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Generalized affine connections applied to a unified field theory

Richard T. Hammond
Vassar College, Poughkeepsie, New York 12601
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The structure of the affine space is generalized by assuming that the change in a vector upon parallel transportation is given by terms containing not only a three-index symbol, but a one-index symbol as well. The relation between the affine connections and the metric tensor is established by the requirement that the length of a vector remains constant under parallel transportation. The one-index symbol is then taken to be proportional to the electromagnetic potential, the curvature tensor is derived, and its contractions are presented. The field equations are derived from an action principle which is formed from scalars derived from the curvature tensor. Maxwell's equations in curved space are obtained and gravitational field equations are obtained that differ from the Einstein equations by the presence of a term bilinear in the electromagnetic potential. It is shown that this term may represent the attractive force capable of balancing the repulsive stress of a sphere of charge.

I. INTRODUCTION

Attempts at the unification of gravitation and electromagnetism have been made ever since the advent of Einstein's general theory of relativity. A review of these attempts will not be given here, but most of these theories share the idea that Einstein's original theory must in some way be generalized to "make room" for the electromagnetic field. The Kaluza-Klein five-dimensional theory and Einstein's nonsymmetric metric tensor theory were motivated by just this reason. In the Weyl[1] unified field theory a four-vector was introduced as a geometric object the effect of which manifested itself as a change in length of a vector under parallel transport. When this four-vector was taken to be the electromagnetic potential, a self-consistent geometrization of the electromagnetic field was established. However, the change-in-length concept seemed to lead to nonphysical consequences.[2] In this paper a four-vector is introduced into the affine connection, which subsequently will be taken to be proportional to the electromagnetic potential; however, the concept of constancy of length upon parallel transport is retained. Owing to the formal (but not physical) similarity to the Weyl theory, comparisons will be made in a few places in the following sections.

As motivation for the present theory consider that the fundamental concept underlying gravitation is geometry. This is immediately evident from the starting point of general relativity when the affine connections are defined; the effect of gravity is to change the components (but not the length) of a vector when it is parallel transported around a closed path. Following this idea, it is assumed here that the effect of electromagnetism manifests itself in an analogous manner. Specifically, it is assumed that there is a change in the components of a vector, beyond that due to a purely gravitational field, which is given by terms linear in a one-index symbol. Thus, the electromagnetic field has a similar geometric foundation as the gravitational field.

In Sec. II the aforementioned ideas are stated in mathematical terms and the structure of the affine space is investigated without regard to the metrical properties of space. In Sec. III the relation between the affine connection symbols and the metric tensor is established by the requirement that the length of a vector does not change under parallel transportation. The results of Sec. II are then rewritten for this case. In Sec. IV the field equations are derived from an action principle which is built from scalars formed from the generalized curvature tensor derived in Sec. III. Maxwell's equations are then derived by varying the action with respect to the potential, and the gravitational field equations are derived by varying the metric tensor. These equations differ from the usual field equations by the presence of an additional term bilinear in the potential. Finally, a short comparison between this and the Weyl theory is given.

II. THE AFFINE SPACE

Following the preceding ideas, a four-vector ϕ_μ is introduced, the effect of which is to generalize the form of the change of a vector δA^σ under parallel transportation. As a guide to the form of this generalization, it is assumed that the effects of ϕ_μ on δA^σ are structured in an analogous way to that of the gravitational field. Specifically, it is assumed that in addition to a term containing a symmetric three-index symbol $\Lambda^\sigma_{\mu\nu}A^\mu dx^\nu$, which contains the gravitational part, there are terms linear in the products of A^σ, dx^ν, and ϕ_μ [see (1)]. Thus, the interpretation may be made that ϕ_μ acts as a one-index symbol (it is, however, defined to be a vector) describing the change of a vector upon parallel transportation due to some agent other than the gravitational field. It is further assumed that δA^σ is given by terms that do not explicitly contain the metric tensor. This last assumption, although not logically necessary, is in keeping with the idea that δA^σ can be defined devoid of the concept of a metric space. Thus, in this section general relations are derived that are valid for any arbitrary relation between the metric tensor and the symbols of the affine connection.

With these ideas in mind, the most general form for the change in a vector upon parallel transportation is given by

$$\delta A^\sigma = -\Omega^\sigma_{\mu\nu}A^\mu dx^\nu \equiv -(\Lambda^\sigma_{\mu\nu}+a\delta^\sigma_\nu\phi_\mu+\delta^\sigma_\mu\phi_\nu)A^\mu dx^\nu , \tag{1}$$

where $\Omega^\sigma_{\mu\nu}$, defined in terms of $\Lambda^\sigma_{\mu\nu}$ and ϕ_μ by (1), will be defined as the generalized affine connection, and a is a dimensionless constant (there is no need to write another constant in the last term because it is absorbed in the definition of ϕ_μ). The condition to be imposed later is that (1) reduces to the usual equation

$$\delta A^\sigma = -\Gamma^\sigma_{\mu\nu}A^\mu dx^\nu , \tag{2}$$

where $\Gamma^\sigma_{\mu\nu}$ are the Christoffel symbols, if and only if $\phi_\mu=0$ (this condition will be used in Sec. III).

With (1), one can define a generalized covariant derivative, denoted by a colon, as

$$A^\sigma_{:\nu}=A^\sigma_{,\nu}+\Omega^\sigma_{\mu\nu}A^\mu . \tag{3}$$

Requiring that $A^\sigma_{:\nu}$ transforms as a tensor defines the transformation rule for the generalized affine connection. It turns out that, since the last two terms in (1) are vectors, the transformation rule for the $\Lambda^\sigma_{\mu\nu}$ is the same as that for the Christoffel symbols. In the next section, when a connection with the metrical space is made, it will be manifestly evident that $A^\sigma_{:\nu}$ is a tensor.

The curvature tensor will be defined by the mathematical form of the statement that if a vector is parallel transported around a closed curve, then δA^σ is zero if and only if the curvature tensor is zero. To obtain an explicit form of this tensor, first the covariant derivative of a vector along a curve is defined, using (3), as

$$\frac{DA^\sigma}{d\tau}=A^\sigma{}_{:\nu}\frac{dx^\nu}{d\tau}=\frac{dA^\sigma}{d\tau}+\Omega^\sigma_{\mu\nu}A^\mu\frac{dx^\nu}{d\tau} , \tag{4}$$

which is the obvious generalization of the ordinary covariant derivative along a curve. Parallel transport is defined such that $DA^\sigma/d\tau$, evaluated along some curve $x^\mu(\tau)$, is zero. Thus, from (4) one has

$$\int_{\tau_1}^{\tau} dA^\sigma = -\int_{\tau_1}^{\tau}(\Lambda^\sigma_{\mu\nu}A^\mu dx^\nu+aA^\beta\phi_\beta dx^\sigma +A^\sigma\phi_\nu dx^\nu) . \tag{5}$$

For small displacements, i.e., $x^\mu(\tau)-x^\mu(\tau_1)\rightarrow 0$, $\Lambda^\sigma_{\mu\nu}$ and ϕ_β are expanded in a Taylor series about $x^\mu(\tau_1)$:

$$\Lambda^\sigma_{\mu\nu}(x(\tau))=\Lambda^\sigma_{\mu\nu}(x(\tau_1)) +\epsilon^\beta\Lambda^\sigma_{\mu\nu,\beta}(x(\tau_1))+\cdots , \tag{6}$$

$$\phi_\nu(x(\tau))=\phi_\nu(x(\tau_1)) +\epsilon^\beta\phi_{\nu,\beta}(x(\tau_1))+\cdots , \tag{7}$$

where $\epsilon^\beta\equiv x^\beta(\tau_1)-x^\beta(\tau)$. Using (6) and (7) in (5) to obtain $A^\sigma(\tau)$ to first order in ϵ^β one has

$$A^\sigma(\tau)=A^\sigma(\tau_1)-[\Lambda^\sigma_{\mu\nu}(\tau_1)A^\mu(\tau_1)\epsilon^\nu+aA^\beta(\tau_1)\phi_\beta(\tau_1)\epsilon^\sigma +A^\sigma(\tau_1)\phi_\nu(\tau_1)\epsilon^\nu] . \tag{8}$$

Equations (8), (6), and (7) can now be used in (5) to obtain an expression for the change in A^σ valid to second order. Considering further that the path is now closed this yields

$$\Delta A^\sigma=\tfrac{1}{2}R^\sigma{}_{\mu\nu\rho}A^\mu\oint x^\rho dx^\nu , \tag{9}$$

where

$$R^\sigma{}_{\mu\nu\rho}=C^\sigma{}_{\mu\nu\rho}+a\delta^\sigma_\rho(\phi_{\mu:\nu}+\phi_\mu\phi_\nu) -a\delta^\sigma_\nu(\phi_{\mu:\rho}+\phi_\mu\phi_\rho)+\delta^\sigma_\mu(\phi_{\rho:\nu}-\phi_{\nu:\rho}) \tag{10}$$

and $\phi_{\mu:\nu}$ is defined by

$$\phi_{\mu:\nu}\equiv\phi_{\mu,\nu}-\phi_\beta\Lambda^\beta_{\mu\nu}-(a+1)\phi_\mu\phi_\nu$$

[see (18)] and also

$$C^\sigma{}_{\mu\nu\rho}\equiv-\Lambda^\sigma_{\mu\nu,\rho}+\Lambda^\sigma_{\mu\rho,\nu}+\Lambda^\beta_{\mu\rho}\Lambda^\sigma_{\beta\nu}-\Lambda^\beta_{\mu\nu}\Lambda^\sigma_{\beta\rho} . \tag{11}$$

The tensorial nature of $R^\sigma{}_{\mu\nu\rho}$ follows from the tranformation rule for the $\Lambda^{\sigma}_{\mu\nu}$. It is evident that, in the case $\phi_\mu=0$, $R^{\sigma}{}_{\mu\nu\rho}$ reduces to the ordinary Riemann curvature tensor.

Although the contracted forms of $R^{\sigma}{}_{\mu\nu\rho}$ will be used only after the metrical connection is made, for completeness the generalized Ricci tensor is given, calling $C_{\mu\nu}=C^{\sigma}{}_{\mu\nu\sigma}$ and $C=g^{\mu\nu}C_{\mu\nu}$, one has

$$R_{\mu\nu}\equiv R^{\sigma}{}_{\mu\nu\sigma} = C_{\mu\nu}+3a(\phi_{\mu:\nu}+\phi_\mu\phi_\nu)+\phi_{\mu:\nu}-\phi_{\nu:\mu} \quad (12)$$

and, for the generalized curvature scalar,

$$R\equiv g^{\mu\nu}R_{\mu\nu}=C+3a\phi_\mu{}^{:\mu}+3a\phi_\mu\phi^\mu \; . \quad (13)$$

III. THE METRICAL CONNECTION

With the exception of (13), all of the preceding work was presented without regard to the metric tensor. At this point a relation between the affine connection $\Omega^{\sigma}_{\mu\nu}$ and the metric tensor will be established.

The metric tensor is defined, as usual, by the equation

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu \; . \quad (14)$$

The contravariant form is defined such that $g^{\mu\sigma}g_{\nu\sigma}=\delta^\mu_\nu$. From this it follows that the metric tensor may raise and lower indices.

It is now assumed that the scalar product of two vectors is invariant when they are parallel transported along a curve, i.e.,

$$\frac{d}{d\tau}(A^\mu B_\mu)=0 \; . \quad (15)$$

It is pointed out that this is not a logically necessary requirement and that relaxation of (15) would lead to a generalized Weyl geometry. However, with (15) the concept of constancy of length is retained and the apparently nonphysical consequences of the Weyl geometry are avoided. Using (4) in (15) and performing the usual tensor manipulations one obtains

$$\Lambda^{\sigma}_{\mu\nu}=\Gamma^{\sigma}_{\mu\nu}-\phi_\mu\delta^\sigma_\nu-\phi_\nu\delta^\sigma_\mu-(a-1)\phi^\sigma g_{\mu\nu} \quad (16a)$$

and

$$\Omega^{\sigma}_{\mu\nu}=\Gamma^{\sigma}_{\mu\nu}+(a-1)\phi_\mu\delta^\sigma_\nu-(a-1)\phi^\sigma g_{\mu\nu} \; , \quad (16b)$$

where the $\Gamma^{\sigma}_{\mu\nu}$ are the usual Christoffel symbols. It can be noted from the above equation that, while $\Lambda^{\sigma}_{\mu\nu}$ is symmetric in its lower indices, $\Omega^{\sigma}_{\mu\nu}$ is not.

Also, from (16a) one can see that if $a=0$, then the relation between $\Lambda^{\sigma}_{\mu\nu}$ and $\Gamma^{\sigma}_{\mu\nu}$ is that which could be obtained through a conformal transformation if $\Lambda^{\sigma}_{\mu\nu}$ represented the entire affine connection. This is not the case, however, as can be seen from (1). Thus, the possibility of an equivalence with the Weyl theory is prohibited.

Equations (16) give the fundamental relationship between the affine connections and the metric tensor. Furthermore, (15) may also be used to derive a formula for the generalized covariant derivative of a covariant vector field. Writing

$$\frac{d}{d\tau}(A^\sigma A_\sigma)=0 \quad (17)$$

for parallel transport along a curve one obtains, for arbitrary A^σ, using (16),

$$A_{\sigma:\nu}=A_{\sigma,\nu}-A_\mu\Omega^{\mu}_{\sigma\nu} \; . \quad (18)$$

This notation was used earlier, for convenience [in (10)], but only with the requirement of constancy of length can it be called a covariant derivative. The covariant derivative of higher-rank tensor fields may be found by writing the tensors as products of vectors. For example, writing $T_{\mu\nu}=A_\mu B_\nu$, one easily finds that

$$T_{\sigma\mu:\nu}=T_{\sigma\mu,\nu}-T_{\beta\mu}\Lambda^{\beta}_{\sigma\nu}-T_{\sigma\beta}\Lambda^{\beta}_{\mu\nu}-2\phi_\nu T_{\sigma\mu} -a(\phi_\sigma T_{\nu\mu}+\phi_\mu T_{\sigma\nu}) \; . \quad (19)$$

Applying (19) to the metric tensor one finds

$$g_{\sigma\mu:\nu}=g_{\sigma\mu;\nu}=0 \; , \quad (20)$$

an important result, which shows that the metric tensor commutes with generalized covariant differentiation as well as with ordinary covariant differentiation.

Since the relation between the $\Lambda^{\sigma}_{\mu\nu}$, ϕ_μ, and $\Gamma^{\sigma}_{\mu\nu}$ is known, it is useful to rewrite some previous results in terms of $\Gamma^{\sigma}_{\mu\nu}$ and ϕ_μ alone. Using (16) to eliminate $\Lambda^{\sigma}_{\mu\nu}$, (1) becomes

$$\delta A^\sigma=-A^\mu dx^\nu\Gamma^{\sigma}_{\mu\nu}+(1-a)A^\mu\phi_\mu dx^\sigma -(1-a)\phi^\sigma A_\nu dx^\nu \; . \quad (21)$$

The condition stated earlier is now imposed that δA^σ reduces to the ordinary form, i.e., to (2), if and only if $\phi_\mu=0$. Obviously the "if" part of the statement is true, but if $a=1$, even if $\phi_\mu\neq 0$, δA^σ reduces to the usual form. Therefore, it is required that $a\neq 1$. With this condition, (21) establishes the very important point that ϕ_μ gives rise to a change in a vector upon parallel transport that cannot be represented by the gravitation field alone. Thus it is postulated that ϕ_μ is associated with a field other than the gravitational field. Specifically, it is assumed that ϕ_μ is proportional to the electromagnetic potential and that a second-rank tensor, proportional to the electromagnetic field tensor, is defined by

$$F_{\mu\nu}\equiv\phi_{\mu,\nu}-\phi_{\nu,\mu} \tag{22}$$

(letting $\phi_\mu=q\psi_\mu$, where q is a constant and ψ_μ is the actual potential, then $F_{\mu\nu}$ is q times the actual electromagnetic field tensor). It is worthwhile to mention that in (22), ordinary covariant or generalized covariant differentiation may be used, the result being the same.

With (22), it is interesting to note that it is the potential of the field, and not the field itself, that manifests itself via the change in a vector. This is not, however, surprising when one considers the $g_{\mu\nu}$ as the potentials of the gravitational field (of course, in the gravitational case, derivatives of $g_{\mu\nu}$ appear as well). With these ideas in mind, therefore, it is not at all unreasonable that the potential effects the change in a vector, and therefore the curvature of space. This point will be discussed further in Sec. IV.

The effects of gravity and electromagnetism can be nicely separated by using (16) to eliminate $\Gamma^\sigma_{\mu\nu}$ from the equations of Sec. II. Equation (18), with (16), becomes [from here on, for convenience, let ϕ_σ be redefined to absorb the constant $1-a$, i.e., let $(1-a)\phi_\sigma\to\phi_\sigma$]

$$A_{\sigma:\nu}=A_{\sigma;\nu}+\phi_\sigma A_\nu-\phi^\beta A_\beta g_{\sigma\nu}\ . \tag{23}$$

Using (16) and (23) in (10), one obtains, after some manipulation,

$$\begin{aligned}R^\sigma{}_{\theta\nu\rho}={}&{}^0R^\sigma{}_{\theta\nu\rho}+\delta^\sigma_\nu\phi_{\theta:\rho}-\delta^\sigma_\rho\phi_{\theta:\nu}\\&+g_{\theta\rho}(\phi^\sigma{}_{:\nu}+\phi_\mu\phi^\mu\delta^\sigma_\nu)\\&-g_{\theta\nu}(\phi^\sigma{}_{:\rho}+\phi_\mu\phi^\mu\delta^\sigma_\rho)\ ,\end{aligned} \tag{24}$$

where the presuperscript stands for the ordinary Riemannian curvature tensor.[3] Consider, for the moment, that ${}^0R^\sigma{}_{\theta\nu\rho}=0$. It is evident that if space is flat, i.e., if $R^\sigma{}_{\theta\nu\rho}=0$, then $F_{\mu\nu}=0$. Thus flat space implies that there is no electromagnetic field. The converse of this statement, however, is not true. Space can be curved ($R^\sigma{}_{\theta\nu\rho}\neq0$) even if the electromagnetic field is zero due to the presence of ϕ_μ in (24). Thus, the actual value of the potential is needed in determining the structure of space and the zero point may not be taken as arbitrary. This result is not surprising since, as previously mentioned, it is the potential, and not the field, that determines the change in a vector upon parallel transportation. The electromagnetic field is, of course, still gauge invariant and unaffected by a shift in the zero point of the potential.

The generalized Ricci tensor is obtained by contraction of (24). Calculation shows that $R^\sigma{}_{\sigma\nu\rho}$ vanishes identically and $R^\sigma{}_{\theta\nu\rho}=-R^\sigma{}_{\theta\rho\nu}$. Therefore, there is only one algebraically independent contraction, which is taken as

$$\begin{aligned}R_{\theta\nu}\equiv{}&R^\sigma{}_{\theta\nu\sigma}\\={}&{}^0R_{\theta\nu}+2(\phi^\sigma\phi_\sigma g_{\theta\nu}-\phi_\theta\phi_\nu)-\phi^\sigma{}_{;\sigma}g_{\theta\nu}\\&-(\phi_{\theta;\nu}+\phi_{\nu;\theta})-F_{\theta\nu}\ ,\end{aligned} \tag{25}$$

where ${}^0R_{\theta\nu}$ represents the ordinary Ricci tensor and (25) is written as a sum of its symmetric and antisymmetric parts. For later convenience, (25) has been written using ordinary covariant differentiation. The usual scalar of curvature is obtained by contracting $R_{\theta\nu}$ with the metric tensor:

$$R\equiv g^{\theta\nu}R_{\theta\nu}={}^0R-6\phi^\sigma{}_{;\sigma}+6\phi^\sigma\phi_\sigma\ . \tag{26}$$

It is remarkable that this is the same scalar as that in the Weyl theory; however, the subsequent formulations are different. As is well known, there are many other curvature scalars that could be formed from the curvature tensor and its contractions. The greatest virtue of the one used in (26) is its simplicity, and the fact that the field equations derived by using (26) in an action principle are of second differential order. However, since $g^{\theta\nu}$ is symmetric, the antisymmetric part of $R_{\theta\nu}$ vanishes when R is formed. This would suggest defining another scalar by multiplication of $R_{\theta\nu}$ with a second-rank, contravariant tensor, that is not symmetric. An obvious choice is $F^{\theta\nu}$. Thus, another scalar R' is defined as

$$R'\equiv F^{\theta\nu}R_{\theta\nu}=-F^{\theta\nu}F_{\theta\nu}\ . \tag{27}$$

Other scalars could be formed, e.g., $R^{\theta\nu}R_{\theta\nu}$, which contains (27), however, (26) and (27) are much simpler and they will yield gravitational equations of second order: Therefore, these are the scalars that will be used.

IV. THE FIELD EQUATIONS

The field equations will be derived from an action principle with a Lagrangian density formed from a linear combination of the scalars of curvature previously derived. To account for the presence of matter or charge, the term $L=L_e+L_m$ will be included, which is defined in the usual way by

$$\frac{\delta(L_m\sqrt{-g})}{\delta g^{\mu\nu}}\equiv A\sqrt{-g}\,T^{\rm mat}_{\mu\nu}\ , \tag{28}$$

$$\frac{\delta(L_e\sqrt{-g})}{\delta\phi_\mu}\equiv B\sqrt{-g}\,j^\mu\ , \tag{29}$$

where $T^{\rm mat}_{\mu\nu}$ is the energy-momentum tensor of matter, j_μ is the current density due to charge, and A and B are constants. Therefore, the action principle takes the form

$$\delta\int\sqrt{-g}\,(R+kR'+L)d^4x=0\ , \tag{30}$$

where k is a constant of dimension length squared. Using (26) and (27) in (30), one has

$$\delta\int\sqrt{-g}\,({}^0R+6\phi_\sigma\phi^\sigma-kF^{\mu\nu}F_{\mu\nu}+\phi^\sigma{}_{;\sigma}+L)d^4x=0\ . \tag{31}$$

Taking the variation with respect to the metric tensor, and assuming that the variations of $g_{\mu\nu}$ and ϕ_μ vanish on a hypersurface surrounding the four-volume of integration, one obtains (the $\phi^\sigma{}_{;\sigma}$ term can be converted to a surface term which vanishes)

$${}^0R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\,{}^0R=-2k(T^{\rm em}_{\mu\nu}+t_{\mu\nu})-AT^{\rm mat}_{\mu\nu}\ , \tag{32}$$

where

$$T^{\rm em}_{\mu\nu}\equiv -F_\mu{}^\sigma F_{\nu\sigma}+\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

and

$$t_{\mu\nu}\equiv\frac{3}{k}(\phi_\mu\phi_\nu-\tfrac{1}{2}g_{\mu\nu}\phi_\sigma\phi^\sigma)\ .$$

Equation (32) is what is usually taken to be the correct gravitational field equation with matter present, except for the new term $t_{\mu\nu}$ on the right-hand side. Taking the trace of (32) gives

$${}^0R=-6\phi_\sigma\phi^\sigma+AT^{\rm mat}\ . \tag{33}$$

Using this in (31) one has

$$\delta\int\sqrt{-g}\,(AT^{\rm mat}-kF^{\mu\nu}F_{\mu\nu}+L)d^4x=0 \tag{34}$$

and, letting the variation be taken with respect to ϕ_μ (assuming the matter terms have no explicit dependence on ϕ_μ), this becomes

$$\delta\int\sqrt{-g}\,(-kF^{\mu\nu}F_{\mu\nu}+L_e)d^4x=0\ . \tag{35}$$

Of course, this form is valid only when (32), and therefore (33), is satisfied. Equation (35) yields

$$F^{\mu\nu}{}_{;\nu}=-Bj^\mu/4k\ . \tag{36}$$

Equation (36) represents two of Maxwell's equations and the other two,

$$\text{permutation}\{F_{\mu\nu;\sigma}\}=0\ , \tag{37}$$

follow from the definition (22).

A final point to be considered is the effect of the Bianchi identities, which imply the covariant divergence of the right-hand side of (32) must be zero. In the presence of a material medium this could be inferred as representing a coupling between matter and charge. However, in free space the Bianchi identities imply

$$t^{\mu\nu}{}_{;\nu}=0\ . \tag{38}$$

However, Maxwell's equations determine the electromagnetic potential only to within an additive gauge term $\lambda_{,\mu}$. Thus the most general form of the electromagnetic potential is $\phi_\mu+\lambda_{,\mu}$. Using this, (38) can be written in the form

$$(\phi_\nu+\lambda_{,\nu})F^{\nu\mu}=(\phi^\mu+\lambda^{,\mu})(\phi^\nu+\lambda^{,\nu})_{;\nu}\ . \tag{39}$$

Once ϕ_ν and therefore $F^{\nu\mu}$, are determined, (39) can be solved for $\lambda_{,\nu}$. Thus, the effect of the Bianchi identities is to impose a certain gauge on the electromagnetic potential.

This completes the derivation of the field equations for electromagnetism and gravitation. All of Maxwell's equations are obtained and the field equations of general relativity are also obtained with the modification provided by the presence of $t_{\mu\nu}$. As mentioned earlier, this term, bilinear in ϕ_σ, represents a contribution to the curvature of space. This term may be viewed as giving rise to a gravitational stress due to the energy of the electromagnetic field, in addition to the usual stress of the field itself. It is interesting to speculate that $t_{\mu\nu}$ may represent the self-stress that holds a charged particle together. To see this, neglect, for the moment, the energy-momentum tensor of matter. Then the stress in the radial direction S_r is given by the 1-1 component of the energy-momentum tensor in mixed form, i.e.,

$$S_r=(T^1{}_1{}^{\rm em}+t^1{}_1)\ . \tag{40}$$

Assuming $\phi_4=\epsilon/r$ and $\phi_i=0$ $(i=1,2,3)$, and assuming $g_{\mu\nu}$ is diagonal, this becomes

$$S_r=-\tfrac{1}{2}g^{11}g^{44}F_{14}F_{14}-\frac{3g^{44}}{2k}\phi_4\phi_4\ . \tag{41}$$

Since the metric signature is $(---+)$, it is seen that the first term is the usual repulsive stress but that the second term represents an attractive stress.

Thus, the term $t_{\mu\nu}$ could account for the forces required to hold together a sphere of charge.

In this paper, it has been the structure of spacetime and the field equations that have been examined so far. In developing the equations of motion of a particle, it should be pointed out that the equations

$$\frac{dv^\sigma}{ds}+\Omega^\sigma_{\mu\nu}v^\mu v^\nu=0\ , \qquad (42)$$

which would be the obvious generalization obtained by replacing normal covariant differentiation with generalized covariant differentiation, difffer from those obtained by assuming geodesic motion defined by

$$\delta\int ds=0\ . \qquad (43)$$

However, charged particles in an electromagnetic field do not follow along geodesics (the force term $ev^\mu F^\sigma{}_\mu$ must be added) and therefore neither (42) nor (43), as they stand, could represent the correct equation. This point bears further investigation but will not be dealt with here.

Owing to some formal similarities between this and the Weyl theory, it is worthwhile to make a final comparison between the two. They are fundamentally different, of course, since in the Weyl theory length is not constant under parallel transportation while in this theory it is. Furthermore, in the Weyl theory the Lagrangian density is made to be gauge invariant whereas this theory is not gauge invariant, and, a different Lagrangian density is used. The greatest similarity is that each theory contains the term $\phi_\sigma\phi^\sigma$ in the Lagrangian density and it is this term that gives rise to $t_{\mu\nu}$. However, the Weyl theory contains other terms as well, and even with the special choice of the natural gauge there are cosmological terms present that do not occur here.

V. CONCLUSION

It has been shown that a generalized affine connection gives emergence to field equations more general than those of ordinary general relativity. By associating ϕ_μ with the electromagnetic potential and defining $F_{\mu\nu}=\phi_{\mu,\nu}-\phi_{\nu,\mu,}$ all of Maxwell's equations are obtained. It is emphasized that the Maxwell equations fall out naturally from an action principle formed from scalars of curvature derived from tensors of the generalized affine space. Specifically, the term $F^{\mu\nu}F_{\mu\nu,}$ which was obtained by contracting $R_{\mu\nu}$ with $F^{\mu\nu}$, was not concocted arbitrarily, but has its roots in the generalized affine space.

In the limiting case of $\phi_\mu=0$, the usual Einstein field equations are obtained and, in the presence of a gravitational field, the Maxwell equations of curved space are obtained. The new feature introduced here is the presence of $t_{\mu\nu}$. It appears that this could represent a stress capable of balancing the electromagnetic repulsion of a sphere of charge.

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[1]H. Weyl, *Spacetime Matter* (Dover, New York, 1952).

[2]See, e.g, R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1975), p. 505.

[3]In this paper the metrical signature is $(---+)$, the subscript 4 represents the time component, the ordinary Riemann curvature tensor is defined as ${}^0R^\alpha{}_{\mu\nu\beta}\equiv-\Gamma^\alpha_{\mu\nu,\beta}+\cdots$, ${}^0R_{\mu\nu}\equiv{}^0R^\alpha{}_{\mu\nu\alpha}$, and the units are chosen such that $c=G=1$.