x)}\right]^4+\lambda f(\vec{x})\left[\frac{\delta}{i\delta K(x)}\right]^3\right]\right\}$$
$$\times D_0^{-1}e^{i\beta}\int[d\phi]\exp\left\{i\int d^nx\,dt[\tfrac{1}{2}\phi(\Box-m^2-Y^{(1)})\phi+(J+K-X)\phi]\right\}\Bigg|_{\substack{J=0\\K=0}}\ , \tag{3.7}$$

where

$$X(\vec{x})=\lambda[g_0(\vec{x})]^3+3\lambda v(\vec{x})[g_0(\vec{x})]^2 \tag{3.8a}$$

and

$$Y^{(1)}(\vec{x})=3\lambda[g_0(\vec{x})]^2+6\lambda v(\vec{x})g_0(\vec{x})=\frac{\partial X}{\partial g_0}\ . \tag{3.8b}$$

The path integration may be performed to obtain

$$G_{\rm MU}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+f(\vec{x}_1)\right]\cdots\exp\left\{-i\int d^nx\,dt\left[\frac{1}{4}\lambda\left[\frac{\delta}{i\delta K(x)}\right]^4+\lambda f(\vec{x})\left[\frac{\delta}{i\delta K(x)}\right]^3\right]\right\}$$
$$\times\frac{D_1}{D_0}e^{i\beta}\exp\left\{-\tfrac{1}{2}i\int d^nx_1dt_1d^nx_2dt_2[J(x_1)+K(x_1)-X(\vec{x}_1)]\right.$$
$$\left.\times\ \Delta^{(1)}(x_1-x_2)[J(x_2)+K(x_2)-X(\vec{x}_2)]\right\}\Bigg|_{\substack{J=0\\K=0}}\ , \tag{3.9}$$

where $\Delta^{(1)}$ satisfies

$$(\Box-m^2-Y^{(1)})\Delta^{(1)}(x-x')=\delta(t-t')\delta^n(\vec{x}-\vec{x}') \tag{3.10}$$

and has the formal representation as an iteration of the equation

$$\Delta^{(1)}(x-x')=\Delta^{(0)}(x-x')+\int d^nx_1dt_1\Delta^{(0)}(x-x_1)Y^{(1)}(\vec{x}_1)\Delta^{(1)}(x_1-x')\ , \tag{3.11}$$

while $D_1$ is the Jacobian given by

$$D_1=[\det\Delta^{(1)}(x-x')]^{1/2}\ . \tag{3.12}$$

The second exponential in (3.9) may be rewritten, modulo terms independent of $J$ and $K$ which are absorbed into $\beta$, as

$$D_1 D_0{}^{-1} e^{i\beta} \exp\left\{-\tfrac{1}{2} i \int d^n x_1 dt_1 d^n x_2 dt_2 [J(x_1)+K(x_1)-X(\vec{x}_1)]\Delta^{(1)}(x_1-x_2)[J(x_2)+K(x_2)-X(\vec{x}_2)]\right\}$$

$$= D_0{}^{-1} e^{i\beta_1} \int [d\phi] \exp\left\{ i \int d^n x\, dt [\tfrac{1}{2}\phi(\Box - m^2 - Y^{(1)})\phi + (J+K)(\phi+g_1)]\right\}, \quad (3.13)$$

where

$$g_1(\vec{x}) = \int d^n x'\, dt' \Delta^{(1)}(x-x') X(\vec{x}') = \int d^n x'\, dt' \Delta^{(1)}(x-x')[\lambda g_0{}^3(\vec{x}') + 3\lambda v(\vec{x}') g_0{}^2(\vec{x}')] . \quad (3.14)$$

Inserting (3.13) into (3.9) gives

$$G_{\mathrm{MU}}(x_1,\ldots) = \left[\frac{\delta}{i\delta J(x_1)} + f(\vec{x}_1)\right] \cdots D_0{}^{-1} e^{i\beta_1}$$
$$\times \int [d\phi] \exp\left\{ i \int d^n x\, dt [\tfrac{1}{2}\phi(\Box - m^2 - Y^{(1)})\phi \right.$$
$$\left. \left. -\tfrac{1}{4}\lambda(\phi+g_1)^4 - \lambda(v+g_0)(\phi+g_1)^3 + J(\phi+g_1)]\right\} \right|_{J=0} . \quad (3.15)$$

Result (3.15) is analogous to (3.6) and gives a representation entirely equivalent to the MU form, but with a new set of Feynman rules.

Steps (3.7)–(3.15) may be repeated $j$ times, yielding

$$G_{\mathrm{MU}}(x_1,\ldots) = \left[\frac{\delta}{i\delta J(x_1)} + f(\vec{x}_1)\right] \cdots D_0{}^{-1} e^{i\beta j}$$
$$\times \int [d\phi] \exp\left\{ i \int d^n x\, dt \left[ \tfrac{1}{2}\phi(\Box - m^2 - Y^{(j)})\phi \right.\right.$$
$$\left.\left.\left. - \tfrac{1}{4}\lambda(\phi+g_j)^4 - \lambda\left[v + \sum_{l=0}^{j-1} g_l\right](\phi+g_j)^3 + J\left[\phi + \sum_{l=1}^{j} g_l\right]\right]\right\}\right|_{J=0} ,$$

where (3.16)

$$g_j(\vec{x}) = \int d^n x'\, dt'\, \Delta^{(j)}(x-x') \left\{ \lambda g_{j-1}{}^3(\vec{x}') + 3\lambda\left[v(\vec{x}') + \sum_{l=0}^{j-2} g_l(\vec{x}')\right] g_{j-1}{}^2(\vec{x}')\right\}, \quad (3.17a)$$

with $\Delta^{(j)}$ satisfying

$$(\Box - m^2 - Y^{(j)})\Delta^{(j)}(x-x') = \delta(t-t')\delta^n(\vec{x}-\vec{x}') , \quad (3.17b)$$

while

$$Y^{(j)}(\vec{x}) = \sum_{l=0}^{j-1}\left[3\lambda g_l{}^2(\vec{x}) + 6\lambda\left[v(\vec{x}) + \sum_{r=0}^{l-1} g_r(\vec{x})\right] g_l(\vec{x})\right] . \quad (3.17c)$$

Making the assumption that

$$\lim_{j\to\infty} g_j = 0 \quad (3.18)$$

in a manner sufficient for the convergence of the series

$$\phi_0^f(\vec{x}) = v(\vec{x}) + \sum_{j=0}^{\infty} g_j(\vec{x}) , \quad (3.19)$$

it can now be shown that $\phi_0^f$ is a solution of

$$(\Box+\alpha^2)\phi_0^f=\lambda(\phi_0^f)^3 . \tag{3.20}$$

By use of (3.17) it is straightforward to find

$$(\Box-m^2)\phi_0^f=-m^2v+\sum_{j=0}^{\infty}\left\{\lambda g_j{}^3+3\lambda g_j{}^2\left[v+\sum_{l=0}^{j-1}g_l\right]+3\lambda g_{j+1}\sum_{l=0}^{j}\left[g_l{}^2+2\left[v+\sum_{r=0}^{l-1}g_r\right]g_l\right]\right\} . \tag{3.21}$$

Using the identity

$$\left[\sum_{l=0}^{n}g_l\right]^3=\sum_{l=0}^{n}\left\{g_l{}^3+3g_l{}^2\left[\sum_{j=0}^{l-1}g_j\right]+3g_l\sum_{j=0}^{l-1}\left[g_j{}^2+2g_j\left[\sum_{r=0}^{j-1}g_r\right]\right]\right\} , \tag{3.22}$$

and the fact that the convergence of the series allows interchange of sum and product, (3.21) can be rewritten

$$(\Box-m^2)\phi_0^f=\lambda(\phi_0^f)^3-3\lambda v^2\phi_0^f-m^2v+2\lambda v^3 . \tag{3.23}$$

This reduces to (3.20) upon use of the relations (2.75b).

Using the second identity

$$\left[\sum_{l=0}^{n}g_l\right]^2=\sum_{l=0}^{n}\left[g_l{}^2+2g_l\left[\sum_{j=0}^{l-1}g_j\right]\right] , \tag{3.24}$$

it is easy to show that

$$\lim_{j\to\infty}Y^{(j)}=3\lambda(\phi_0^f)^2-3\lambda v^2 , \tag{3.25}$$

so that, from (2.75b),

$$m^2+\lim_{j\to\infty}Y^{(j)}=3\lambda(\phi_0^f)^2-\alpha^2=\frac{\partial^2}{\partial\phi_0^{f\,2}}\mathscr{H}_I(\phi_0^f) . \tag{3.26}$$

Thus, in the limit $j\to\infty$, expression (3.16) becomes

$$G_{\rm MU}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+f(\vec{x}_1)\right]\cdots D_0{}^{-1}e^{i\tilde{\beta}}$$
$$\times\int[d\phi]\exp\left\{i\int d^nx\,dt\left[\tfrac{1}{2}\phi\left[\Box-\frac{\partial^2}{\partial\phi_0^{f\,2}}\mathscr{H}_I(\phi_0^f)\right]\phi\right.\right.$$
$$\left.\left.\left.-\tfrac{1}{4}\lambda\phi^4-\lambda\phi_0^f\phi^3+J(\phi+\phi_0^f-f)\right]\right\}\right|_{J=0} . \tag{3.27}$$

It is obvious that this could be rewritten as

$$G_{\rm MU}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+\phi_0^f(\vec{x}_1)\right]\cdots D_0{}^{-1}e^{i\tilde{\beta}}$$
$$\times\int[d\phi]\exp\left\{i\int d^nx\,dt\left[\tfrac{1}{2}\phi\left[\Box-\frac{\partial^2}{\partial\phi_0^{f\,2}}\mathscr{H}_I(\phi_0^f)\right]\phi\right.\right.$$
$$\left.\left.\left.-\tfrac{1}{4}\lambda\phi^4-\lambda\phi_0^f\phi^3+J(\phi+\phi_0^f-f)\right]\right\}\right|_{J=0} . \tag{3.28}$$

Expression (3.28) can be written in terms of a set of Feynman rules by separating the cubic and quartic terms as a functional derivative series. If this is done (3.28) becomes

$$G_{\rm MU}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+\phi_0^f(\vec{x}_1)\right]\cdots D_0{}^{-1}\tilde{D}e^{i\tilde{\beta}}$$
$$\times\exp\left\{-i\int d^nx\,dt\left[\tfrac{1}{4}\lambda\left[\frac{\delta}{i\delta J(x)}\right]^4+\lambda\phi_0^f(\vec{x})\left[\frac{\delta}{i\delta J(x)}\right]^3\right]\right\}$$
$$\times\exp\left\{-\tfrac{1}{2}i\int d^nx_1dt_1d^nx_2dt_2[J(x_1)\tilde{\Delta}(x_1-x_2)J(x_2)]\right\}\Bigg|_{J=0}, \tag{3.29}$$

where

$$\tilde{D}=[\det\tilde{\Delta}(x_1-x_2)] \tag{3.30}$$

and

$$\left[\Box-\frac{\partial^2}{\partial\phi_0^{f\,2}}\mathscr{H}_I(\phi_0^f)\right]\tilde{\Delta}(x_1-x_2)=\delta(t_1-t_2)\delta^n(\vec{x}_1-\vec{x}_2)\,. \tag{3.31}$$

It is clear that (3.29) coincides with the CL representation of the unrenormalized Green's functions, apart from the irrelevant phase factor $D_0{}^{-1}\tilde{D}e^{i\tilde{\beta}}$, *if* $\phi_0^f$ *and* $\phi_0$ *are the same function.* Because many choices are available for $f$, it is clear that $\phi_0^f$ must be calculated by (3.19) before it is known which solution it converges to, and if this function gives a set of stable solutions to (2.16).

The equivalence has been proved for the unrenormalized Green's functions, and the natural extension is to the renormalized Green's functions. It has been shown[19] that the MU Green's functions are renormalized by the same counterterms and wave-function constants as the same theory without a soliton present. It would be straightforward to include these in the initial representation of the Green's function (3.2) and then to resum this series to obtain the renormalized CL series to which it is equivalent. Such a program will be followed elsewhere.

As a final remark it is obvious that the argument presented in this section can be generalized to arbitrary interactions which are polynomial in nature by using the generalizations of (3.22) and (3.24). Of course, there are a denumerable infinity of representations which are passed through on the way to CL form. These are entirely equivalent to both forms, although considerably more difficult to calculate with.

## IV. THE LEHMANN-SYMANZIK-ZIMMERMANN REDUCTION FORMULAS

In this section the Lehmann-Symanzik-Zimmermann reduction formulas[20] for particle scattering in the static soliton sector will be derived. These will be used to show that the MU method of switching leaves asymptotic particles which scatter off the extended object even in the tree approximation.

The basic ingredient in deriving the reduction formulas is the asymptotic limit of the field operator. For the respective methods these are given by (2.9a) and (2.9b). A particle state can then be defined in each approach by

$$|k;{\rm in}\rangle_{\rm MU}=V^{-1}(t_-)a_{\vec{k}}^{\dagger}|0\rangle \tag{4.1a}$$

and

$$|k;{\rm in}\rangle_{\rm CL}=\tilde{V}^{-1}(t_-)\alpha_{\vec{k}}^{\dagger}|\tilde{0},t_-\rangle\,, \tag{4.1b}$$

where all objects have been defined in Sec. II. In essence the definitions of (4.1) have picked asymptotic *particle* operators as

$$a_{\vec{k}}^{\dagger{\rm in}}=V^{-1}(t_-)a_{\vec{k}}^{\dagger}V(t_-) \tag{4.2a}$$

and

$$\alpha_{\vec{k}}^{\dagger{\rm in}}=\tilde{V}^{-1}(t_-)\alpha_{\vec{k}}^{\dagger}\tilde{V}(t_-)\,. \tag{4.2b}$$

These particle operators also satisfy the usual algebra of annihilation and creation operators. The many-particle extensions of (4.1) are obvious.

The states of (4.1) can be written

$$|k,A;{\rm in}\rangle_{\rm MU}=-i\lim_{t\to t_-}V^{-1}(t)\int d^nx[\chi(x,k)\overleftrightarrow{\partial}_t\psi(x)]V(t)|A;{\rm in}\rangle_{\rm MU} \tag{4.3a}$$

and

$$|k,A;{\rm in}\rangle_{\rm CL}=-i\lim_{t\to t_-}\tilde{V}^{-1}(t)\int d^nx[\tilde{\chi}(x,k)\overleftrightarrow{\partial}_t\psi(x)]\tilde{V}(t)|A;{\rm in}\rangle_{\rm CL}\,, \tag{4.3b}$$

where

$$\chi(x,k)=(2\omega_{\vec{k}})^{-1/2}e^{ikx} \tag{4.4a}$$

and

$$\tilde{\chi}(x,k)=(2\tilde{\omega}_{\vec{k}})^{-1/2}u_{\vec{k}}(\vec{x})e^{-i\tilde{\omega}_{\vec{k}}t} . \tag{4.4b}$$

Of course, any discrete mode except the translation mode is allowed in (4.4b), as well as the scattering solutions. Using the standard replacement

$$\lim_{t\to t_-} = -\int_{t_-}^{t_+} dt\frac{\partial}{\partial t}+\lim_{t\to t_+} , \tag{4.5}$$

the definition of time ordering, and dropping any forward scattering terms, it follows that

$$_{\rm MU}\langle B;{\rm out}\,|\,T\{\cdots\}\,|\,k,A;{\rm in}\rangle_{\rm MU}$$
$$=-i\int d^nx\int_{t_-}^{t_+} dt\chi(x,k)(\Box_x-m^2)_{\rm MU}\langle B;{\rm out}\,|\,T\{\cdots[\psi(x)-f(\vec{x})]\}\,|\,A;{\rm in}\rangle_{\rm MU} , \tag{4.6a}$$

while

$$_{\rm CL}\langle B;{\rm out}\,|\,T\{\cdots\}\,|\,k,A;{\rm in}\rangle_{\rm CL}$$
$$=-i\int d^nx\int_{t_-}^{t_+} dt\tilde{\chi}(x,k)\left[\vec{\Box}_x-\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\right]{}_{\rm CL}\langle B;{\rm out}\,|\,T\{\cdots[\psi(x)-\phi_0(\vec{x})]\}\,|\,A;{\rm in}\rangle_{\rm CL} , \tag{4.6b}$$

where use has been made of the equations of motion for (4.4). These formulas may be continued until all particles are reduced from the asymptotic states, leaving only the coherent-state operator. It is important to note that, in the limit $t_+$ and $t_-$ become arbitrarily large, the terms proportional to $f(\vec{x})$ and $\phi_0(\vec{x})$ in (4.6) will disappear due to the time integration and the fact that there are no zero-frequency modes associated with particle states. The amplitudes are thus reduced to a set of differential operators acting upon Green's functions of the form (2.34). Implicit in relations (4.6) are the respective adiabatic-switching assumptions.

At this point the two-particle amplitude may be examined. For the MU approach this is given by

$$_{\rm MU}\langle k;{\rm out}\,|\,p;{\rm in}\rangle_{\rm MU}=\int d^nx\,dt\,d^nx'\,dt'\chi^*(x,k)\chi(x',p)$$
$$\times(\vec{\Box}_x-m^2)(\vec{\Box}_{x'}-m^2)\langle f,t_+\,|\,T\{\psi(x)\psi(x')\}\,|\,f,t_-\rangle , \tag{4.7a}$$

while the CL method gives

$$_{\rm CL}\langle k;{\rm out}\,|\,p;{\rm in}\rangle_{\rm CL}=\int d^nx\,dt\,d^nx'\,dt'\tilde{\chi}^*(x,k)\chi(x',p)$$
$$\times\left[\vec{\Box}_x-\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\right]\left[\vec{\Box}_{x'}-\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)\right]$$
$$\times\langle\phi_0,t_+\,|\,T\{\psi(x)\psi(x')\}\,|\,\phi_0,t_-\rangle , \tag{4.7b}$$

where it is assumed that $k\neq p$, that the ground states in both expressions have been normalized, and that the terms proportional to $\phi_0$ and $f$ may be dropped. The results of Sec. III show that the unrenormalized Green's functions appearing in (4.7a) and (4.7b) are, modulo an irrelevant phase factor, identical. Thus, in the limit $\hbar\to 0$ both must reduce to $\tilde{\Delta}$, given by (2.51), where terms proportional to $\phi_0$ and $f$ are again dropped. It is apparent that (4.7b) vanishes in this limit. However, expression (4.7a) becomes

$$\lim_{\hbar\to 0} {}_{\rm MU}\langle k;\text{out}\,|\,p;\text{in}\rangle_{\rm MU} = \int d^n x\, dt\, \chi^*(x,k)\chi(x,p)\left[\frac{\partial^2}{\partial\phi_0{}^2}\mathscr{H}_I(\phi_0)-m^2\right]$$
$$+\int d^n x\, dt\, d^n x'\, dt'\, \chi^*(x,k)\chi(x',p)\left[\frac{\partial^2}{\partial\phi_0{}^2(\vec{x})}\mathscr{H}_I-m^2\right]$$
$$\times\left[\frac{\partial^2}{\partial\phi_0{}^2(\vec{x}')}\mathscr{H}_I-m^2\right]\tilde{\Delta}(x-x')\,, \tag{4.8}$$

which does not necessarily vanish. Result (4.8) shows that the asymptotic particles of the MU method scatter off a *classical* potential created by the extended object. This is, within the constant $m^2$, the same potential which was used to generate the interaction picture in the CL approach, where the phase shifts and bound states are incorporated into the particle spectrum and normal modes. In the MU approach they must become part of the perturbation series, hence result (4.8).

In this respect it is interesting to note that the vacuum graphs of the MU representation of the $\phi^4$ theory, given by (2.7a), are given by a phase with an essential singularity in the coupling constant $\lambda$. This is because the solutions to (2.6) for the theory are proportional to $\lambda^{-1/2}$. When solving (2.71) the function $g_0$ will then also be proportional to $\lambda^{-1/2}$. In the expansion of $I(\phi+g_0,v)$, there will occur vertices again with negative powers of $v$, and these will lead to a phase whose argument is singular in $\lambda$. The CL representation of this theory does not have this feature. This is symptomatic of the presence of the classical potential which causes (4.8), and which does not vanish for the $\phi^4$ theory if $\lambda$ is zero.

As a final note, it is possible to express the particles of the CL approach as superpositions of the particles used in the MU method. This is only a formal relation since the two sets of operators are defined over different Fock spaces. The nontranslation mode operators are written

$$\alpha_{\vec{k}}=\int d^n x \frac{d^n p}{(2\pi)^{n/2}}\, u_{\vec{k}}(\vec{x})e^{i\vec{p}\cdot\vec{x}}a_{\vec{p}} \tag{4.9a}$$

and

$$\alpha^\dagger_{\vec{q}}=\int d^n y \frac{d^n r}{(2\pi)^{n/2}}\, u^*_{\vec{q}}(\vec{y})e^{-i\vec{r}\cdot\vec{y}}a^\dagger_{\vec{r}}\,, \tag{4.9b}$$

with similar expressions for the bound state operators. It is trivial to show, using the commutator (2.14c) and the orthonormality conditions (2.19), that these definitions satisfy the relations (2.21).

The collective coordinate and its canonically conjugate operator are written

$$\vec{Q}(t=0)=\int d^n x \frac{d^n p}{(2\pi)^{n/2}}\, 2^{-1/2}\vec{u}_T(\vec{x})\times(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{\vec{p}}e^{-\vec{p}\cdot\vec{x}}) \tag{4.10a}$$

and

$$\vec{P}=i\int d^n y \frac{d^n k}{(2\pi)^{n/2}}\, 2^{-1/2}\vec{u}_T(\vec{y})\times(a^\dagger_{\vec{k}}e^{-i\vec{k}\cdot\vec{y}}-a_{\vec{k}}e^{i\vec{k}\cdot\vec{y}})\,. \tag{4.10b}$$

These clearly satisfy (2.22b), are Hermitian, and can be extended to arbitrary times by relation (2.25). Of course, relations (4.9) and (4.10) implicitly assume that the CL modes possess well-defined Fourier transforms, a condition usually satisfied for any well-behaved potential.

## V. CONCLUSIONS

It is relevant to summarize the major results of this paper. The first is that both field-theoretic formulations currently in use in the literature for calculating in the presence of an extended object may be derived by using the same operator techniques, but by selecting different adiabatic-switching conditions. The CL switching condition allowed perturbations to be performed around the exact classical solution at the cost of first solving the potential scattering problem (2.16), introducing the collective coordinate, and calculating with an interaction-picture field that is not manifestly a Lorentz scalar. The MU switching condition allowed a simple particle spectrum, but led to more complicated Feynman rules and to the necessity of introducing the function $f$ which is not a classical solution. The second major result is that, for the class of theories studied in this paper, the equation of motion for $f$ differs from that used by Matsumoto, Umezawa, *et al.*, and possesses solutions with Fourier transforms, so that coherent-state representations of the theory are possible. The third major result is that the two representations yield, modulo an irrelevant phase factor, identical results for the unrenormalized Green's functions of the theory, an equivalence obtained by using functional techniques. In addition, it was seen that there exist a denumerable infinity of calculationally equivalent representations intermediate to these two. The fourth major result is that the MU representation manifests particle-soliton scattering in the static soliton sector, whereas the CL representation does not. However, it was seen that the same particle-soliton scattering information is carried in the normal modes of the CL representation, and thus the exact propagator would be the same for both methods.

It is useful to discuss the advantages and drawbacks of both approaches. The first asset of the CL approach is that the particle spectrum in the soliton sector is known if the equation (2.16) can be solved and thus the question of stability is immediately resolved. A second advantage is that the Feynman rules are simpler, containing no vertices lower than cubic. The major drawback to the CL representation is the need to input an exact classical solution, an item which may not be available. As a result, calculations of multisoliton effects by this method have been difficult, and nearly impossible in two- and three-spatial dimensions. The MU approach has the asset that the equation for $f$ is much easier to solve, having solutions in any spatial dimension for multivalued $v(\vec{x})$. Thus, the MU representation in the multisoliton sector may be obtained easily. However, a drawback of the MU approach is that the stability and the particle spectrum of such a sector is not readily apparent. In addition, the convergence of the series analogous to (3.19) remains to be determined on a case-by-case basis.

The stability of the sector represented in the MU approach may be inferred from the form of the function $f$ used for the asymptotic ground state. Should the inner product of $|f,t\rangle$ and some other possible ground state $|f',t\rangle$ vanish, then the two represent unitarily inequivalent sectors of the theory.[21] In particular, the translationally invariant ground state $|v,t\rangle$, given by

$$|v,t\rangle=\exp\left[-i\int d^n x\, v\dot{\phi}(\vec{x},t)\right]|0\rangle\ , \tag{5.1}$$

where $v$ is a constant solution to (2.6), can be used to ascertain whether the extended object is stable against collapse into the translationally invariant sector. Initial results show that the MU representations of multikink states reflect the instability determined by other arguments, while spherically symmetric single-kink states in higher dimensions are not unitarily inequivalent to the state (5.1). More complete analysis of this will be presented elsewhere.

A further extension of the work presented here would be in analyzing energy-momentum eigenstates in the soliton sector,[22] and to evaluating the energy of multikink states using the MU function $f$ as a first-order approximation for the multikink classical solution.[23] Finally, the MU method has been applied to the Abelian Higgs model as a phenomenological approximation to the superconductor in order to investigate the formation of vortices.[24] It would be of interest to evaluate the possible equivalence between this approach and other treatments[25] of this problem.

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## APPENDIX

Attributing the derivation of the perturbation series (2.48) with propagator (2.51) for the time-ordered products to the paper by Christ and Lee[4] stems from several statements made therein [viz. Eq. (2.8) and remarks following] regarding expansion of the interaction-picture field with a *complete* set of eigenfunctions consistent with the classical solution. However, it is of some concern to examine the form of the perturbation expansion developed in Sec. II of this paper to ensure that it is equivalent to that developed and used by Christ and Lee elsewhere [viz. Eq. (2.13)] in their paper. The primary concern is the zero-frequency mode present in the interaction-picture fields which could potentially cause infrared divergences to appear in the perturbation series. This mode is cleverly excluded from contributing in the CL formalism, but appears to contribute to (2.48)–(2.51). This problem is directly related to the question of equivalence of the Hamiltonians used in the respective approaches.

These questions can be resolved simultaneously. To begin with the interaction-picture field (2.20) is rewritten as

$$\tilde{\phi}(x)=\eta(x)+\vec{Q}(t)\cdot\vec{u}_T(\vec{x})\ . \tag{A1}$$

Clearly, for the static case considered in this paper, $\eta$ is identical to the small fluctuation field employed by Christ and Lee, and its time-ordered products will be free of the zero-frequency modes. The problem is then to resum the perturbation series (2.48) in such a way as to fold $\vec{Q}$ into the $\eta$ fields. This is possible because the representation of the perturbation series used in Sec. II has terms linear and quadratic in the $\eta$ fields when (A1) is inserted, and so will cause a resummation process to occur. As in Sec. III this resummation is easier to accomplish with functional methods. Only the salient features of this procedure will be presented since the details are essentially identical to those of Sec. III. Again, for simplicity only the $\phi^4$ case will be presented.

The resummation is begun by noting that (2.48) can be represented functionally (see Sec. III of this paper):

$$\begin{aligned}G_{\rm CL}(x_1,\dots)=&\left[\frac{\delta}{i\delta J(x_1)}+\phi_0(\vec{x}_1)\right]\cdots\\&\times\exp\left\{-i\int dx\left[\frac{1}{4}\lambda\left[\frac{\delta}{i\delta J(x)}\right]^4+\lambda\phi_0(\vec{x})\left[\frac{\delta}{i\delta J(x)}\right]^3\right]\right\}\\&\times\exp\left[-\tfrac{1}{2}i\int dx\,dy\,J(x)\tilde{\Delta}(x-y)J(y)\right]\Big|_{J=0}\ .\end{aligned} \tag{A2}$$

If the second exponential is replaced by a path-integral representation (A2) becomes

$$G_{\rm CL}(x_1,\dots)=\left[\frac{\delta}{i\delta J(x_1)}+\phi_0(\vec{x}_1)\right]\cdots D_0\int[d\phi]\exp\left[i\int dx\,\tfrac{1}{2}\phi\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}\right]\phi-\tfrac{1}{4}\lambda\phi^4-\lambda\phi_0\phi^3+J\phi\right]\Big|_{J=0}\ . \tag{A3}$$

In order to show the resummation it is necessary to split the measure in (A3) into a direct product of the $\eta$ and $\vec{Q}$ measures. This is done by the replacement

$$[d\phi]=[d\eta][dq]\delta\left[\int\vec{u}_T\eta\right]\ , \tag{A4}$$

where the δ function over space-time prevents the $q$ modes from being integrated by the $\eta$ measure, and there is an implicit sum over all translation modes. This δ function can be written

$$\delta\left[\int\vec{u}_T\eta\right]=\int[d\vec{\pi}]\exp\left[i\int dx\,\vec{\pi}\cdot\vec{u}_T\eta\right]\ , \tag{A5}$$

which is much easier to implement in (A3). Direct substitution of (A4) and (A1) into (A3) gives

$$\begin{aligned}G_{\rm CL}(x_1,\dots)=&\left[\frac{\delta}{i\delta J(x_1)}+\phi_0(\vec{x}_1)\right]\cdots D_0\\&\times\int[dq][d\pi][d\eta]\ \exp\left\{i\int dx\left[\tfrac{1}{2}\eta\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}-3\lambda(\vec{q}\cdot\vec{u}_T)^2-6\lambda(\vec{q}\cdot\vec{u}_T)\right]\eta\right.\right.\\&\qquad+\tfrac{1}{2}\vec{q}\cdot\vec{u}_T\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}\right]\vec{q}\cdot\vec{u}_T+\vec{\pi}\cdot\vec{u}_T\eta\\&\qquad+(J-\lambda(\vec{q}\cdot\vec{u}_T)^3-3\lambda\phi_0(\vec{q}\cdot\vec{u}_T)^2)\eta\\&\left.\left.\qquad+J\vec{q}\cdot\vec{u}_T-\tfrac{1}{4}\lambda\eta^4-\tfrac{1}{4}\lambda(\vec{q}\cdot\vec{u}_T)^4-\lambda\phi_0(\vec{q}\cdot\vec{u}_T)^3-\lambda\,(\phi_0+\vec{q}\cdot\vec{u}_T)\eta^3\right]\right\}\Big|_{J=0}\ .\end{aligned} \tag{A6}$$

As in Sec. III, the terms cubic and quartic in $\eta$, as well as the term proportional to $\vec{\pi}$, may be considered as interactions and the remaining path integral, linear, and quadratic in $\eta$, may be calculated exactly. The $q$ tegrations are *not* performed during this process. This functional is in turn replaced by an equivalent path integral, as in step (3.13) of Sec. III. Leaving the details as an exercise to the reader, the result is

$$G_{\rm CL}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+\phi_0(\vec{x}_1)\right]\cdots D_0$$
$$\times\int[dq][d\pi][d\eta]\,\exp\left\{i\int dx\left[\tfrac{1}{2}\eta\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}-6\lambda\phi_0(\vec{q}\cdot\vec{u}_T)-3\lambda\chi^2-6\lambda(\phi_0+\vec{q}\cdot\vec{u}_T)\chi\right]\eta\right.\right.$$
$$+[J-\lambda\chi^3-3\lambda(\phi_0+\vec{q}\cdot\vec{u}_T)\chi^2]\eta-\tfrac{1}{4}\lambda\eta^4-\lambda(\phi_0+\vec{q}\cdot\vec{u}_T+\chi)\eta^3$$
$$+(\vec{q}\cdot\vec{u}_T+\chi)J+\vec{\pi}\cdot\vec{u}_T(\eta+\chi)+\tfrac{1}{2}(\vec{q}\cdot\vec{u}_T)\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}\right](\vec{q}\cdot\vec{u}_T)$$
$$-\tfrac{1}{4}\lambda(\vec{q}\cdot\vec{u}_T)^4-\lambda\phi_0(\vec{q}\cdot\vec{u}_T)^3-\tfrac{1}{4}\lambda\chi^4-\lambda(\phi_0+\vec{q}\cdot\vec{u})\chi^3$$
$$\left.\left.-\tfrac{1}{2}\chi\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}-3\lambda(\vec{q}\cdot\vec{u}_T)-6\lambda\phi_0(\vec{q}\cdot\vec{u}_T)\right]\chi\right]\right\}\Bigg|_{J=0}, \tag{A7}$$

where

$$\chi(x)=\int dy\,\Delta'(x-y)[\lambda(\vec{q}\cdot\vec{u}_T)^3+3\lambda\phi_0(\vec{q}\cdot\vec{u}_T)^2] \tag{A8}$$

and $\Delta'$ satisfies

$$\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}-3\lambda(\vec{q}\cdot\vec{u}_T)^2-6\lambda\phi_0(\vec{q}\cdot\vec{u}_T)\right]\Delta'=\delta\,. \tag{A9}$$

This process of resummation can be repeated an arbitrary number of times, but the form will not be exhibited since it is possible to see the trend from (A7). Recalling that the translation modes are given by

$$\vec{u}_T=c\vec{\nabla}\phi_0\,, \tag{A10}$$

where $c$ is simply a normalization constant, and that for the $\phi^4$ theory

$$\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}=-\alpha^2+3\lambda\phi_0{}^2\,, \tag{A11}$$

it is easy to see that

$$\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}+6\lambda\phi_0(\vec{q}\cdot\vec{u}_T)+3\lambda(\vec{q}\cdot\vec{u}_T)^2=-\alpha^2+3\lambda(\phi_0+c\vec{q}\cdot\vec{\nabla}\phi_0)^2\,. \tag{A12}$$

This is the first step in the expansion

$$\phi_0(\vec{x}+c\vec{q})=\phi_0(\vec{x})+c\vec{q}\cdot\vec{\nabla}\phi_0(\vec{x})+\cdots\,. \tag{A13}$$

This process is contained by the next term $\chi$, as the diligent reader will verify. The latter terms in (A7), all of which are independent of $\eta$, are the first terms in the expansion of

$$\phi_0(\vec{x}+c\vec{q})\Box\phi_0(\vec{x}+c\vec{q})+\tfrac{1}{2}\alpha^2\phi_0{}^2(\vec{x}+c\vec{q})-\tfrac{1}{4}\lambda\phi_0{}^4(\vec{x}+c\vec{q})=\mathscr{L}(\phi_0(\vec{x}+c\vec{q}))\,, \tag{A14}$$

the Lagrangian density of the quantum-mechanical extended object with $c\vec{q}$ acting as the center-of-mass coordinate of this object.[4]

The final form of the generator, after an infinite number of iterations, is

$$G_{\rm CL}(x_1,\ldots)=\left[\frac{\delta}{i\delta J(x_1)}+\phi_0(\vec{x})\right]\cdots D_0$$
$$\times\int[dq][d\pi][d\eta]\exp\left\{i\int dx\left[\tfrac{1}{2}\eta\left[\Box-\frac{\partial^2\mathscr{H}_I}{\partial\phi_0{}^2}\Big|_{\phi_0=\phi_0(\vec{x}+c\vec{q})}\right]\eta-\tfrac{1}{4}\lambda\eta^4\right.\right.$$
$$-\lambda[\phi_0(\vec{x}+c\vec{q})]\eta^3+J[\eta+\phi_0(\vec{x}+c\vec{q})-\phi_0(\vec{x})]$$
$$+\vec{\pi}\cdot\vec{u}_T(\vec{x})\,[\eta+\phi_0(\vec{x}+c\vec{q})-\vec{q}\cdot\vec{u}_T(\vec{x})-\phi_0(\vec{x})]$$
$$\left.\left.+\mathscr{L}(\phi_0(\vec{x}+c\vec{q}))\right]\right\}\Big|_{J=0}\,. \tag{A15}$$

This generates the same Feynman rules derived elsewhere in the Christ-Lee paper and discussed in path-integral form by Callan and Gross.[13] It is apparent that the zero-frequency modes have canceled in a consistent way. This completes the proof of equivalence.

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