Axially symmetric, static self-dual SU(3) gauge fields and stationary Einstein-Maxwell metrics

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The stationary axially symmetric (AS) Einstein-Maxwell equations are shown to be a special case of the static AS self-dual SU(3) Yang-Mills equations; they can become identical by imposing certain functional relations on the Yang-Mills fields. Certain techniques developed for the Einstein-Maxwell equations are translated into the SU(3) Yang-Mills equations, and a linear eigenvalue problem for the stationary AS Einstein-Maxwell equations is formulated.

I. INTRODUCTION

The main objective of the present paper is to demonstrate that the stationary axially symmetric (AS) solutions of the Einstein-Maxwell equations are also solutions of the static AS classical self-dual SU(3) Yang-Mills equations. Our work extends the result of Witten[1] who showed that the stationary AS vacuum Einstein equations can be identified with a subclass of the static AS self-dual SU(2) Yang-Mills equations.

The self-dual Yang-Mills equations are more conveniently described in the R gauge, first introduced by Yang[2] for the SU(2) case and subsequently extended by Prasad,[3] Ardalan,[4] and Brihaye *et al.*[5] to the SU(n) group. For this gauge group the self-duality equations are

$$\bar{D}_a(P^{-1}D_aP)=0,\quad a=y,\, z\ , \tag{1}$$

where P is an $n\times n$ Hermitian matrix with unit determinant,

$$D_a=\left[\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right],\quad \bar{D}_a=\left[\frac{\partial}{\partial \bar{y}},\frac{\partial}{\partial \bar{z}}\right],$$

and y and z are complex coordinates related to the four-dimensional Euclidean-space Cartesian coordinates by

$$y=\frac{1}{\sqrt{2}}(x_1+ix_2)\ ,\quad z=\frac{1}{\sqrt{2}}(x_3-ix_4)\ .$$

Here and in the sequel the bar operation denotes complex conjugation. Equation (1) is invariant under the gauge transformation

$$\tilde{P}=APA^{\dagger}\ , \tag{2}$$

where A is a constant SL(n,C) matrix and the dagger denotes Hermitian conjugation.

For the SU(3) equations a convenient parametrization of the matrix P is given by

$$P=\begin{bmatrix}\phi_1^{-1} & \bar{\rho}_1\phi_1^{-1} & \bar{\rho}_3\phi_1^{-1}\\ \rho_1\phi_1^{-1} & \rho_1\bar{\rho}_1\phi_1^{-1}+\phi_1\phi_2^{-1} & \rho_1\bar{\rho}_3\phi_1^{-1}+\bar{\rho}_2\phi_1\phi_2^{-1}\\ \rho_3\phi_1^{-1} & \bar{\rho}_1\rho_3\phi_1^{-1}+\rho_2\phi_1\phi_2^{-1} & \rho_3\bar{\rho}_3\phi_1^{-1}+\rho_2\bar{\rho}_2\phi_1\phi_2^{-1}+\phi_2\end{bmatrix}, \tag{3}$$

where ϕ_1, ϕ_2, ρ_1, ρ_2, and ρ_3 are scalar fields in Euclidean space. In terms of these fields Eqs. (1) have already been given by Prasad[3] who also gave some SU(2) fields imbedded into the SU(3) gauge group. As an example of one of these embedded SU(2) gauge fields we give the following

$$\phi_1=\phi_2=\phi\ ,\quad \rho_3=\rho\ ,\quad \rho_1=\rho_2=0\ . \tag{4}$$

Next we review the stationary AS Einstein-Maxwell equations. These equations are usually expressed in terms of the Ernst complex potentials ϵ and Φ. The equations read[6]

$$(\epsilon+\bar{\epsilon}+2\Phi\bar{\Phi})\nabla^2\epsilon=2(\vec{\nabla}\epsilon)(\vec{\nabla}\epsilon+2\bar{\Phi}\vec{\nabla}\Phi)\ ,$$
$$(\epsilon+\bar{\epsilon}+2\Phi\bar{\Phi})\nabla^2\Phi=2(\vec{\nabla}\Phi)(\vec{\nabla}\epsilon+2\bar{\Phi}\vec{\nabla}\Phi)\ , \tag{5}$$

where $\vec{\nabla}$ and ∇^2 are, respectively, the three-dimensional flat-space gradient and Laplacian operators, both acting on fields with azimuthal symmetry (signature $++\,+-$). In fact, for any linear combination of the Killing fields of the stationary AS spacetime, a pair (ϵ,Φ) can be constructed which satisfies Eqs. (5). The gravitational complex potential ϵ is conveniently parametrized[7] as

$$\epsilon=-f-\Phi\bar{\Phi}+i\psi ,$$

where f is the square of the norm of the chosen Killing field and ψ is its twist potential; the electromagnetic complex potential Φ is constructed from certain components of the vector potential.

II. THE IDENTIFICATION

The basic observation which leads to the identification of the Einstein-Maxwell and the SU(3) self-dual Yang-Mills equations is the following. First, we consider the 3×3 Hermitian matrix with unit determinant

$$P_e=f^{-1}\begin{bmatrix} 1 & \sqrt{2}\Phi & -\frac{i}{2}(\epsilon-\bar{\epsilon}-2\Phi\bar{\Phi}) \\ \sqrt{2}\bar{\Phi} & -\frac{1}{2}(\epsilon+\bar{\epsilon}-2\Phi\bar{\Phi}) & -i\sqrt{2}\bar{\Phi}\epsilon \\ \frac{i}{2}(\bar{\epsilon}-\epsilon-2\Phi\bar{\Phi}) & i\sqrt{2}\Phi\bar{\epsilon} & \epsilon\bar{\epsilon} \end{bmatrix} , \tag{6}$$

and we write the stationary AS Einstein-Maxwell equations (5) in terms of the matrix (6) alone; they read

$$\vec{\nabla}\cdot(P_e{}^{-1}\vec{\nabla}P_e)=0 . \tag{7}$$

Second, we consider only the static, $\partial P/\partial x_4=0$, and axisymmetric,

$$\left[x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right]P=0 ,$$

solutions of the SU(3) self-dual Yang-Mills equations. With these assumptions Eq. (1) becomes

$$\vec{\nabla}\cdot(P^{-1}\vec{\nabla}P)=0 , \tag{8}$$

where in both Eqs. (7) and (8) $\vec{\nabla}$ and $\vec{\nabla}\cdot$ are the three-dimensional flat-space gradient and divergence operators, respectively. Finally, to complete the identification we specialize the matrix P in (3) by setting

$$\begin{aligned} &\phi_1=\phi_2=f=-\tfrac{1}{2}(\epsilon+\bar{\epsilon}+2\Phi\bar{\Phi}) ,\\ &\rho_1=-i\bar{\rho}_2=\sqrt{2}\Phi ,\\ &\rho_3=\frac{i}{2}(\bar{\epsilon}-\epsilon-2\Phi\bar{\Phi}) . \end{aligned} \tag{9}$$

Witten's[1] result corresponds to the imbedding (4), or equivalently to the $\Phi=0$ (vacuum) case in Eqs. (9).

It is well known in general relativity that stationary AS Einstein-Maxwell Eqs. (5) are invariant under the action of an eight-parameter group of transformations.[8,9] This group is generated by the following transformations: (1) the Ehlers transformations (one real parameter), (2) the Harrison transformations (two real parameters), (3) the gravitational gauge transformations (one real parameter), (4) the electromagnetic gauge transformations (two real parameters), and (5) the combination of coordinate and gauge transformations (two real parameters). These transformations can be cast into the form

$$\tilde{P}_e=AP_eA^{\dagger} , \tag{10}$$

where the constant SL(3,C) matrices A are given, respectively, by

$$\begin{aligned} &A_1=\begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad A_2=\begin{bmatrix} 1 & a\sqrt{2} & -ia\bar{a} \\ 0 & 1 & -i\bar{a}\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} ,\\ &A_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix} , \quad A_4=\begin{bmatrix} 1 & 0 & 0 \\ \bar{b}\sqrt{2} & 1 & 0 \\ ib\bar{b} & ib\sqrt{2} & 1 \end{bmatrix} ,\\ &A_5=\begin{bmatrix} \beta^{-1}e^{i\sigma} & 0 & 0 \\ 0 & e^{-2i\sigma/3} & 0 \\ 0 & 0 & \beta e^{-i\sigma/3} \end{bmatrix} , \end{aligned} \tag{11}$$

where α, β, and γ are arbitrary real and a and b are arbitrary complex constants. The Ehlers (A_1), the Harrison (A_2), and combinations of these with each of the other transformations generate distinct solutions of the Einstein-Maxwell equations from old solutions. For instance, the Kerr-Newman metric, which apparently presents the most general black-hole solution of general relativity, can be constructed from the Kerr metric by an application of the Harrison followed by an electromagnetic gauge transformation.[6] In view of the identification (9), the transformations (11) are nothing but the gauge transformations of Eq. (2). Therefore, although they have been used to construct new solutions in relativity, they do not generate physically distinct solutions in the Yang-Mills theory. In particular, the self-duality equations do not distinguish the Kerr and the charged Kerr black-hole metrics.

In addition to the above continuous group, there exist two discrete transformations, the inversion and the Bonnor transformation, which preserve Eqs. (5). In terms of the potentials (ϵ,Φ) the inversion is given by $(\tilde{\epsilon},\tilde{\Phi})=(\epsilon^{-1},\Phi\epsilon^{-1})$. The action of the inversion on P_e, Eq. (6), is $\tilde{P}_e=P_e{}^{-1}$, which justifies the word "inversion" and makes the transformation immediately verifiable. For the Yang-Mills fields it corresponds to Prasads "I" transformation,[3] which turns out to be a gauge transformation.

Starting from any stationary AS vacuum solution of the Einstein equation with Ernst potential E, the Bonnor transformation provides an electrovacuum solution described by

$$\epsilon=-E\bar{E}\ ,\qquad \Phi=\frac{i}{2}(\bar{E}-E)\ . \tag{12}$$

In this case, the resulting Yang-Mills matrix P_e is not related by a global gauge transformation (2) to the one constructed from the vacuum pair (E,O). Moreover, the topological density[10] of the SU(3) solution (12) equals four times the topological density of the SU(2) solution (E,O). We conclude, therefore, that among all the above algebraic transformations the Bonnor transformation is the unique one which provides physically distinct Yang-Mills solutions.

A potentially more powerful technique for the generation of solutions of both the Einstein-Maxwell and the Yang-Mills equations which we have not mentioned above could be the use of the Kinnersley-Chitre[11] transformations. So far, this technique has been proved very useful for constructing physically interesting solutions of the vacuum Einstein[12] and of the corresponding SU(2) Yang-Mills equations.[13] It would be very interesting to extend these techniques to the Einstein-Maxwell and the SU(3) Yang-Mills equations.

Since the expressions for the Yang-Mills potentials and fields constructed from P involve the terms $\sqrt{\phi_1}$ and $\sqrt{\phi_2}$, the function f in (9), which is the scalar product $\xi^a\xi_a$ of the Killing field (that we started with to construct the Ernst potentials) with itself, must be positive. This constraint allows only those regions of the spacetime where the Killing field is spacelike. The regions of the applicability of the technique therefore are bounded by the Killing horizons. In the case of the Kerr metric the rotational Killing field can be used in the entire spacetime except the axis of rotation. However, the Killing field which asymptotically becomes a time translation can only be used in the inside of the Killing horizon (the ergosphere) region. Killing horizons and the essential singularities of the spacetime are in fact the singularities of the Yang-Mills fields.

III. CONCLUDING REMARKS

In this paper we have attempted to transfer the knowledge accumulated in general relativity into the self-dual SU(3) Yang-Mills fields. We would like to remark that some insight can be gained for the Einstein-Maxwell equations from the techniques developed for the Yang-Mills equation (1). For instance, recently Manakov and Zakharov[14] have written a linear eigenvalue problem for Eq. (1) in 2 + 1 dimensions. Their prescription for the SU(3) group can be used for the stationary AS Einstein-Maxwell equations. The relevant eigenvalue problem is

$$[\lambda\partial_{\bar{y}}-\lambda^{-1}(\partial_y+B)+A]\Psi=0\ ,\qquad \left[\frac{1}{2i}\partial_z+\lambda\partial_{\bar{y}}+A\right]\Psi=0\ , \tag{13}$$

where $\Psi=\Psi(\lambda,z,\rho)$ is a 3×3 matrix,

$$A=\frac{1}{2i}P_e{}^{-1}P_{e,z}\ ,\quad B=P_e{}^{-1}P_{e,y}\ ,$$

λ is the constant eigenvalue of the system, and

$y=\rho e^{i\phi}$, $z=\bar{z}$ are equivalent to the cylindrical coordinates. The integrability conditions of (13) are

$$\begin{aligned} &A_z = 2iB_{\bar{y}} \ , \\ &B_z = 2i(A_y - [A,B]) \ , \end{aligned} \tag{14}$$

which are equivalent to the Einstein-Maxwell equations (7).

The linear eigenvalue equations (13) are simpler than those given by Aleksejev,[15] and the Belinski-Zakharov[16] eigenvalue equation for the vacuum is formally identical to the one corresponding to the embedding (4).

Note added in proof: It has been shown[17] that a slight modification of the Belinski-Zakharov method of integrating the stationary AS vacuum field equations can be applied to the stationary AS Einstein-Maxwell equations.

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$$\begin{aligned} \mathrm{Tr}(A_\mu \bar{A}_{\bar{\nu}} + d_\mu \bar{d}_{\bar{\nu}}) = -\frac{1}{f^2}[\tfrac{1}{2}(\epsilon_\mu \bar{\epsilon}_{\bar{\nu}} + \bar{\epsilon}_\mu \epsilon_{\bar{\nu}}) + \bar{\Phi}(\epsilon_\mu \bar{\Phi}_{\bar{\nu}} &+ \epsilon_{\bar{\nu}} \bar{\Phi}_\mu) + \Phi(\bar{\epsilon}_\mu \Phi_{\bar{\nu}} + \bar{\epsilon}_{\bar{\nu}} \Phi_\mu) \\ &- (\epsilon + \bar{\epsilon})(\bar{\Phi}_\mu \Phi_{\bar{\nu}} + \Phi_\mu \bar{\Phi}_{\bar{\nu}})] \ . \end{aligned}$$

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