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Diagrammatic methods for spinors in Feynman diagrams

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We introduce new graphical methods which simplify the manipulation of group-theoretic factors involving spinors which arise, for example, in the evaluation of Feynman diagrams. These methods are applicable to both internal and Poincaré symmetries, and form a natural extension of previous techniques for fundamental and adjoint representations, which are also reviewed in this paper.

I. INTRODUCTION

In this paper we shall introduce a new diagrammatic method for simplifying the calculation of group-theoretic weight factors involving spinors, such as those arising from Feynman diagrams. This is a generalization of Cvitanović's work.[1] We use the term group-theoretic weights to mean the factors arising from the group-theoretic matrices in the Feyman rules for a Lagrangian field theory with some symmetry, such as a gauge theory. These weights are not to be confused with the weight diagrams of Cartan. In order to explain the motivation for this we must answer three questions: Why do we use a diagrammatic notation? What are the advantages of our new algorithm? Why are we interested in spinorial group-theory weights?

The first question is simply answered by noting that diagrams satisfy the criteria for a good notation to a greater extent than the usual tensor notation does. For one example, diagrams do not require quantities such as dummy indices, so the operation of relabeling such indices becomes unnecessary; for another it is far easier to identify topologically identical diagrams than it is to recognize the equivalence between the corresponding tensor expressions. A further advantage follows from the universal use of Feynman diagrams to represent terms in field-theory perturbation expansions (which themselves are used for much the same reasons as those outlined above) since often the "group space" graph can be read directly from the Feynman diagram, and our notation will be chosen to emphasize this similarity.

The second question requires an analysis of what the alternative methods for simplifying spinorial expressions are. Apart from a set of somewhat *ad hoc* trace identities and contraction theorems, these consist of a variety of methods based on Chisholm identities,[2] such as the Kahane algorithm.[3,4] The latter are unfortunately only applicable for SO(4) spinors [or their analytic continuation to the Lorentz group SO(3,1)], and therefore fail for many of the interesting applications to be discussed below. Our method, based on Fierz identities,[5] is a synthesis and simplification of the former identities which enables them to be used more easily and in a more efficient manner.

Finally we address the third question as to why such a simplification method is useful. For internal symmetries our method completes the graphical methods introduced by Cvitanović[1] for fundamental and adjoint representations of simple Lie groups (and in principle all tensor products of such representations) by extending them to spinor representations of the SO(n) groups. The analysis of the weights arising from such internal-symmetry groups has several uses apart from the obvious one of just evaluating Feynman diagrams, where the parameter integrals are often a harder problem anyway. Such an analysis may be useful in searching for patterns of cancellations of infrared singularities, or in analyzing the behavior of a theory when it is dominated by group-theoretic factors, as in the $1/n$ expansion.[6] So far our discussion of the need for spinorial simplification has been concerned with internal symmetries, whose importance with the rise to popularity of non-Abelian gauge theories is not to be underestimated: Spinor representations of internal symmetries are used for instance in SO(10) grand unified models. There is however another use for the method, namely in simplifying the Dirac algebra arising from the Poincaré invariance of relativistic field theories involving fermions.

With the use of dimensional regularization the treatment of spinor algebra in n dimensions has become a subject of practical importance, and we shall show how our methods may be applied directly in $4+\epsilon$ dimensions. We consider mainly even-dimension orthogonal groups, but the generalization of our results to odd dimensions is easy: We shall not consider it explicitly as its complications[7] appear to add only minor technical difficulties without revealing any essentially new properties, and also it currently has no important applications.

For the sake of mathematical correctness we point out that throughout we shall follow the usual abuse of notation by referring to spinors as transforming according to representations of SO(n), rather than those of its universal covering group spin(n). As we shall be concerned with the properties of the associated Lie algebras, the distinction is in any case unimportant for our purposes.

The paper is organized as follows: In Sec. II we give a review of Cvitanović's techniques for simplifying group-theoretic weights involving adjoint and fundamental representation of SO(n), together with various additional results which follow easily from them. Section III explains the basic rules for dealing with spinor representations by the use of Fierz transformations, and Sec. IV is concerned with the analytic continuation of our explicit expression for the Fierz coefficients to complex dimensions.

Apart from Sec. IV, the paper is based on a large number of graphical proofs and examples to which the text is meant to be an adjunct, which we believe is in keeping with the nature of the subject matter. In particular, Fig. 1 gives a summary of the elements of our diagrammatic notation and is, we hope, fairly self-explanatory: The way in which the various forms are combined and used will be explained in the following sections.

II. A REVIEW OF THE FUNDAMENTAL AND ADJOINT REPRESENTATIONS

In this section we briefly explain Cvitanović's[1] techniques for calculating the group-theoretic weights of Feynman diagrams for the group SO(n), and we will try to illuminate their significance and origin. We use the term "fundamental" to describe the defining representations of the groups, and "adjoint" to describe the representation induced by the natural action of the generators on themselves, corresponding to Cvitanović's use of the terms "quark" and "gluon" representations, respectively.

The first set of rules (shown in Fig. 2) is merely a restatement in graphical notation of the general relations which hold for all Lie algebras, such as the commutation relations for the generators $[T_i, T_j]=ic_{ijk}T_k$ [Fig. 2(a)], the Jacobi identity [Fig. 2(e)], and so on. We choose to introduce an arbitrary constant a into the normalization condition on the generators $\mathrm{tr}(T_iT_j)=a\delta_{ij}$ [Fig. 2(d)], to make the notation more flexible.

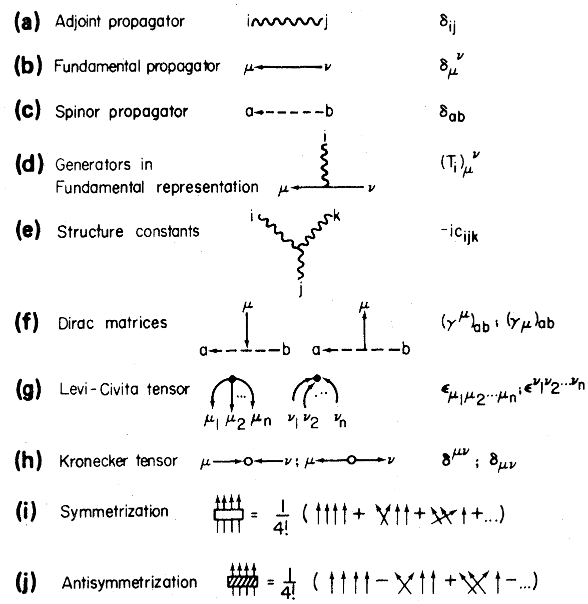

FIG. 1. Diagrammatic notation for group-theory factors in adjoint, fundamental, and spinor representations. Square brackets [· · ·] around indices indicate antisymmetrization, parentheses (· · ·) indicate symmetrization.

The rules specifying the allowed deformations of a diagram—that is those which do not affect the value of the expression to which it corresponds—are conveniently summarized by imagining each line ("propagator") to be an elastic string and each vertex to be a block onto which the strings are attached. Valid deformations are those in which no strings are broken or detached from the blocks they were initially attached to. The blocks may be arbitrarily rotated or translated, but care must be taken if they are turned upside down (the vertices are oriented). The external lines correspond to strings whose end points are fixed throughout the deformations. Finally, any subdiagram may be replaced by another subdiagram equivalent to it. The significance of these rules may be made clear by studying Fig. 2(b).

The last two relations, Figs. 2(h) and 2(i), require a little more explanation. The basis of Cvitanović's method is the classification of the various simple Lie algebra according to their invariant ten-

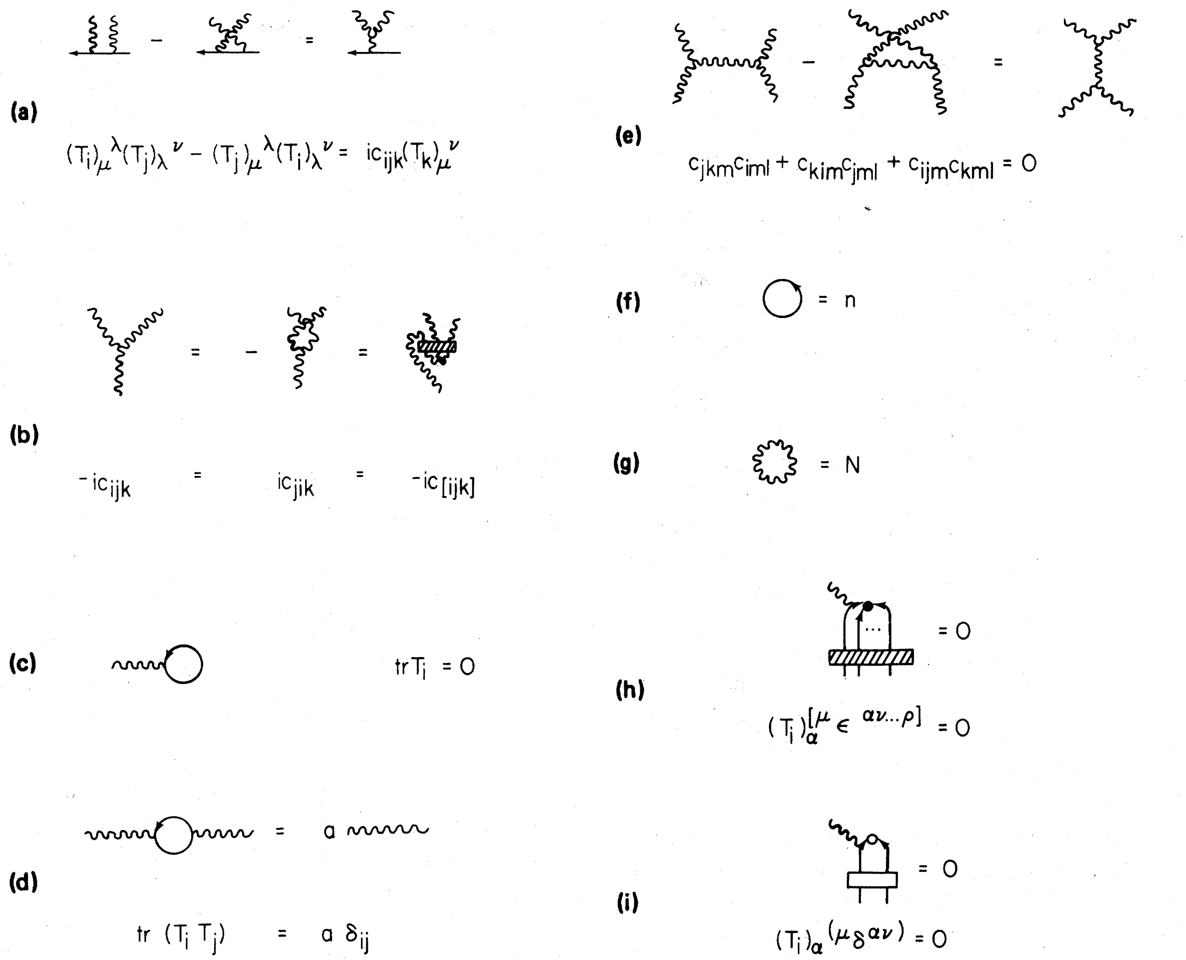

FIG. 2. General identities for the fundamental and adjoint representations: (a) the commutation relations for the generators of a Lie group; (b) the antisymmetry of the structure constants; (c) the tracelessness of the generators; (d) the normalization for the generators; (e) the Jacobi identity obeyed by the structure constants; (f) the dimension of the fundamental representation; (g) the dimension of the adjoint representation; (h) the invariance condition for the Levi-Civita ϵ tensor; (i) the invariance condition for the Kronecker δ tensor for SO(n).

sors (or equivalently their invariant polynomials). For example, SU(n) preserves the tensor δ_μ^ν (i.e., a sesquilinear metric) and the n-dimensional volume element derived from the Levi-Civita tensor $\epsilon_{\mu_1\mu_2\cdots\mu_n}$; SO($n$) preserves in addition to these $\delta_{\mu\nu}$ (i.e., a bilinear metric); similar invariants characterize the simplectic and exceptional algebras as well. To express the invariance of these tensors in a convenient and compact form it is useful to consider the action of an infinitesimal group transformation $\delta\omega_i T_i$ on some antisymmetric tensor $f^{\alpha\beta\cdots\omega}$:

$$\delta f^{\alpha\beta\cdots\omega} = \delta\omega_i (T_i)_\mu{}^\alpha f^{\mu\beta\cdots\omega} + \delta\omega_i (T_i)_\mu{}^\beta f^{\alpha\mu\cdots\omega} + \cdots + \delta\omega_i (T_i)_\mu{}^\omega f^{\alpha\beta\cdots\mu} .$$

If f is an invariant, then $\delta f = 0$, and using f's antisymmetry [as well as the tracelessness of the generators, $(T_i)_\mu{}^\mu = 0$] we obtain[8]

$$(T_i)_\mu{}^{[\alpha} f^{\mu\beta\cdots\omega]} = 0 .$$

The corresponding condition for symmetric invariant tensors follows analogously.

Given the basic diagrammatic rules of Fig. 2, we can derive a family of identities which are particularly useful for simplifying graphs, and these are given as Fig. 3. The first relation is just the well-known identity relating the product of two Levi-Civita tensors to the generalized Kronecker δ tensor, and it follows by noting the definition of ϵ:

$$\epsilon_{\mu_1 \cdots \mu_n} = \begin{cases} 1 \text{ if } \mu_1, \ldots, \mu_n \text{ is an even permutation of } 1, \ldots, n \\ -1 \text{ if } \mu_1, \ldots, \mu_n \text{ is an odd permutation of } 1, \ldots, n \\ 0 \text{ otherwise ,} \end{cases}$$

$$\epsilon_{\mu_1 \cdots \mu_n} \epsilon^{\nu_1 \cdots \nu_n} = \left\{ \begin{array}{l} 1 \text{ if } \mu_1, \ldots, \mu_n \text{ is an even permutation of } \nu_1, \ldots, \nu_n \\ -1 \text{ if } \mu_1, \ldots, \mu_n \text{ is an odd permutation of } \nu_1, \ldots, \nu_n \\ 0 \text{ otherwise} \end{array} \right\} \equiv \delta^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_n} .$$

The second relation, Fig. 3(b), is just a straightforward combinatorial identity.[9] Figure 3(c) gives a relation which enables the "three-adjoint vertex" to be rewritten in terms of traces in the fundamental representation; its proof will be given as an example of graphical techniques in Fig. 4(a).

Figures 3(d) and 3(e) give the diagrammatic forms of the quadratic Casimir invariant, whose derivations will also be given later. Notice that C_F, the value of the Casimir invariant in the fundamental representation, may be expressed as aN/n for all simple groups, whereas the adjoint Casimir invariant C_A has a form dependent on the particular group in question.

Finally we come to the "gluon projection operators" which play the central role in Cvitanović's algorithm for simplifying group-theoretic weights. The simplest case occurs for SU(n), where the relation of Fig. 3(f) holds: From the graphical form it is obvious how this may be used to eliminate adjoint lines (e.g., gluons) sandwiched between fundamental ones (e.g., quarks). This is slightly less

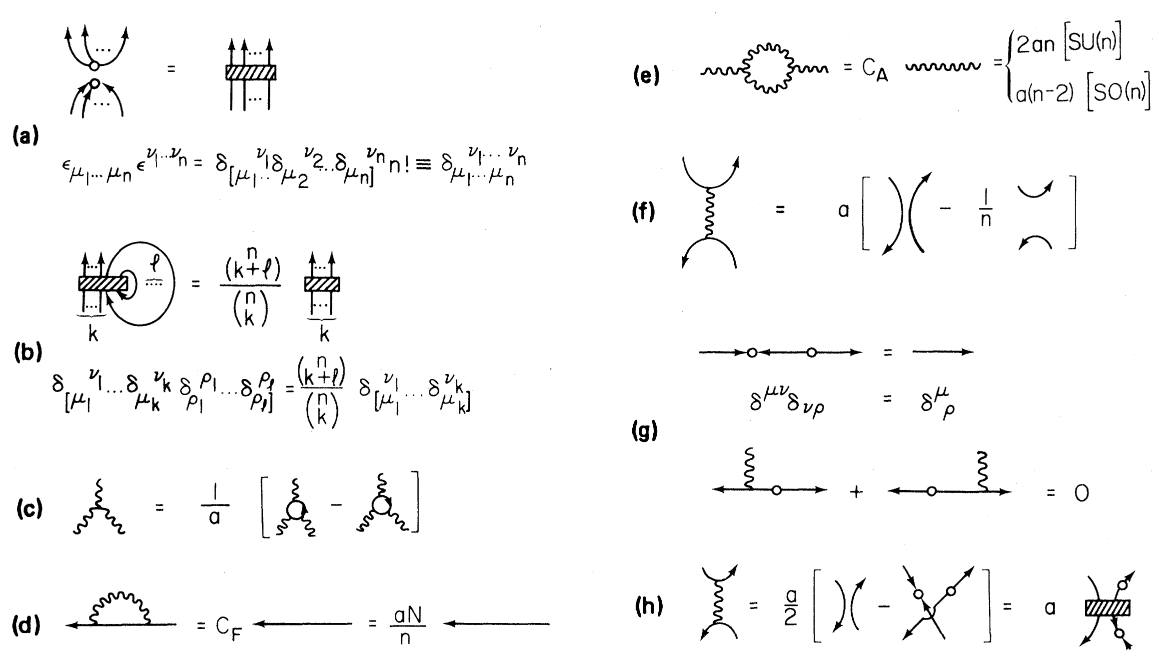

FIG. 3. Reduction identities for the fundamental and adjoint representations: (a) the relationship between the Levi-Civita tensor and the generalized Kronecker tensor; (b) a useful combinatorial identity; (c) reduction of the three-adjoint vertex into canonical form in terms of fundamental representation traces; (d) the quadratic Casimir invariant for the fundamental representation and its value for an arbitrary Lie algebra; (e) the quadratic Casimir invariant for the adjoint representation, and its value for SU(n) and SO(n) algebras, respectively; (f) the gluon projection operator for SU(n), this is the basic formula for removing internal adjoint lines (gluons in the terminology of Ref. 1); (g) the identities obeyed by the Kronecker δ tensor; (h) the gluon projection operator for SO(n).

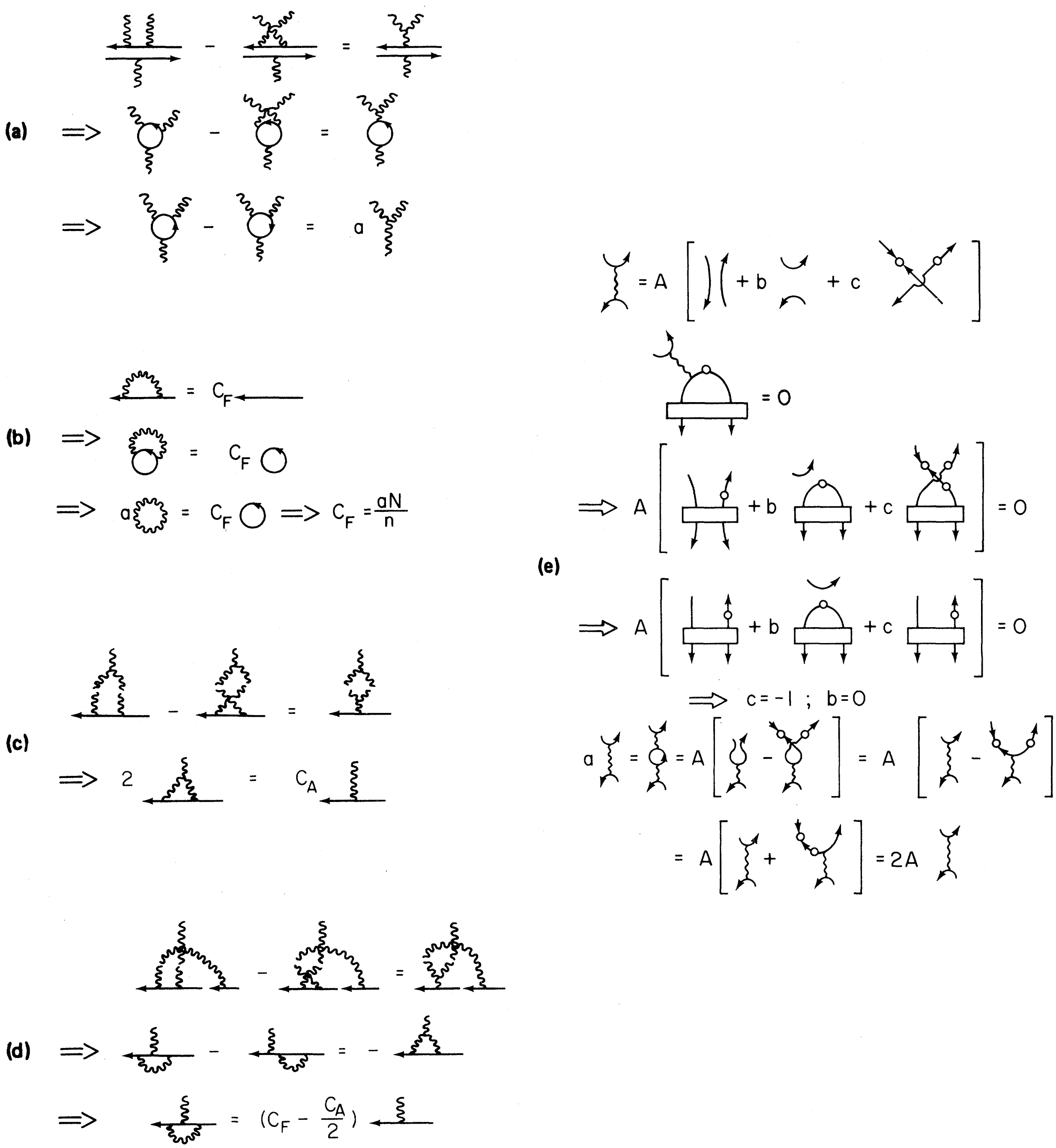

FIG. 4. Examples of diagrammatic reductions for the fundamental and adjoint representations: (a) proof of the identity of Fig. 3 (c) from the commutation relation of Fig. 2(a); (b) derivation of the general form of the quadratic Casimir invariant C_F of Fig. 3(d); (c) simplification of a "vertex correction" in terms of the adjoint quadratic Casimir invariant C_A [Fig. 3(e)]; (d) simplification of another vertex correction group weight factor in terms of Casimir invariants; (e) derivation of the SO(n) gluon projection operator of Fig. 3(h); (f) the relationship between the dimensions of the adjoint and fundamental representations for SO(n); (g) derivation of the form of the adjoint quadratic Casimir invariant C_A for SO(n) [Fig. 3(e)]; (h) proof that reversing the sense of a fundamental loop with an odd number of external adjoint lines changes its sign, making use of the identity of Fig. 3(g); (i) reduction of the three-adjoint vertex into canonical form in terms of a fundamental representation trace for SO(n), utilizing the last result [Fig. 4(h)] in Fig. 3(c).

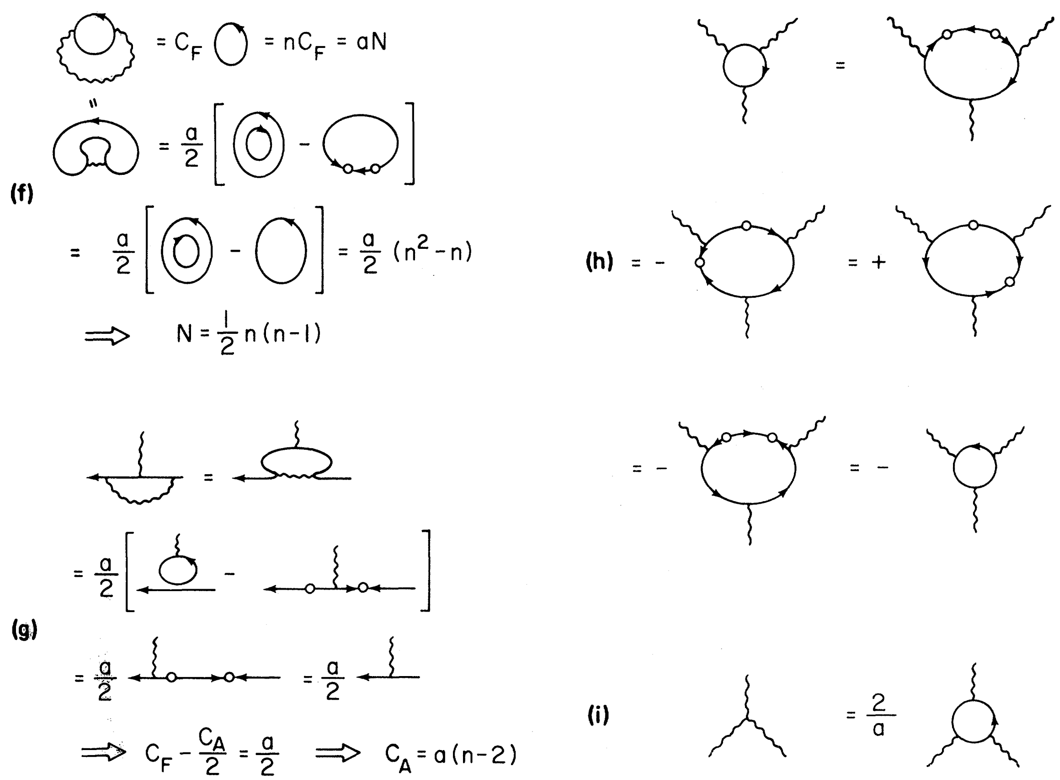

FIG. 4. *(Continued.)*

clear from the corresponding expression in terms of generators, viz.,

$$(T_i)_\mu{}^\nu (T_i)_\rho{}^\sigma = a\left[\delta^\sigma_\mu \delta^\nu_\rho - \frac{1}{n}\delta^\nu_\mu \delta^\sigma_\rho\right] .$$

While we shall not give proof of this result here, we want to point out here that it corresponds to the fact that the adjoint representation of SU(n) is equivalent to that acting on the traceless part of the tensor product of two C^n vectors, one transforming according to the fundamental representation and the other according to its conjugate.

The SO(n) analog in Fig. 3(h) is slightly more complicated since it involves the extra Kronecker tensor invariant of that group. This tensor has the basic properties summarized in Fig. 3(g), namely the trivial identity $\delta^{\mu\nu}\delta_{\nu\rho} = \delta^\mu_\rho$ and the relation

$$(T_i)_\mu{}^\nu \delta_{\nu\rho} + \delta_{\mu\nu}(T_i)_\rho{}^\nu = 0 ,$$

or equivalently $(T_i)_{\mu\rho} = -(T_i)_{\rho\mu}$, which follows from the invariance of the Kronecker tensor through Fig. 2(i). We shall give a derivation of the SO(n) gluon projection operator in Fig. 4(e), and we shall also develop further the idea which it encapsulates—namely that the adjoint representation is equivalent to that acting upon the antisymmetric product of two fundamental SO(n) vectors—in Fig. 5.

In order to explain the rather abstract formulation of the identities above we give a variety of examples of their application in Fig. 4. We include diagrammatic proofs of some of the identities of Fig. 3 among our examples. A particularly simple example illustrating the power of graphical methods is given in Fig. 4(a), in which we prove the result of Fig. 3(c). The reader is urged to reconstruct the proof in tensor notation to see how the operations of reordering terms and relabeling dummy indices are made redundant by the implicit structure of the graphical notation.

The example of Fig. 4(b) derives the general expression for the quadratic Casimir invariant of the fundamental representation given in Fig. 3(d). The key point illustrated here is how, after taking the trace of the expression defining C_F, we have a graph which may not only be viewed as a "self-energy correction" to the fundamental line but also as a "self-energy" bubble in the adjoint line, which may then be simplified using Fig. 2(d).

The ease of distorting a graph from one form to another, and of recognizing some graphs which have been drawn in different ways actually to be

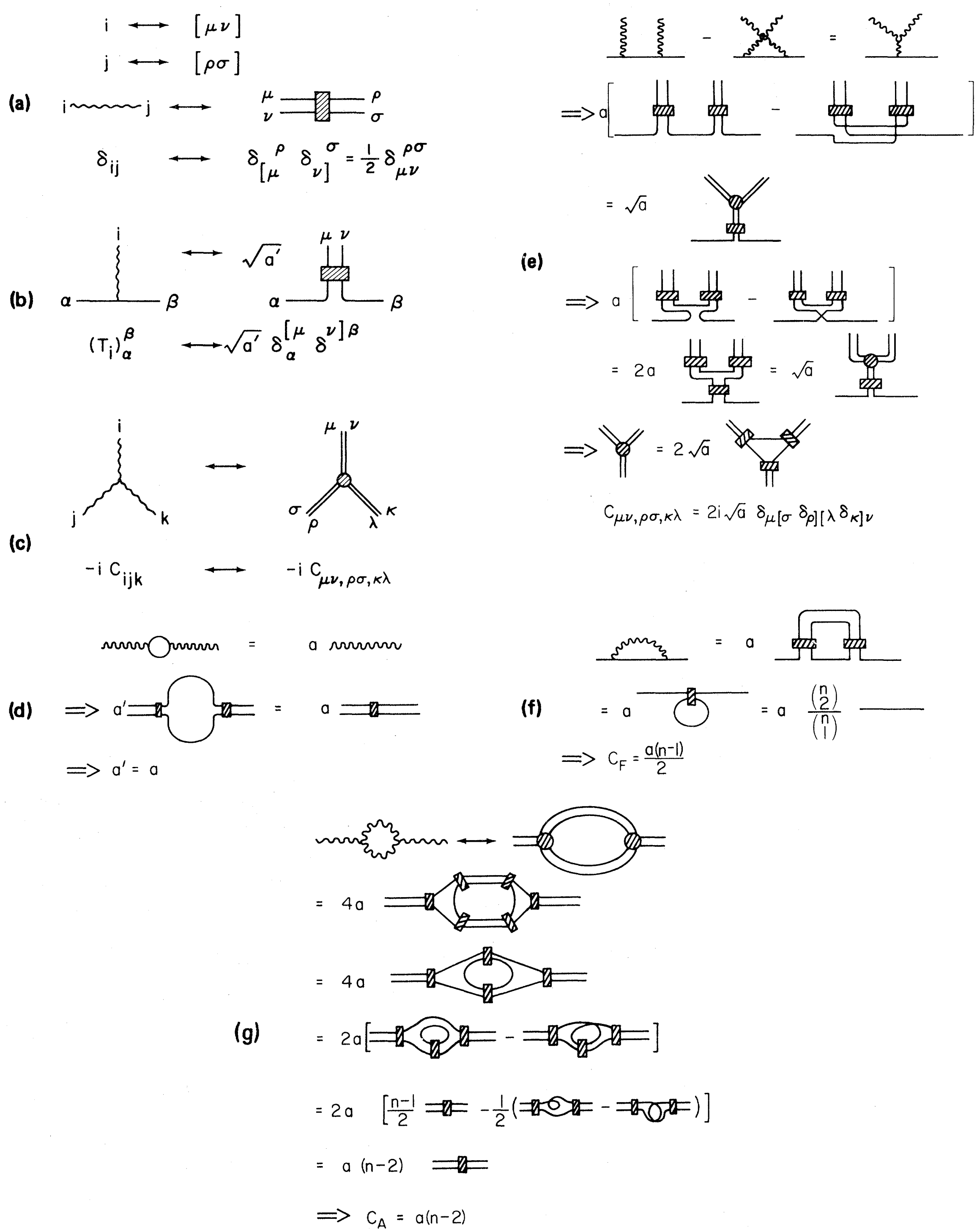

FIG. 5. An example of tensor product representations: The SO(n) adjoint propagator as an antisymmetric pair of fundamental propagators. (a) The correspondence between the previous notation for the adjoint propagator and the notation for it as an antisymmetric product of two fundamental propagators; (b) the notational correspondence for the fundamental generators; (c) the notational correspondence for the structure constants; (d) the normalization condition in the tensor product notation; (e) simplification of the commutation relations for the tensor product representation giving an explicit form for the structure functions in terms of Kronecker tensors; (f) derivation of the form of the quadratic Casimir invariant C_F of Fig. 3(d) for SO (n) [cf. Fig. 4(b)]; (g) derivation of the form of the adjoint quadratic Casimir invariant C_A of Fig. 3(e) for SO(n) [cf. Fig. 4(g)].

identical, is at the heart of the graphical method. This point is clearly demonstrated in Fig. 4(d), which derives an expression for the "vertex correction" graph in terms of the quadratic Casimir invariants, using the result of Fig. 4(c) which is in turn derived in a manner quite similar to our first example. Although the translation from diagrams to equations should be quite clear by now, we write the tensor form of the result of Fig. 4(d) below just to show that not only the proof but also the statement of such identities is simplified by graphical methods:

$$(T_i)_\mu{}^\rho (T_j)_\rho{}^\lambda (T_i)_\lambda{}^\nu = (C_F - \tfrac{1}{2} C_A)(T_j)_\mu{}^\nu \, .$$

We are now prepared to tackle the marginally more involved example of deriving the expressions for the gluon projection operators. As usual SU(n) provides the simplest example, but the SO(n) case can be easily dealt with too. The analogous results for the other simple groups are remarkably similar, and we refer the reader to Cvitanović's paper for details. The derivation involves three steps: First we write the most general form of the answer in terms of the available invariant tensors, second we insert the appropriate invariance relations of Figs. 2(h) or 2(i) for the invariants of the group in question in order to find the relative weights of each term present from the first step, and finally we find the overall normalization constant using Fig. 2(d).

As we do not use the SU(n) gluon projection operator in this paper, and it has been adequately explained elesewhere,[1] we shall omit its derivation here. The procedure is shown for SO(n) in Fig. 4(e). For the first step the only available tensors are the Kronecker tensors δ_μ^ν and $\delta_{\mu\nu}$, leading to three terms and three arbitrary parameters, A, b, and c. Notice that even for SO(2) where one might have expected an extra term involving the Levi-Civita tensor $\epsilon_{\mu\nu}$ this form is correct, as Fig. 3(a) shows that the extra term is not in fact independent of those already included. For the second step this form is inserted into the invariance relation for $\delta_{\mu\nu}$ (multiplied by a generator in the fundamental representation in the obvious way), and on connecting the appropriate fundamental lines, in the same way for each term of course, we find that b must equal 0 and $c=-1$. Finally an application of the normalization relation of Fig. 2(d), and the tracelessness condition of Fig. 2(c) shows that $A=a/2$, giving the desired result of Fig. 3(h).

Figures 4(f) and 4(g) derive the relation between the dimensions of the adjoint and fundamental representations, and the expression for the quadratic Casimir invariant C_A. The remaining two examples, Figs. 4(h) and 4(i), illustrate how the properties of the Kronecker tensor given by Fig. 3(g) may be used to prove that reversing the direction of an SO(n) fundamental loop changes its sign if it has an odd number of adjoint legs, and leaves it unchanged if it is has an even number of them; and further how this may be used to simplify the reduction rule of Fig. 3(c) for the SO(n) three-adjoint vertex.

This completes our review of Cvitanović's diagrammatic techniques. Before proceeding to discuss an extension of the method to deal with spinor representations, however, we want to study briefly tensor product representations, partly to throw some light on the significance of the gluon projection operator and partly to show how diagrammatic techniques can be extended to this case.

As is well known, higher-dimensional irreducible representations of SO(n) may be constructed from tensor products of the fundamental representation by reducing the tensors carrying the representation with respect to the action of the symmetric group on their tensor indices, and then removing all possible traces. In particular, the adjoint representation may be constructed on the antisymmetric product of two fundamental vectors, as is shown by the correspondences of Fig. 5(a). We have introduced a streamlined notation in which both Kronecker tensors and arrows on fundamental lines have been dropped as both may be reinserted at will (the reader is advised, however, to observe Fig. 4(h) and use this notation with due care). This correspondence induces the form of the fundamental-adjoint vertex of Fig. 5(b), where a' is an as-yet-undetermined normalization constant; this may also be viewed equivalently as an explicit realization of the generators $(T_i)_\alpha{}^\beta$ in terms of Kronecker tensors. The correspondence also induces an expression [Fig. 5(c)] for the structure constants (three-adjoint vertex), whose explicit form we shall deduce shortly. To fix the constant a' of Fig. 5(b), we insert the expressions of Figs. 5(a) and 5(b) into the Cartan normalization condition of Fig. 2(d) to obtain $a' = a$ [Fig. 5(d)]. In a similar fashion the commutator of Fig. 2(a) gives the explicit form

$$C_{\mu\nu,\rho\sigma,\kappa\lambda} = 2i\sqrt{a} \;\; \delta_{\mu[\sigma}\delta_{\rho][\lambda}\delta_{\kappa]\nu}$$

for the SO(n) structure constants as is shown in Fig. 5(e): notice the similarity of the graphical form of this result with the three-adjoint vertex of Fig. 4(i). Now that we have found out how to translate problems involving the adjoint representation into ones involving only the fundamental representation we may proceed to re-solve some previous examples in the new notation. Figures 5(f) and 5(g) show how the computation of C_F and C_A, respectively, are reduced to mere combinatorial manipulations, and although they are more cumbersome than our previous methods, they do illustrate the significance of the techniques.

III. SPINOR REPRESENTATIONS AND USE OF FIERZ TRANSFORMATIONS

A. Spinors

The methods of the previous section illustrate the rules for computing group-theoretic weights in all the representations of SO(n) which can be constructed from tensor products of the defining fundamental representation. There is one very important representation which is not included in this class, namely the double-valued[10] spinor representation, and it is the purpose of this section to show how graphical methods may be developed to deal with this case also.[11]

The properties of spinors are particularly simply derived by considering a set $\{\gamma_\mu\}$ of generators of a Clifford algebra (that is, they obey the anticommutation relations $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$) which transform into each other under the action of the fundamental representation of SO(n):

$$\gamma_\mu \to \tilde{\gamma}_\mu \equiv \Omega_\mu{}^\nu \gamma_\nu \ .$$

$\Omega_\mu{}^\nu$ is an $n\times n$ matrix representing a finite SO(n) transformation, which therefore obeys the relation

$$\Omega_\mu{}^{\mu'}\Omega_\nu{}^{\nu'}\delta_{\mu'\nu'}=\delta_{\mu\nu} \ .$$

The important observation is that the structure of the Clifford algebra is preserved by this transformation, since

$$\{\gamma_\mu,\gamma_\nu\}\to\{\tilde{\gamma}_\mu,\tilde{\gamma}_\nu\}=\Omega_\mu{}^{\mu'}\Omega_\nu{}^{\nu'}\{\gamma_{\mu'},\gamma_{\nu'}\}$$
$$=\Omega_\mu{}^{\mu'}\Omega_\nu{}^{\nu'}\delta_{\mu'\nu'}=\delta_{\mu\nu} \ .$$

As the algebra can be represented faithfully by the complete algebra of $2^{n/2}\times 2^{n/2}$ complex matrices (for n even[12]), a simple theorem[7] tells us that every automorphism of the algebra is necessarily an inner automorphism: In other words there is an (invertible) element $S(\Omega)$ of the Clifford algebra such that the structure-preserving SO(n) transformation above may also be written as

$$\tilde{\gamma}_\mu \equiv \Omega_\mu{}^\nu\gamma_\nu = S^{-1}(\Omega)\gamma_\mu S(\Omega) \ .$$

Up to a factor these quantities $S(\Omega)$ provide a representation of SO(n), for

$$S(\Omega_1)S(\Omega_2)=CS(\Omega_1\Omega_2) \ ,$$

and therefore we have a new representation given by the matrices representing the quantities $S(\Omega)$ carried by vectors in $C^{2^{n/2}}$.[7]

B. Spinor diagrams

To represent these quantities graphically we introduce the propagator δ_{ab} for a spinor and draw it as a dashed directed line [Fig. 1(c)], and the vertex $(\gamma_\mu)_{ab}$ [Fig. 1(f)] connecting the spinor and fundamental lines. The basic rules for manipulating such quantities are given in Fig. 6.

Most properties of the Clifford algebra follow from the basic recursion relation illustrated in Fig. 6(d),

$$\mathrm{tr}[\gamma_{\mu_1}\cdots\gamma_{\mu_{i-1}}\gamma_{\mu_i}\gamma_{\mu_{i+1}}\cdots\gamma_{\mu_{p-1}}\gamma_{\mu_p}]=\sum_{i=1}^{p-1}(-)^{i+1}\delta_{\mu_i\mu_p}\mathrm{tr}[\gamma_{\mu_1}\cdots\gamma_{\mu_{i-1}}\gamma_{\mu_{i+1}}\cdots\gamma_{\mu_{p-1}}] \ .$$

This last formula may be expanded out to give the trace of an even number of γ's (the trace of an odd number vanishes) as the sum over all possible ways of connecting the fundamental lines with a minus sign for every time two such connecting lines cross and an overall factor of tr[1]. This is shown for four γ's in Fig. 6(d) and for six γ's in Fig. 6(e), and this latter example makes the disadvantages of the naive application of the formula especially clear: The number of terms involved in taking the trace of p γ's is $(p-1)!!$ which grows exponentially with p. As we showed previously[11] we can introduce an algorithm for the reduction of traces which is only of polynomial order in the number of γ's occurring, provided we make greater use of the symmetries of the problem under consideration. Before stating the necessary results we wish to introduce a few useful elementary graphical rules.

The first of these [Fig. 6(f)] shows that we are free to reverse the direction of a closed spinor loop whenever we please. This follows from the observation that the mapping $\gamma\to\gamma^T$ leaves the algebraic structure of the γ's unchanged,

$$\{\gamma_\mu,\gamma_\nu\}\to\{\gamma_\mu^T,\gamma_\nu^T\}=\{\gamma_\nu,\gamma_\mu\}^T=\delta_{\mu\nu} \ ,$$

which in more mathematical terminology means

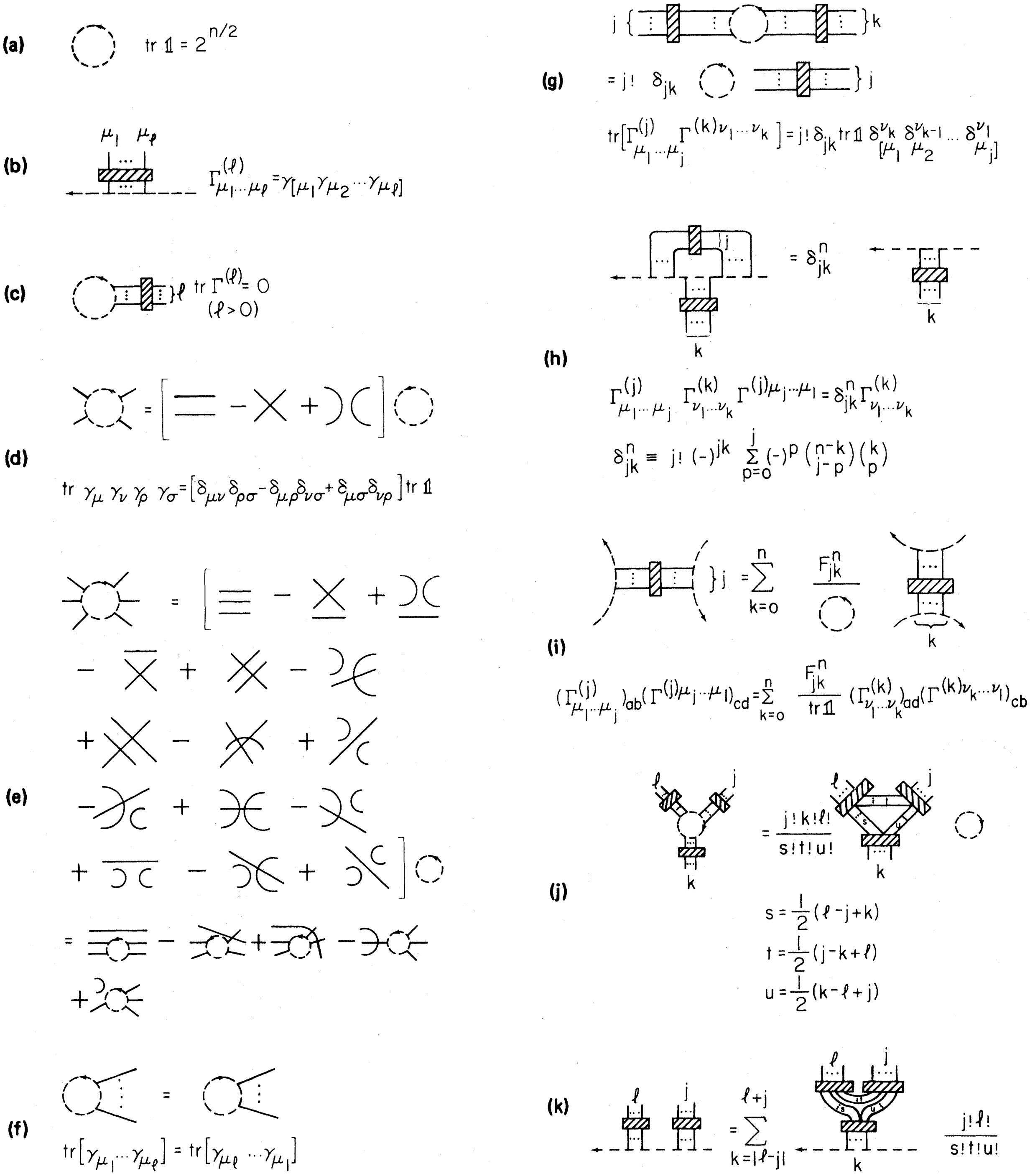

FIG. 6. Basic identities for the spinor representation: (a) notation for the trace of the unit matrix; (b) definition of the totally antisymmetric basic tensors Γ; (c) tracelessness condition for the Γ's; (d) an example of the basic reduction identity for traces, we connect the external fundamental lines in all possible ways with a minus sign for each time the lines cross and multiply by tr[1]; (e) an example of the basic reduction identity applied to a trace of six γ matrices, showing how the number of terms proliferates and also showing the relationship to the recursive reduction formula for traces given in the text; (f) the reversibility of the sense of a spinorial trace; (g) the orthogonality condition satisfied by the Γ's; (h) the general contraction identity for the Γ's [this generalizes the familiar result $\gamma_\mu \gamma_\nu \gamma^\mu = (2-n)\gamma_\nu$]; (i) the Fierz identity (the Fierz coefficients F^n_{jk} are derived in Fig. 7); (j) reduction formula for the trace of three Γ's; (k) "Clebsch-Gordan" series for Γ's.

that it furnishes an outer automorphism of the Clifford algebra, and by the previously stated theorem on complete matrix algebras[7] there must also be an inner automorphism which performs the same transformation,[13] $\gamma^T=\eta^{-1}\gamma\eta$.

We next introduce the orthogonality theorem for the basis elements Γ of the Clifford algebra in Fig. 6(g); notice that the fundamental line indices $\mu_1,\ldots,\mu_j,\nu_1,\ldots,\nu_k$ are written in cyclic order around the diagram in accordance with the previously stated rules. This result is helpful in that if allows us to project out a particular basis element by multiplying by a Γ and taking the trace.

The next result, shown in Fig. 6(h), plays a key role in deriving the important Fierz transformation identity. It is a purely combinatorial contraction theorem whose proof may be found in Ref. 11. The coefficients

$$\delta^n_{jk}\equiv j!(-)^{jk}\sum_{p=0}^{j}(-)^p\begin{bmatrix}n-k\\j-p\end{bmatrix}\begin{bmatrix}k\\p\end{bmatrix}$$

have some interesting relationships as we shall see in the following, demonstrating their underlying group-theoretic significance. From this result we shall derive (Fig. 7) the general form for the Fierz transformation[5] shown in Fig. 6(i). This equation will provide the basis for our treatment of spinors in the diagrammatic approach to group-theory weight factors. The reader will immediately see the similarities between the Fierz transformation and the gluon projection operator of the previous section, and indeed they will be used in very similar ways; nevertheless their derivations are quite different and they have some quite distinct differences as well as similarities. The Fierz coefficients are given by the following formula:

$$F^n_{jk}\equiv\frac{\delta^n_{jk}}{k!}$$
$$=(-)^{jk}\frac{j!}{k!}\sum_{p=0}^{j}(-)^p\begin{bmatrix}n-k\\j-p\end{bmatrix}\begin{bmatrix}k\\p\end{bmatrix}.$$

The remaining two results are again combinatorial, the first being based on the observation that if we connect together three basis tensors $\Gamma^{(j)}$, $\Gamma^{(k)}$, and $\Gamma^{(l)}$ in all possible ways in order to evaluate the trace $\mathrm{tr}[\Gamma^{(j)}\Gamma^{(k)}\Gamma^{(l)}]$ on the left-hand side of the diagrammatic identity of Fig. 6(j) according to the recursion formula illustrated in Figs. 6(d) and 6(e), then there is only one structure possible for the right-hand side which is consistent with the symmetry. A simple calculation then shows that the number of lines connecting the generalized Kronecker tensors must be s, t, and u as shown in

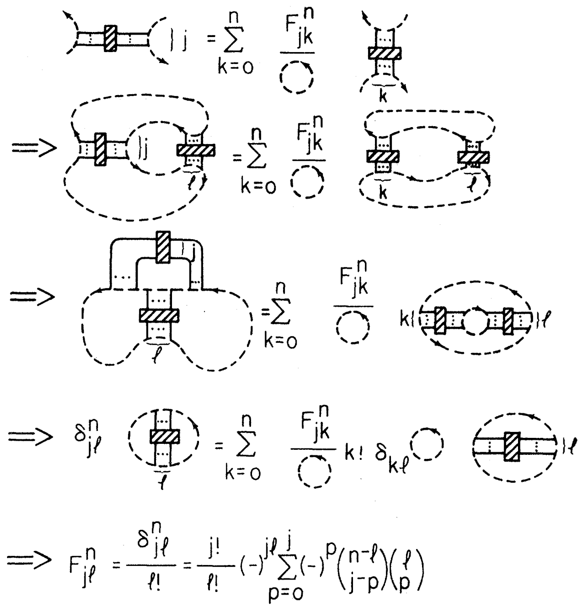

FIG. 7. Diagrammatic derivation of the Fierz transformation coefficients.

the figure, and furthermore the combinatorial coefficient for the whole tensor on the right-hand side must be $j!k!l!/s!t!u!$.

This leads easily to the result of Fig. 6(k), which is the spinorial analog of the Clebsch-Gordan series, expressing $\Gamma^{(l)}\Gamma^{(j)}$ as a sum over the single basis elements $\Gamma^{(k)}$. To derive this result we expand the product of basis elements in terms of the complete set of basis tensors $\Gamma^{(k)}$ and then determine the coefficents in this expansion by projecting them out using Figs. 6(g) and 6(j).

This last formula [Fig. 6(k)] may be used to reduce any γ-matrix expression into a sum of combinations of generalized Kronecker tensors of the form appearing on the right-hand side of the equation,[11] and furthermore this reduction may be performed for an expression involving p factors in approximately p^2 steps. Naturally we are still left with the problem of simplifying the tensor expression this procedure leaves us with, which may be quite a nontrivial problem, but sensible use of the known symmetries of the original expression can make this task much easier (for an approach to how this may be done see the paper by Canning[6]). We shall not follow this approach further here; although it is an eminently suitable method for use on a computer it very rapidly becomes very cumbersome for hand calculation. Instead we shall introduce a new method based on the Fierz transformation formula exhibited in Fig. 6(i).

C. Application of Fierz transformations

Let us now show how the Fierz transformation provides a convenient method for performing diagrammatic spinor calculations. As a simple example, let us consider the expression $\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho]$ shown in Fig. 8(a). As this is the first example of the use of the method we shall go through all the stages involved in detail, although it is usually possible to combine many of the steps together in actual use. The first step is to redraw the figure so as to make things clearer, and we have done so by moving a couple of spectator lines to the outside of the spinor loop and by pinching the loop near the fundamental line to which we are going to apply the Fierz transformation. We now apply the Fierz identity [Fig. 6(i)] for the case where $j=1$ (there is a single fundamental line of the left-hand side of the identity), and we obtain two terms from the sum on the right-hand side. This is because the traces remaining after the Fierz reordering vanish if we introduce more than two antisymmetrized fundamental lines between them and also if we introduce an odd number of such lines. For simplicity we shall assume that $n=4$, so F_{12}^4 vanishes, and only the first term survives. We now use the orthogonality theorem of Fig. 6(g) for the bubble on the left, and again redraw the remaining graph to make it clear how it can be reduced once again by the Fierz identity. Only one term survives this time, as can be seen by using the identity of Fig. 6(c) (tracelessness of the Γ's), and this can be reduced yet again by the Fierz identity to remove the final fundamental line from the graph to give the result $(F_{10}{}^4)^3$ tr[1], which is easily calculated to be 64 tr[1]. The only real difficulty with this technique is in deciding which fundamental line should be removed by an application of the Fierz transformation, as there are often several inequivalent choices, but once this decision has been made the application of the identity is quite straightforward.

An interesting example of some of the other techniques which may be of use in simplifying γ-matrix expressions is the commutation relation of Fig. 8(b). It is well known that the generators of SO(n) transformations for the spinor representation are proportional to the tensors $\Gamma^{(2)}_{ab}$, and this calculation will therefore explicitly check whether the necessary commutation relations are obeyed in this case. Using the fact that the external fundamental lines are pairwise antisymmetric we can write the commutator in terms of the three possible Γ tensors with three undetermined coefficients A, B, and C: The tensor form of this graphical expansion is

$$\Gamma^{(2)}_{\mu\nu}\Gamma^{(2)}_{\rho\sigma}-\Gamma^{(2)}_{\rho\sigma}\Gamma^{(2)}_{\mu\nu}=\frac{A}{2}(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})\Gamma^{(0)}+\frac{B}{4}(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})(\delta_{\rho\beta}\delta_{\sigma\gamma}-\delta_{\rho\gamma}\delta_{\sigma\beta})\Gamma^{(2)}_{\alpha\gamma}+C\Gamma^{(4)}_{\mu\nu\rho\sigma}\ .$$

In order to determine these coefficients we use the orthogonality property of Fig. 6(g) to project out each term in turn. In step (1) we project out the $\Gamma^{(0)}$ component by multiplying by $\Gamma^{(0)}$ (which being the unit matrix has no effect) and then taking the trace over the spinor indices. When we have untwisted the second term from the commutator we get the difference of two terms which differ only in the direction of the spinor loop, and using the result of Fig. 6(f) we therefore find that $A=0$. In step (2) we project out the $\Gamma^{(4)}$ component similarly, and again we get the vanishing difference of two terms differing only in the sense of a spinor loop; this time we must make use of the fact that reversing the order of the lines on $\Gamma^{(4)}$ has no effect, because

$$\Gamma^{(4)}_{\mu\nu\rho\sigma}=\Gamma^{(4)}_{\sigma\rho\nu\mu}\ .$$

In step (3) we turn to the determination of the coefficient B, and this time we get a nonzero answer because there is an additional minus sign arising from reversing the order of the lines on $\Gamma^{(2)}$. This does indeed give the expected result for the commutator, namely that it is a multiple of the generator $\Gamma^{(2)}$, and furthermore the form of the structure constants obtained in terms of generalized Kronecker tensors is consistent with that found using the fundamental representation in Fig. 5(e). Indeed, we extend the correspondences of Figs. 5(a)–5(c) by identifying the generators in the spinor representation $(T_i)_{ab}$ with a particular multiple of $\Gamma^{(2)}$, namely,

$$(T_i)_{ab}\leftrightarrow\frac{\sqrt{a}}{4}\Gamma^{(2)}_{ab}\ ,$$

as is shown in Fig. 8(c).

The next three examples are related to the quadratic Casimir invariants C_A and C_F introduced in Sec. II. We introduce the analogous quantities for the spinor representation C_{SF} and C_{SA} defined in Figs. 8(d) and 8(e), respectively:

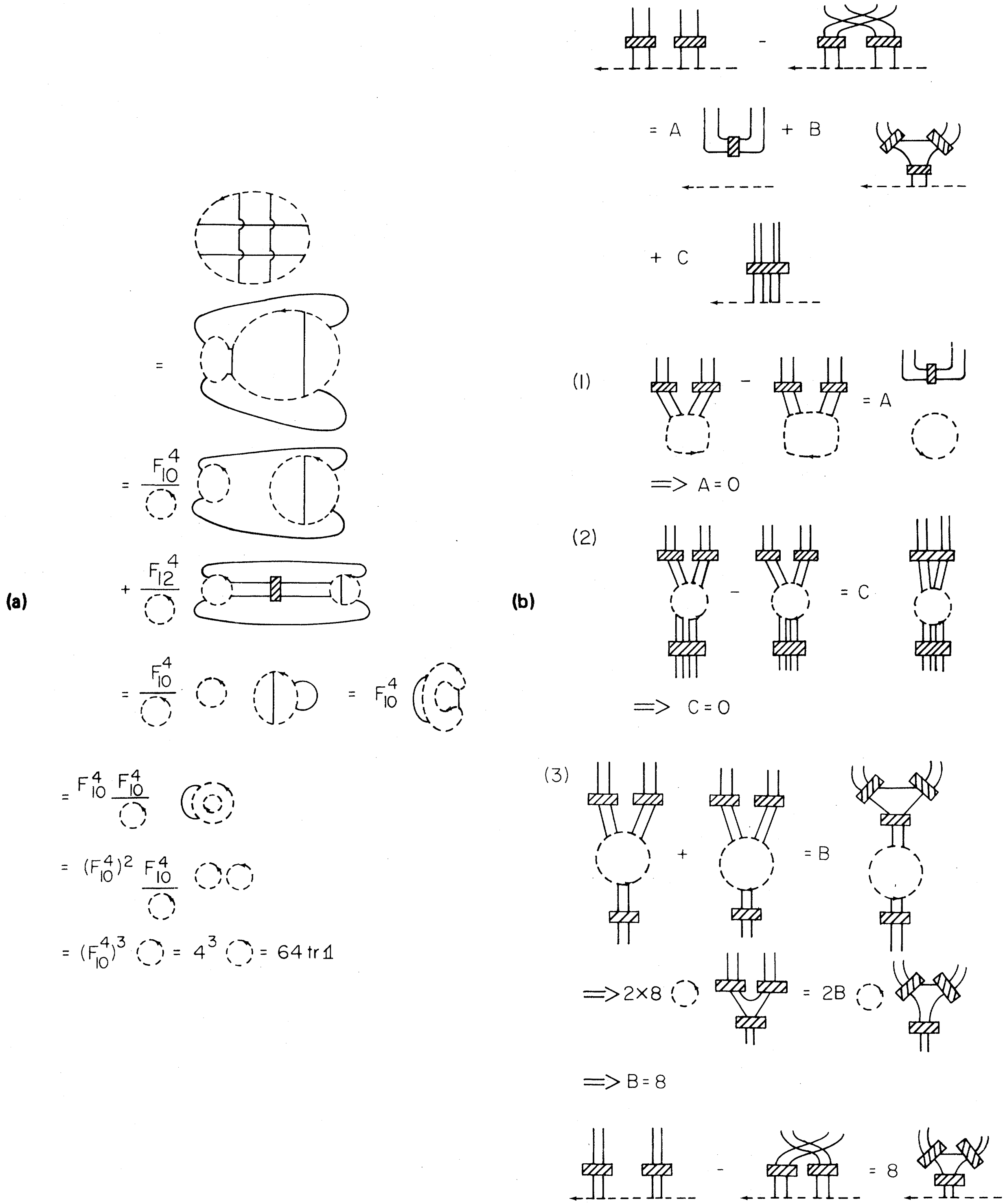

FIG. 8. Examples of diagrammatic methods for spinors in (even) integer-dimensional spaces: (a) four-dimensional reduction of $\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho]$ using Fierz reordering; (b) derivation of the commutation relations for the generators in the spinor representation [cf. Fig. 5(e)]; (c) correspondence between Fig. 2(a) and the results of Fig. 8(b); (d) spinor fundamental quadratic Casimir invariant C_{SF}; (e) spinor adjoint quadratic Casimir invariant C_{SA}; (f) general spinor quadratic Casimir invariant C_{Sj}; (g) a spinor vertex correction to a γ matrix; (h) a spinor vertex correction to a generator in the spinor representation [cf. Fig. 6(h)].

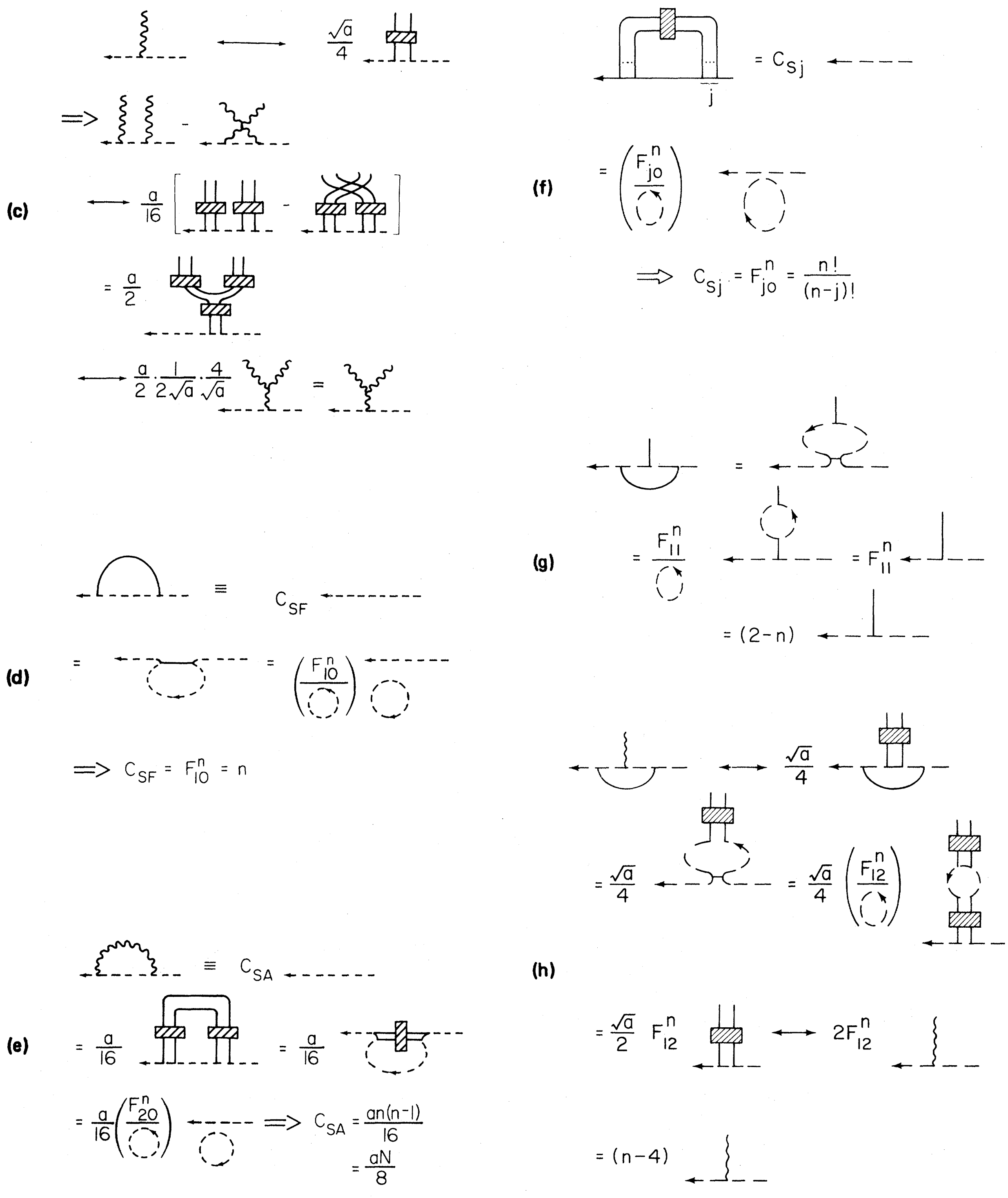

FIG. 8. *(Continued.)*

$$(\gamma_\mu\gamma_\mu)_{ab} = C_{SF}\delta_{ab} ,$$

$$(T_iT_j)_{ab} = \frac{a}{16}(\Gamma^{(2)}_{\mu\nu}\Gamma^{(2)}_{\nu\mu})_{ab} = C_{SA}\delta_{ab} ,$$

and the simple manipulations shown in the figures gives the result that $C_{SF}=n$ and $C_{SA}=aN/8$. Both of these calculations were of course straightforward anyhow, as formulas such as $\gamma_\mu\gamma_\mu=n$ are

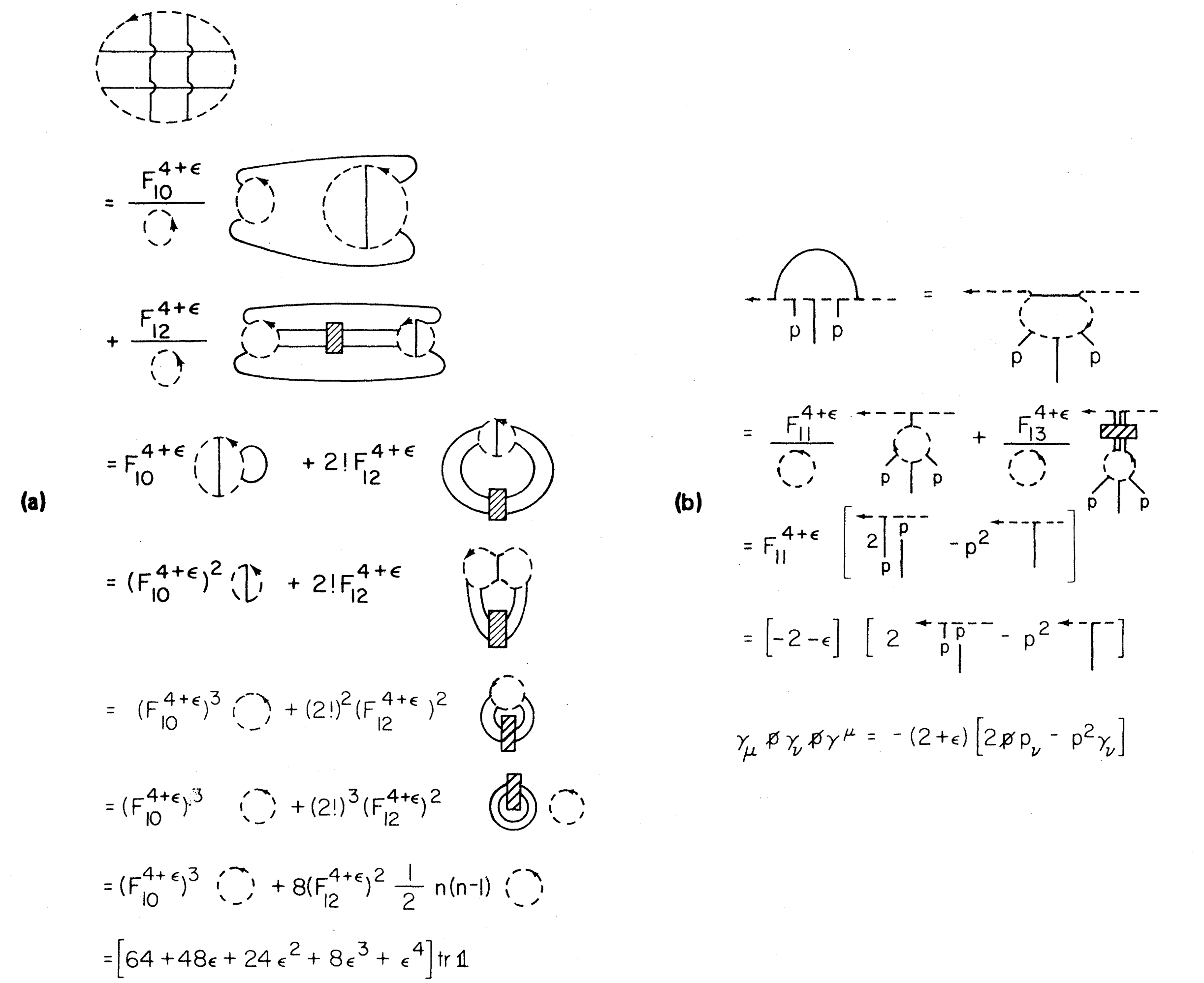

FIG. 9. Examples of diagrammatic method for spinors in $4+\epsilon$ dimensions: (a) reduction of $\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho]$ using Fierz reordering [cf. Fig. 8(a)]; (b) simplification of the expression $\gamma_\mu \not{p} \gamma_\nu \not{p} \gamma^\mu$ using Fierz transformations.

quite familiar, but they show the way in which the Fierz coefficients F^n_{jk} summarize all the usual identities. Many other similar identities are shown by the natural generalization to the family of spinorial quadratic Casimir invariants C_{Sj} of Fig. 8(f).

Figure 8(g) shows how to compute the group-theoretic weight associated with a vertex correction diagram. Once again it is a simple application of a Fierz transformation and the orthogonality property [Fig. 6(g)]. Figure 8(h) derives a similar result for a correction to the adjoint-spinor vertex, and quickly leads to a simple answer. We should not be surprised at the simplicity of these results as they are only special cases of the general spinor vertex correction graph of Fig. 6(h) from which we derived the expression for the Fierz coefficients in the first place.

IV. THE ANALYTIC CONTINUATION OF FIERZ TRANSFORMATIONS

A. Definition

So far we have been primarily concerned with spinorial representations of internal SO(n) symmetry groups, but of course one of the most important applications of spinors is that they carry the representation of the Poincaré group which describes fermions. Within the context of dimensional regularization it is not sufficient to perform the Dirac algebra in four dimensions—it becomes necessary to calculate in its n-dimensional generalization to a Clifford algebra. The techniques explained in the previous section apply to all[14] integer values of n, and therefore they are quite ade-

TABLE I. The Fierz coefficients $F_{jk}^{(0)}$ (defined in the Appendix) arising in the expansion of $F_{jk}^{4+\epsilon}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$
1	4	-2	0	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{20}$	$-\frac{1}{90}$
2	12	0	-2	0	$\frac{1}{2}$	$\frac{4}{15}$	$\frac{1}{12}$
3	24	12	0	-2	-1	$\frac{13}{10}$	$-\frac{3}{5}$
4	24	-24	12	-4	1	$\frac{31}{5}$	$\frac{43}{10}$
5	0	0	0	0	0	32	-32
6	0	0	0	0	0	192	256

quate to compute the group-theoretic weights for a Feynman diagram as a function of n. These can then be "analytically continued" into the complex n plane, but it is not clear that this is the best way of setting about the calculation.

One reason for doubting this is that we calculate an expression $f(n)$ whose Taylor expansion[15] about $n=4$ often involves high powers of the new variable $\epsilon \equiv n-4$, whereas we usually are only interested in the first few terms of this expansion. The obvious solution to this problem is to calculate f directly as a function of ϵ, and then not to generate any powers of ϵ higher than those we need during the calculation. For this reason we shall show how to analytically continue the expression for the Fierz coefficients in n dimensions,

$$F_{jk}^{n} \equiv \frac{\delta_{jk}^{n}}{k!} = (-)^{jk}\frac{j!}{k!}\sum_{p=0}^{j}(-)^{p}\begin{bmatrix} n-k \\ j-p \end{bmatrix}\begin{bmatrix} k \\ p \end{bmatrix} .$$

The simplest way to do this is based upon the definition of the Jacobi polynomials[16] (which are the most general classical orthogonal polynomials with a quadratic weight function)[17]:

$$P_{n}^{(\alpha,\beta)}(x) \equiv \frac{1}{2^n}\sum_{m=0}^{n}\begin{bmatrix} n-\alpha \\ m \end{bmatrix}\begin{bmatrix} n+\beta \\ n-m \end{bmatrix}(x-1)^{n-m}(x+1)^{m} \quad (\alpha > -1, \beta > -1) .$$

From this we immediately see that

$$F_{jk}^{n} \equiv \frac{j!}{k!}(-)^{jk}2^{j}P_{j}^{(n-k-j,k-j)}(0) \quad (n > k+j-1, j < k+1) ,$$

which we shall take as the definition of the Fierz coefficients for noninteger values of n when the Jacobi polynomials are analytically continued into the complex n plane.

TABLE II. The Fierz coefficients $F_{jk}^{(1)}$ (defined in the Appendix) arising in the expansion of $F_{jk}^{4+\epsilon}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$
2	7	3	$-\frac{1}{2}$	$-\frac{5}{6}$	$-\frac{3}{8}$	$-\frac{13}{120}$	$-\frac{17}{720}$
3	26	4	-5	$-\frac{4}{3}$	$\frac{25}{12}$	$-\frac{29}{30}$	$\frac{103}{360}$
4	50	-38	9	$\frac{13}{3}$	$-\frac{103}{12}$	$-\frac{29}{4}$	$-\frac{1079}{360}$
5	24	36	32	26	31	$-\frac{1531}{30}$	$\frac{443}{15}$
6	-24	48	-60	72	-129	$-\frac{1846}{5}$	$-\frac{8963}{30}$

TABLE III. The Fierz coefficients $F^{(2)}_{jk}$ (defined in the Appendix) arising in the expansion of $F^{4+\epsilon}_{jk}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$
3	9	-3	$-\frac{3}{2}$	$\frac{3}{2}$	$-\frac{5}{8}$	$\frac{7}{40}$	$-\frac{3}{80}$
4	35	-13	$-\frac{13}{2}$	$\frac{35}{6}$	$\frac{131}{24}$	$\frac{55}{24}$	$\frac{467}{720}$
5	50	60	35	$\frac{20}{3}$	$-\frac{145}{4}$	$\frac{145}{6}$	$-\frac{661}{72}$
6	-26	34	-13	$-\frac{103}{3}$	$\frac{2627}{12}$	$\frac{13\,937}{60}$	$\frac{42\,467}{360}$

In practice it is easy to calculate any particular Fierz coefficient. For example, the value of F^{n}_{35} follows at once from the above expressions:

$$F^{n}_{35}=-\frac{n^3-33n^2+332n-1020}{120}=-\frac{\epsilon^3}{120}+\frac{7\epsilon^2}{40}-\frac{29\epsilon}{30}+\frac{13}{10}\ ,$$

where $n=4+\epsilon$. The Appendix and Tables contain most of the Fierz coefficients likely to be of practical use in $(4+\epsilon)$-dimensional calculations.

We may also find the generic form of the expansion of the Fierz coefficients $F^{4+\epsilon}_{jk}$ in powers of ϵ for arbitrary j and k, although this is less useful in practice. After expanding the Γ functions we find[18]

$$\begin{aligned}F^{4+\epsilon}_{jk}=\Gamma(j+1)\Gamma(5-k)(-)^{jk}\sum_{p=0}^{j}&\frac{(-)^p}{\Gamma(j-p+1)\Gamma(k-p+1)\Gamma(p+1)\Gamma(p-k-j+5)}\\&\times\Bigg\{1-[\psi_0(p-j-k+5)-\psi_0(5-k)]\epsilon\\&\quad+[\psi_0{}^2(p-j-k+5)-2\psi_0(5-k)\psi_0(p-j-k+5)\\&\quad-\psi_1(p-j-k+5)+\psi_0{}^2(5-k)+\psi_1(5-k)]\frac{\epsilon^2}{2}+O(\epsilon^3)\Bigg\}\ ,\end{aligned}$$

so, for example, setting $j=3$ and taking the limit as $k\to 5$ we find

TABLE IV. The Fierz coefficients $F^{(3)}_{jk}$ (defined in the Appendix) arising in the expansion of $F^{4+\epsilon}_{jk}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$
4	10	2	-3	$-\frac{7}{3}$	$-\frac{11}{12}$	$-\frac{1}{4}$	$-\frac{19}{360}$
5	35	25	$-\frac{5}{2}$	$-\frac{95}{6}$	$\frac{275}{24}$	$-\frac{107}{24}$	$\frac{175}{144}$
6	15	-45	$\frac{135}{2}$	$-\frac{135}{2}$	$-\frac{875}{8}$	$-\frac{499}{8}$	$-\frac{1063}{48}$

TABLE V. The Fierz coefficients $F_{jk}^{(4)}$ (defined in the Appendix) arising in the expansion of $F_{jk}^{4+\epsilon}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$
5	10	0	-5	$\frac{10}{3}$	$-\frac{5}{4}$	$\frac{1}{3}$	$-\frac{5}{72}$
6	25	-35	$\frac{25}{2}$	$\frac{205}{6}$	$\frac{505}{24}$	$\frac{185}{24}$	$\frac{293}{144}$

$$F_{35}^{4+\epsilon}=\lim_{k\to 5}\left[6\Gamma(5-k)(-)^{3k}\left[\frac{1}{6\Gamma(2-k)\Gamma(k+1)}-\frac{1}{2\Gamma(3-k)\Gamma(k)}+\frac{1}{2\Gamma(4-k)\Gamma(k-1)}-\frac{1}{6\Gamma(5-k)\Gamma(k-2)}\right]\right]+O(\epsilon)=\tfrac{13}{10}+O(\epsilon)\ .$$

B. Properties and Applications

The Fierz coefficients have several interesting symmetries which were discussed in Ref. 11 and these are related to symmetries of the Jacobi polynomials. For example, the identity

$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)2^{\beta}}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)(x+1)^{\beta}}\times P_{n+\beta}^{(\alpha,-\beta)}(x)$$

follows from the Fierz symmetry

$$F_{jk}^{n}=\frac{(n-k)!j!}{(n-k)!k!}F_{kj}^{n}\ ,$$

and this result may be checked directly with the use of the simple identity

$$\binom{n+\alpha}{m}\binom{n+\beta}{n-m}\binom{2n+\alpha+\beta}{n+\beta}=\binom{n+\alpha+\beta}{m+\beta}\binom{n}{n-m}\binom{2n+\alpha+\beta}{n}\ .$$

With the $(4+\epsilon)$-dimensional Fierz coefficients given in the Appendix, spinor graphs can be simplified in noninteger dimensions just as easily as they could for integer dimensions; for example, Fig. 9(a) repeats the calculation of Fig 8(a) except taking $n=4+\epsilon$ rather than 4. In the first step only the terms involving $F_{10}^{4+\epsilon}$ and $F_{12}^{4+\epsilon}$ are included as further terms in the Fierz expansion vanish (they involve traces of γ matrices which are an-

TABLE VI. The Fierz coefficients $F_{jk}^{(5)}$ (defined in the Appendix) arising in the expansion of $F_{jk}^{4+\epsilon}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$
6	9	-3	$-\frac{15}{2}$	$-\frac{9}{2}$	$-\frac{13}{8}$	$-\frac{17}{40}$	$-\frac{7}{80}$

TABLE VII. The Fierz coefficients $F_{jk}^{(6)}$ (defined in the Appendix) arising in the expansion of $F_{jk}^{4+\epsilon}$ in powers of ϵ.

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$

tisymmetrized on more than half their indices). The identity shown in Fig. 3(b) proves useful in the penultimate step in the derivation. Although we have given the exact result, which involves terms up to order ϵ^4, it is of course only necessary to calculate the result to the order in ϵ appropriate to the probelm under consideration; for instance, one could drop all but constant terms and terms linear in ϵ for a one-loop Feynman diagram calculation where dimensional regularization can give at most a simple pole in ϵ.

This last example is of course somewhat unrealistic insofar as the fermion propagators in Feynman diagrams are of the form $(\not p + m)/(p^2 - m^2)$ and are not just proportional to the spinorial unit matrix. A more realistic example is given in Fig. 9(b) which performs the Dirac algebra for the one-loop vertex correction in massless quantum electrodynamics. We have labeled some of the external fundamental lines with a momentum p to indicate the quantity $\not p$ in a natural way. For such calculations the graphs become a little cumbersome— and even more so when massive fermion propagators are included—but this is only to be expected as in general the result itself is of a comparable degree of complexity. In the cases where this is not so, such as when there are only one or two momenta and masses from which to form invariants which can appear in the answer, it is often possible to exploit the symmetries in the momenta to simplify the manipulations.

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APPENDIX

We give the values of the Fierz coefficients F_{jk}^{n} in $4+\epsilon$ dimensions for small values of j and k. We write

$$F_{jk}^{4+\epsilon} \equiv \sum_{n=0}^{\infty} F_{jk}^{(r)} \epsilon^r ,$$

where the values of $F_{jk}^{(r)}$ are given in Tables I–VII.

[1]P. Cvitanović, Phys. Rev. D 14, 1536 (1976).

[2]J. S. R. Chisholm, Nuovo Cimento 30, 426 (1963); Ph. D. thesis, Cambridge, 1952 (unpublished); Proc. Cambridge Philos. Soc. 48, 300 (1952).

[3]J. Kahane, J. Math. Phys. 9, 1732 (1968).

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[7]H. Weyl and R. Brauer, Am. J. Math. 57, 425 (1935).

[8]Square brackets [· · ·] around indices indicate an-

tisymmetrization, parentheses (· · ·) indicate symmetrization.

[9]It is easily derived by considering a specific term in the l-fold summation. Each μ or ν index is fixed (it is not summed over) and furthermore the total antisymmetry means that they must all be distinct from each other and from the summed indices ρ. The index ρ_1 can therefore take $n-k$ values, ρ_2 can take $(n-k-1)$ values, and so on. This means there are $(n-k)(n-k-1)\cdots(n-k-l+1)=(n-k)!/(n-k-l)!$ terms in the summation for each value of the external indices μ and ν. The symmetry of the original expression clearly makes it proportional to the tensor on the right-hand side, and the result therefore follows by observing that the antisymmetrized products of Kronecker tensors have normalization factors of $1/(k+l)!$ and $1/k!$ on the left- and right-hand sides, respectively.

[10]As we mentioned in the Introduction, this is more properly viewed as a single-valued representation of the universal covering group of SO(n).

[11]A. D. Kennedy, J. Math. Phys. 22, 1330 (1981).

[12]The reader will recall that we stated in the Introduction that we would only explicitly consider the even-dimensional case.

[13]For even-dimensional groups at least.

[14]Compare the comments in the Introduction about odd values of n.

[15]No inverse powers of ϵ arise from the Dirac structure of a diagram.

[16]N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations* (Nauka, Moscow, 1965) (Translation by the American Mathematical Society, Rhode Island, 1968); I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980); M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[17]The binomial coefficient is defined by

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}\equiv\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}$$

for α and β arbitrary complex numbers.

[18]ψ_0 and ψ_1 are polygamma functions: $\psi_0\equiv(\ln\Gamma)'$, $\psi_1\equiv\psi_0'$.