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Comments on conformal Killing vector fields and quantum field theory

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We give a comprehensive analysis of those vacuums for flat and conformally flat space-times which can be defined by timelike, hypersurface-orthogonal, conformal Killing vector fields. We obtain formulas for the difference in stress-energy density between any two such states and display the correspondence with the renormalized stress tensors. A brief discussion is given of the relevance of these results to quantum-mechanical measurements made by noninertial observers moving through flat space.

I. INTRODUCTION

In this paper we give a systematic analysis of the natural vacuum states for conformally flat space-times. In flat space these vacuums are those defined by the eight independent, timelike, curl-free (hypersurface-orthogonal), conformal Killing vector fields; the Minkowski and Rindler vacuum states are two well-known examples. The definition and description of these vector fields may be found in Sec. IV.

The symmetries of the theories that we discuss are such that it is possible to give simple, analytic expressions for all quantities of interest. Indeed it is possible to go further and give a coordinate-free description. This is not always the most direct approach and we do not always adhere to it, but the absence of coordinates does have the desirable effect of focusing attention upon the geometrical significance of key elements of the theory. This is particularly useful given the current status of generally covariant quantum field theory.

In Sec. II we develop those aspects of the quantum theory that we require in Secs. V and VI. In particular, we define for general curved spaces the concept of a "normalized" stress tensor. For the spaces we subsequently discuss this normalized stress tensor can be shown to equal the more frequently encountered renormalized stress tensor. By working throughout with normalized stress tensors we are able to avoid all problems of divergences and operator-ordering ambiguities: we deal with those elements of the quantum theory that are finite and independent of the ordering. It is also possible to make plain the interrelationship of renormalized stress tensors, their conformal transformation law, and the states of the field theory.

The expectation values of the stress tensor operator in the various vacuum states are derived and discussed in Sec. V. In flat space they all have the character of stress-energy densities for perfect fluids in thermal equilibrium, with the flow lines being the trajectories of the vector fields. The temperature is either constant or variable along given flow lines depending on whether the vector field is Killing or conformal Killing. This picture is very similar to the kinetic theory of massless gasses[1] where, again, thermal equilibrium is possible not only when there exists a Killing vector (constant temperature) but also when there exists a conformal Killing vector (variable temperature along trajectories).

Section VI describes the physics of observers in flat space whose world lines are the trajectories of vector fields described in Sec. IV. All such observers have constant acceleration but their observations of a given state are shown to depend critically on the nature of the measuring apparatus and the world lines of its component parts.

To conclude this introduction there are a few points concerning the language and notation[2] used in the text: We say that a space is conformally flat if the Weyl tensor vanishes and flat if the Riemann tensor vanishes.

Minkowski space is the manifold R^4 with the flat Lorentz metric η. In terms of the natural Cartesian coordinates (t, x, y, z) on R^4 the line element can be expressed

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 .$$

We shall also use spherical polar coordinates (t, r, θ, ϕ) and cylindrical polar coordinates (t, x, ρ, ψ). In terms of these coordinates the line element is

$$ds^2 = -dt^2 + dr^2 + r^2\, d\Omega^2$$

and

$$ds^2 = -dt^2 + dx^2 + d\rho^2 + \rho^2\, d\psi^2 ,$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 .$$

We say that a vector field K is curl-free (or hypersurface-orthogonal) if

$$\mathrm{curl} K = 0 , \tag{1.1}$$

where

$$(\mathrm{curl} K)^a = \eta^{abcd} K_b K_{c;d} .$$

Equation (1.1) is equivalent to $K_{[a}K_{b;c]} = 0$.

II. QUANTUM FIELD THEORY

In quantum field theory the absolute energy of a given state is seldom an *a priori* well defined quantity. When space-time is flat this problem is avoided by measuring energies relative to the Minkowski vacuum. It is important to note that this use of the Minkowski vacuum state is quite distinct from its role in Minkowski space where it is the natural global vacuum. In any other flat space it serves as a local reference state enabling one to define normalized (or renormalized) expectation values. The main objective of this section is to generalize this process of normalization to curved space-times. We shall define a local geometrical vacuum $|0,\Delta\rangle$ with respect to which curved-space stress tensor operators can be normal ordered, or, equivalently, their expectation values normalized. We begin with some necessary groundwork.

We shall discuss properties of the simplest possible field theory, that described by the classical action functional

$$S(g_{ab};\Phi) \equiv -\tfrac{1}{2}\int d^4x\, g^{1/2}[\Phi_{;c}\Phi^{;c} + (R/6)\Phi^2] . \tag{2.1}$$

The theory is conformally invariant in the sense that

$$S(g_{ab};\Phi) = S(\Omega^2 g_{ab};\Omega^{-1}\Phi) - \tfrac{1}{2}\int d^4x\, g^{1/2}(\Phi^2\Omega^{-1}\Omega^{;c})_{;c} \tag{2.2}$$

for all scalars Φ and Ω. This is a simple consequence of the definition (2.1). The classical stress-energy tensor for the scalar field Φ is defined in terms of S by the equation

$$T^{ab}(g_{cd};\Phi) \equiv 2g^{-1/2}\frac{\delta}{\delta g_{ab}} S(g_{cd};\Phi) . \tag{2.3}$$

Functional differentiation of Eq. (2.2) with respect to the metric g_{ab} yields the transformation law for the stress tensor,

$$T^{ab}(g_{cd};\Phi) = \Omega^6 T^{ab}(\Omega^2 g_{cd};\Omega^{-1}\Phi) . \tag{2.4}$$

Differentiation of Eq. (2.2) with respect to Φ yields

$$(\Box - R/6)\Phi = \Omega^3(\tilde{\Box} - \tilde{R}/6)(\Omega^{-1}\Phi) , \tag{2.5}$$

where $\tilde{\Box}$ and $\tilde{R}$ are defined with respect to the conformal metric $\Omega^2 g_{cd}$.

The quantum theory can be constructed in the usual way: Let $\{u_k(x)\}$ be a complete set of complex solutions to the wave equation

$$(\Box - R/6)u_k(x) = 0 . \tag{2.6}$$

Corresponding to the field-operator decomposition (summation over repeated indices)

$$\hat{\Phi}(x) = \hat{a}_k u_k(x) + \hat{a}_k^\dagger u_k^*(x) , \tag{2.7}$$

the u vacuum $|0,u\rangle$ is defined by

$$\hat{a}_k|0,u\rangle = 0, \quad \langle 0,u|0,u\rangle = 1 . \tag{2.8}$$

Normalizations can be chosen so that the creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$ satisfy the commutation relations

$$[\hat{a}_k,\hat{a}_{k'}^\dagger] = \delta_{kk'}, \quad [\hat{a}_k,\hat{a}_{k'}] = 0 . \tag{2.9}$$

If $\{v_k(x)\}$ is another complete set of solutions to Eq. (2.6), then we can also write

$$\hat{\Phi}(x) = \hat{b}_k v_k(x) + \hat{b}_k^\dagger v_k^*(x) , \tag{2.10}$$

and define the v vacuum $|0,v\rangle$ by

$$\hat{b}_k|0,v\rangle = 0, \quad \langle 0,v|0,v\rangle = 1 . \tag{2.11}$$

The $\hat{b}$'s can be taken to satisfy the commutation relations

$$[\hat{b}_k,\hat{b}_{k'}^\dagger] = \delta_{kk'}, \quad [\hat{b}_k,\hat{b}_{k'}] = 0 . \tag{2.12}$$

The requirement of completeness implies that

$$v_k(x) = \alpha_{kk'} u_{k'}(x) + \beta_{kk'}^* u_{k'}^*(x) . \tag{2.13}$$

The matrices α and β then satisfy the usual Bogoliubov relations

$$\alpha^{\dagger}_{kk'}\alpha_{k'l}-\beta^{\dagger}_{kk'}\beta_{k'l}=\delta_{kl} ,$$
$$\tilde{\beta}_{kk'}\alpha_{k'l}=\tilde{\alpha}_{kk'}\beta_{k'l} . \qquad (2.14)$$

We also have the operator equations

$$\hat{a}_k=\tilde{\alpha}_{kk'}\hat{b}_{k'}+\tilde{\beta}_{kk'}\hat{b}^{\dagger}_{k'} \qquad (2.15)$$

and

$$\hat{b}_k=\alpha^{*}_{kk'}\hat{a}_{k'}-\beta_{kk'}\hat{a}^{\dagger}_{k'} . \qquad (2.16)$$

When Φ satisfies the field equation (2.6) the stress tensor operator derived from definition (2.3) can be written

$$T^{ab}(g_{cd};\hat{\Phi})=\tfrac{1}{6}\{R^{ab}\hat{\Phi}^2+4\hat{\Phi}^{;a}\hat{\Phi}^{;b}-2\hat{\Phi}\hat{\Phi}^{;ab}$$
$$-g^{ab}[\hat{\Phi}_{;c}\hat{\Phi}^{;c}+(R/6)\hat{\Phi}^2]\} . \qquad (2.17)$$

We shall require some easily established properties of this operator: Let $|S_1\rangle$ and $|S_2\rangle$ be arbitrary states and let ${}^*_*\hat{T}^{ab*}_{*}$ denote any ordering of the operator $\hat{T}^{ab}$. Then the quantity

$$\frac{\langle S_1|{}^*_*\hat{T}^{ab*}_{*}|S_1\rangle}{\langle S_1|S_1\rangle}-\frac{\langle S_2|{}^*_*\hat{T}^{ab*}_{*}|S_2\rangle}{\langle S_2|S_2\rangle} \qquad (2.18)$$

is independent of the ordering (${}^*_*$). This follows because $\hat{T}^{ab}$ is a quadratic function of annihilation and creation operators. Thus any given ordering of it differs from any other by, at most, a c-number and this does not affect the value of the difference (2.18).

If we now take $|S_1\rangle$ to be $|0,u\rangle$ and $|S_2\rangle$ to be $|0,v\rangle$, we have the equations

$$\langle 0,u|{}^*_*\hat{T}^{ab*}_{*}|0,u\rangle-\langle 0,v|{}^*_*\hat{T}^{ab*}_{*}|0,v\rangle$$
$$=\langle 0,u|{:_v}\hat{T}^{ab}{:_v}|0,u\rangle \qquad (2.19)$$
$$=-\langle 0,v|{:_u}\hat{T}^{ab}{:_u}|0,v\rangle , \qquad (2.20)$$

where ($:_v$) denotes normal ordering with respect to the v vacuum and ($:_u$) normal ordering with respect to the u vacuum; for example,

$${:_u}\hat{a}_k\hat{a}^{\dagger}_{k'}{:_u}\equiv\hat{a}^{\dagger}_{k'}\hat{a}_k . \qquad (2.21)$$

The operator ${:_u}\hat{T}^{ab}{:_u}$ measures energy and momentum relative to the u vacuum.

It is frequently convenient to represent the stress tensor operator as the limit of a "point separated" operator,

$$\lim_{x'\to x} D^{ab}(x,x')[G_u(x,x')-G_v(x,x')] , \qquad (2.22)$$

where D^{ab} is a c-number, differential operator. By so doing it is possible to derive properties of stress tensors from the Green's functions of the theory. Clearly there are many representations such as Eq. (2.22); they can all be thought of as different orderings. They are most useful when calculating quantities that are ordering independent. In particular, we can write

$$\langle 0,u|{:_v}\hat{T}^{ab}(x){:_v}|0,u\rangle=\lim_{x'\to x} D^{ab}(x,x')[G_u(x,x')-G_v(x,x')] , \qquad (2.23)$$

where

$$G_u(x,x')\equiv\langle 0,u|T\{\Phi(x)\Phi(x')\}|0,u\rangle , \qquad (2.24)$$

and T denotes time ordering.

There is considerable flexibility in the choice of the functions D^{ab} and G that appear on the right-hand side of Eq. (2.23). We have chosen the Feynman Green's function [Eq. (2.24)] for convenience. $D^{ab}(x,x')$ can be any differential operator that yields Eq. (2.17) in the limit $x'\to x$.

Point separation is commonly associated with the notions of regularization[3] and renormalization. Notice that this is not its role in Eq. (2.23): it merely provides a representation of a well-defined quantity that is already finite. By contrast, renormalization is necessary when one seeks to assign a finite, absolute value to an otherwise ill-defined expectation value such as $\langle 0,u|\hat{T}^{ab}|0,u\rangle$. Let us call this renormalized value $T^{ab}_R(u)$, "the renormalized expectation value of the stress tensor operator in the u vacuum." In subsequent sections of this paper we shall be concerned with computing quantities such as (2.23) and consequently we shall not require renormalization techniques. However it is of considerable interest, and a motivation for this paper, to study properties of renormalized stress tensors in relation to the states of the quantum field theory. To assist in this discussion we give the following schematic definition for $T^{ab}_R(u)$:

$$T^{ab}_R(u)=\lim_{x'\to x} D^{ab}(x,x')$$
$$\times[G_u(x,x')-G_s(x,x')] , \qquad (2.25)$$

where, loosely speaking, $G_s(x,x')$ is the state-independent singular part of $G_u(x,x')$. $G_s(x,x')$ can be, and usually is, taken to be the first few terms of the locally defined series solution[4] for any symmetric Feynman Green's function $G(x,x')$:

$$G(x,x')\equiv(4\pi^2)^{-1}[\Delta^{1/2}(\sigma+i\epsilon)^{-1} + V\ln(\sigma+i\epsilon)+W] , \tag{2.26}$$

where

$$\Delta(x,x')\equiv g^{-1/2}(x)D(x,x')g^{-1/2}(x')$$

and $D=-\det(-\sigma_{;ab'})$ is the Van Vleck-Morette determinant; $\sigma(x,x')$ is one-half the square of the distance along the geodesic joining x and x',

$$V=\sum_0^\infty a_n(x,x')\sigma^n , \tag{2.27}$$

$$W=\sum_0^\infty b_n(x,x')\sigma^n . \tag{2.28}$$

The coefficients $a_n(x,x')$ and $b_n(x,x')$ are determined from the recursion relations

$$(n+1)(n+2)a_{n+1}+(n+1)a_{n+1;c}\sigma^{;c}-(n+1)a_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;c}\sigma^{;c}+\tfrac{1}{2}(\Box-R/6)a_n=0 , \tag{2.29}$$

$$(n+1)(n+2)b_{n+1}+(n+1)b_{n+1;c}\sigma^{;c}-(n+1)b_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;c}\sigma^{;c}+\tfrac{1}{2}(\Box-R/6)b_n + (2n+3)a_{n+1}+a_{n+1;c}\sigma^{;c}-a_{n+1}\Delta^{-1/2}\Delta^{1/2}_{;c}\sigma^{;c}=0 , \tag{2.30}$$

and the boundary conditions

$$a_0+a_{0;c}\sigma^{;c}-a_0\Delta^{-1/2}\Delta^{1/2}_{;c}\sigma^{;c}+\tfrac{1}{2}(\Box-R/6)\Delta^{1/2}=0 .$$

b_0 must be chosen so that $G(x,x')=G(x',x)$.[5] When $b_0=0$ satisfies this condition we call the resulting Green's function "elementary." The series solutions for V and W can be shown to be analytic in those regions for which σ is single valued.[6]

In general, the singular part of G, G_s, will not itself be a Green's function. When this happens T_R^{ab} need no longer possess those symmetries of the classical (unquantized) theory that require the field equations to be satisfied. For example, the conformal invariance of the theory will be broken; there will be a conformal anomaly.[7] In other words, T_R^{ab} cannot, in general, be represented as an inner product in the Hilbert space of solutions to the classical wave equation.

However, in all cases that we shall discuss the spaces are such that the anomaly vanishes and G_s either is, or can be taken to be, a Green's function. In these circumstances the renormalized stress tensor can be represented as a normalized expectation value of a normal-ordered operator. For this reason we shall not attempt to make the definition (2.25) more precise. Instead we make the notion of normalization more precise: Consider a local Feynman Green's function which is a purely geometrical quantity that respects local symmetries of the space. As such it can be thought of as implicitly defining a local vacuum state; the local character of the state being characterized by the distance between the singularities of $\Delta(x,x')$. If Δ has no singularities this local vacuum also serves as a global vacuum in that it now respects all symmetries of the space. This would be the case for empty Minkowski space, for example, where $\Delta=1$ everywhere.

Let us denote by $|0,\Delta\rangle$ this local vacuum state. It can be defined implicitly by the equation

$$\langle 0,\Delta\,|\,T\{\hat\Phi(x)\hat\Phi(x')\}\,|\,0,\Delta\rangle\equiv G(x,x') , \tag{2.31}$$

where $\langle 0,\Delta\,|\,0,\Delta\rangle=1$ and $G(x,x')$ is given by Eq. (2.26). We can now define the "normalized expectation value of the stress tensor operator in the state $|0,u\rangle$," $T_N^{ab}(u)$, as

$$T_N^{ab}(u)\equiv\langle 0,u\,|{}^*_*\hat T^{ab}{}^*_*|\,0,u\rangle - \langle 0,\Delta\,|{}^*_*\hat T^{ab}{}^*_*|\,0,\Delta\rangle . \tag{2.32}$$

It is necessary for the purposes of definition to proceed, as we have done, via the state $|0,\Delta\rangle$. However, now that we have Eq. (2.32) we can express T_N^{ab} in a form that most closely resembles the tensor T_R^{ab}, namely,

$$T_N^{ab}(u)=\lim_{x'\to x} D^{ab}(x,x') \times[G_u(x,x')-G(x,x')] . \tag{2.33}$$

T_N^{ab} coincides with T_R^{ab} whenever

$$\lim_{x'\to x} D^{ab}(x,x')[G(x,x')-G_s(x,x')]=0 . \tag{2.34}$$

It is much easier to work with T_N^{ab} rather than T_R^{ab} and in so far as they are equivalent for the spaces we consider we shall continue to neglect T_R^{ab}.

We next describe how the quantum theory behaves under conformal transformations. We denote by M and $\widetilde{M}$ two spaces that are conformal to one another, having metrics g_{ab} and $\widetilde{g}_{ab}$, respectively, where

$$\widetilde{g}_{ab} \equiv \Omega^2 g_{ab} \; . \tag{2.35}$$

If Φ is a solution to the wave equation on M then, by Eq. (2.5), $\widetilde{\Phi}$ is a solution on $\widetilde{M}$ where

$$\widetilde{\Phi} \equiv \Omega^{-1}\Phi \; . \tag{2.36}$$

Thus corresponding to the u vacuum $|0,u\rangle$ on M there is the conformal $\widetilde{u}$ vacuum $|0,\widetilde{u}\rangle$ on $\widetilde{M}$. The Feynman Green's functions for these conformal vacuums are related by the equation

$$G_{\widetilde{u}}(x,x') = \Omega^{-1}(x) G_u(x,x') \Omega^{-1}(x') \; . \tag{2.37}$$

In more general terms, let $|U\rangle$ be a state in the Hilbert space of solutions to the wave equation on M and let $|\widetilde{U}\rangle$ be a state in the Hilbert space of solutions to the wave equation on $\widetilde{M}$. We shall say that $|\widetilde{U}\rangle$ is conformal to $|U\rangle$ ($|\widetilde{U}\rangle \simeq |U\rangle$) if and only if

$$\frac{\langle \widetilde{U} | T\{\hat{\widetilde{\Phi}}(x)\hat{\widetilde{\Phi}}(x')\} | \widetilde{U}\rangle}{\langle \widetilde{U} | \widetilde{U}\rangle} = \Omega^{-1}(x) \frac{\langle U | T\{\hat{\Phi}(x)\hat{\Phi}(x')\} | U\rangle}{\langle U | U\rangle} \Omega^{-1}(x') \; .$$

The equivalence relation ($\simeq$) will prove useful in describing the vacuums of Sec. VI.

It is a simple consequence of Eqs. (2.4), (2.19), (2.23), and (2.37) that

$$\langle 0,\widetilde{u} \,|\, :_{\widetilde{v}} \hat{T}^{ab}(\widetilde{g}_{cd}) :_{\widetilde{v}} |\, 0,\widetilde{u}\rangle = \Omega^{-6} \langle 0,u \,|\, :_v \hat{T}^{ab}(g_{cd}) :_v |\, 0,u\rangle \; . \tag{2.38}$$

The behavior of $T_N^{ab}(u)$ is more interesting. The normalized stress tensor for the $\widetilde{u}$ vacuum in the space $\widetilde{M}$, $\widetilde{T}_N^{ab}(\widetilde{u})$ is given by

$$\widetilde{T}_N^{ab}(\widetilde{u}) \equiv \langle 0,\widetilde{u} \,|\, {}^*_*\hat{T}^{ab}{}^*_* \,|\, 0,\widetilde{u}\rangle - \langle 0,\widetilde{\Delta} \,|\, {}^*_*\hat{T}^{ab}{}^*_* \,|\, 0,\widetilde{\Delta}\rangle \; . \tag{2.39}$$

It is related to $T_N^{ab}(u)$ by the equation

$$\widetilde{T}_N^{ab}(\widetilde{u}) = \Omega^{-6} T_N^{ab}(u) + \Omega^{-6} \langle 0,\Delta \,|\, {}^*_*\hat{T}^{ab}{}^*_* \,|\, 0,\Delta\rangle - \langle 0,\widetilde{\Delta} \,|\, {}^*_*\hat{T}^{ab}{}^*_* \,|\, 0,\widetilde{\Delta}\rangle \; . \tag{2.40}$$

The crucial point is that $|0,\widetilde{\Delta}\rangle$ need not be conformal to the state $|0,\Delta\rangle$. In terms of Green's functions we can have

$$\widetilde{G}(x,x') \neq \Omega^{-1}(x) G(x,x') \Omega^{-1}(x') \; ,$$

where by $\widetilde{G}$ we mean the same function of the geometry as G, evaluated now at $\widetilde{g}_{ab}$ rather than g_{ab}. Equation (2.40) can be written

$$\widetilde{T}_N^{ab}(\widetilde{u}) = \Omega^{-6} T_N^{ab}(u) + \lim_{x'\to x} D^{ab}(\widetilde{g}_{cd}) [\Omega^{-1}(x) G(x,x') \Omega^{-1}(x') - \widetilde{G}(x,x')] \; . \tag{2.41}$$

Of course, both $\widetilde{G}$ and $\Omega^{-1} G \Omega^{-1}$ are Green's functions on $\widetilde{M}$; they simply need not be the same.

Equation (2.41) is the conformal transformation law for normalized stress tensors that is to be compared with the conformal transformation law for renormalized stress tensors[8] which, for conformally flat spaces, can be written

$$\widetilde{T}_R^{ab}(\widetilde{u}) = \Omega^{-6} T_R^{ab}(u) + (2880\pi^2)^{-1} [R R^{ab} - R^a{}_c R^{cb} - \tfrac{1}{3} R^{;ab} + \widetilde{g}^{ab} (\tfrac{1}{3} \widetilde{\Box} R - \tfrac{1}{3} R^2 + \tfrac{1}{2} R_{cd} R^{cd})] \; , \tag{2.42}$$

where R_{ab} is the Ricci tensor for the space $\widetilde{M}$. When Eq. (2.34) is satisfied Eqs. (2.41) and (2.42) will be identical.

Notice that the operation of normalization preserves conformal invariance whereas that of renormalization, as may be seen in Eq. (2.42), does not; the trace of the normalized stress tensor, for conformally invariant field theories, will always be zero.

III. CONFORMAL KILLING VECTOR FIELDS

A space-time may be said to possess a "natural" vacuum state either if it has or if it is conformal to a space that has a globally timelike, curl-free, Killing vector field. One then has an unambiguous definition for positive-and negative-frequency solutions to the conformally invariant wave equation. We show by means of the following three theorems that the natural vacuums for any given conformally flat space are conformal to certain vacuums of three particular space-times, namely, Minkowski space, the Einstein static universe, and the open Einstein universe. (In Sec. V we shall see that these vacuums can be taken to be the elementary vacuums $|0,\Delta\rangle$.)

Theorem 1. If K^a is a Killing vector field of the metric $\Omega^2 g_{ab}$, then K^a is also a conformal Killing vector of g_{ab}.

Proof. Killing's equations for the metric $\Omega^2 g_{ab}$ are

$$0=\mathscr{L}_K(\Omega^2 g_{ab})\equiv\Omega^2\mathscr{L}_K(g_{ab})+g_{ab}\mathscr{L}_K(\Omega^2)\ , \tag{3.1}$$

where $\mathscr{L}_K$ is the Lie derivative with respect to K. These are equivalent to the conformal Killing equations for g_{ab},

$$\mathscr{L}_K(g_{ab})=2\lambda g_{ab}\ , \tag{3.2}$$

when λ is taken to be $-\frac{1}{2}(\ln\Omega^2)_{,c}K^c$. Thus if K^a satisfies (3.1) it clearly satisfies (3.2).

Theorem 2. If K^a is a timelike conformal Killing vector field of the metric g_{ab}, then it is a globally timelike Killing vector field on the space with metric $(-K^c g_{cd} K^d)^{-1} g_{ab}$. Further, if K^a is curl-free, then this space is ultrastatic[9] (i.e., admits a covariantly constant timelike vector field).

Proof. If K^a is a conformal Killing vector of g_{ab} we have, writing $f\equiv(-K^a g_{ab} K^b)^{1/2}$ and using (3.2),

$$\begin{aligned}\mathscr{L}_K(f^{-2}g_{ab})&=f^{-2}\mathscr{L}_K g_{ab}-f^{-4}g_{ab}\mathscr{L}_K f^2\\&=2f^{-2}\lambda g_{ab}+f^{-4}g_{ab}K^cK^d\mathscr{L}_K g_{cd}\\&=0\ ,\end{aligned}$$

since $\mathscr{L}_K K^a=0$. Thus K^a is a Killing vector of $f^{-2}g_{ab}$. Also

$$f^{-2}g_{ab}K^aK^b=-1\ , \tag{3.3}$$

so K^a is globally timelike on the new space.

To prove the last part, note that the condition of being curl free persists in the conformal space since conformal transformations preserve "angles." Using (3.3) and Killing's equations it is easy to show that

$$\tilde{K}_{a;b}=0\ , \tag{3.4}$$

where $\tilde{K}_a\equiv(f^{-2}g_{ab})K^b$ and the covariant derivative is with respect to $f^{-2}g_{ab}$.

Theorem 3. Any conformally flat, ultrastatic space-time is locally Minkowski space, the Einstein static universe, or the open Einstein universe.

Proof. The existence of a covariantly constant vector field (3.4) implies that

$$R_{abcd}\tilde{K}^d=0\ . \tag{3.5}$$

For a conformally flat metric $\tilde{g}_{ab}$,

$$R_{abcd}=\tilde{g}_{a[c}R_{d]b}+\tilde{g}_{b[d}R_{c]a}+\tfrac{1}{3}R\tilde{g}_{a[d}\tilde{g}_{c]b}\ ,$$

and hence

$$R_{ab}=(R/3)(\tilde{g}_{ab}+\tilde{K}_a\tilde{K}_b)\ , \tag{3.6}$$

where, by the Bianchi identities, R is constant. Thus the three-surfaces orthogonal to $\tilde{K}^a$ have constant curvature[10] which can only be zero, positive, or negative corresponding to the three space-times of the theorem.

Any connected component of a conformally flat, ultrastatic space-time can therefore always be analytically extended to one of the three space-times of Theorem 3.

These theorems reduce the problem of finding the natural vacuums for conformally flat spaces to the simpler problem of finding the natural vacuums on the conformally flat ultrastatic spaces. These spaces are of the form $R\times M^3$ where M^3 is a space of constant curvature. We shall describe their vacuums in Sec. V.

To find the natural vacuum for any given conformally flat space with a timelike, curl-free, conformal Killing vector field one first identifies the corresponding ultrastatic space. This identification will depend upon the structure of the vector field. As an example, in the next section, we shall describe in detail the timelike, curl-free conformal Killing vector fields in flat space and explicitly exhibit the correspondence with the ultrastatic spaces.

IV. CONFORMAL KILLING VECTORS IN FLAT SPACE

In this section we list and describe all timelike curl-free, conformal Killing vectors in flat space. We first find all solutions to Eq. (3.2) when g_{ab} is a flat metric. Let us define

$$\lambda\equiv\tfrac{1}{4}K^c_{;c}\ . \tag{4.1}$$

Equation (3.2) implies that

$$K_{a;bc}=(\lambda_{;b}g_{ac}+\lambda_{;c}g_{ab}-\lambda_{;a}g_{bc})\ , \tag{4.2}$$

and hence

$$K_{a;bcd}=0\ . \tag{4.3}$$

If we now choose Minkowski coordinates ($ds^2=\eta_{ab}dx^a dx^b$), we can easily integrate Eq. (4.3). The resulting general solution to Eq. (3.2) is provided by[11]

$$K^a=k^a+\Omega^a{}_b x^b+lx^a+2x^a(m\cdot x)-m^a x^2\ , \tag{4.4}$$

where l, k^a, m^a, and $\Omega^a{}_b$ are constants and

$$\Omega_{ab}=-\Omega_{ba}\ . \tag{4.5}$$

We now determine those solutions that have vanishing curl,

$$\eta^{abcd}K_b K_{c;d}=0\ . \tag{4.6}$$

This is straightforward. Up to constant scalings and shifts of origin of the coordinates, the distinct solutions that are timelike somewhere can be listed as follows:

$$K_0^a=k^a,\ \ k^2<0\ , \tag{4.7}$$

$$K_1^a=l^a(m\cdot x)-m^a(l\cdot x),\ \ l^2<0, m^2>0, l\cdot m=0\ , \tag{4.8}$$

$$K_2^a=x^a\ , \tag{4.9}$$

$$K_3^a=2x^a(m\cdot x)-m^a x^2,\ \ m^2<0\ , \tag{4.10}$$

$$K_4^a=nm^a+2x^a(m\cdot x)-m^a x^2,\ \ m^2=0, n<0\ , \tag{4.11}$$

$$K_5^a=nm^a+2x^a(m\cdot x)-m^a x^2,\ \ m^2<0, n>0\ , \tag{4.12}$$

$$K_6^a=nm^a+2x^a(m\cdot x)-m^a x^2,\ \ m^2>0, n<0\ , \tag{4.13}$$

$$K_7^a=nm^a+2x^a(m\cdot x)-m^a x^2,\ \ m^2<0, n<0\ . \tag{4.14}$$

Before describing the subspaces of Minkowski space where these vector fields are timelike it is useful first to determine the correspondence with the relevant ultrastatic space.

The constant curvature scalar R of the ultrastatic space with metric

$$\tilde{g}_{ab}=(-K^2)^{-1}g_{ab}\ , \tag{4.15}$$

where g_{ab} is flat, can be written

$$R=\tfrac{3}{2}(\Box K^2-3K^2\Box \ln K^2)$$
$$=-6f^3\Box f^{-1} \tag{4.16}$$

(covariant derivatives are with respect to g). For $i\in\{3,4,\ldots,7\}$, $R=24nm^2$. It follows that

R is negative for K_1, K_2, K_5, K_6 ,

R is zero for K_0, K_3, K_4 ,

R is positive for K_7 .

The conformally flat, ultrastatic space with negative curvature is the open Einstein universe with line element

$$ds^2=-d\tau^2+\xi^{-2}(d\xi^2+dy^2+dz^2) \tag{4.17}$$

for $-\infty<\tau,y,z<\infty$; $0\le\xi<\infty$. Alternatively, we can choose coordinates so that

$$ds^2=-d\tau^2+d\chi^2 + \sinh^2\chi(d\theta^2+\sin^2\theta\, d\phi^2)\ , \tag{4.18}$$

where $-\infty<\tau<\infty$; $0\le\chi<\infty$; $0\le\theta\le\pi$; $0\le\phi<2\pi$. The conformally flat, ultrastatic space with positive curvature is the Einstein static universe with line element

$$ds^2=-dt'^2+d\chi^2 + \sin^2\chi(d\theta^2+\sin^2\theta\, d\phi^2)\ , \tag{4.19}$$

for $-\infty<t'<+\infty$; $0\le\chi,\theta\le\pi$; $0\le\phi<2\pi$. The conformally flat, ultrastatic space with zero curvature is Minkowski space.

We shall now describe the eight vector fields in turn. Let $\tau(K)$ denote the set $K^2<0$, $\tau^+(K)$ the set $K^2<0$, $K^0>0$, $\tau^-(K)$ the set $K^2<0$, $K^0<0$. The trajectories are the curves

$$\dot{x}^a=(-K^2)^{-1/2}K^a\ , \tag{4.20}$$

where an overdot denotes differentiation with respect to proper time.

1. $K_0^a=k^a$ (Minkowski space).

Take $k^a=(1,0,0,0)$. $\tau(K_0)=\tau^+(K_0)$ is the whole of Minkowski space. The trajectories of K_0 are the parallel straight lines $x=$constant, $y=$constant, $z=$constant.

2. $K_1^a=l^a(m\cdot x)-m^a(l\cdot x)$; $l^2<0$, $m^2>0$, $l\cdot m=0$ (Rindler space)

Take $l^a=(1,0,0,0)$ and $m^a=(0,1,0,0)$, then $K_1^a=(x,t,0,0)$. $\tau^+(K_1)$ is the set $x^2>t^2$, $x>0$; $\tau^-(K_1)$ is the set $x^2>t^2$, $x<0$. In the coordinates defined by $x=\xi\cosh\tau$, $t=\xi\sinh\tau$, $\tau^+(K_1)$ is the region $0<\xi<\infty$, $-\infty<\tau<\infty$, and the line element becomes

$$ds^2=\xi^2[-d\tau^2+\xi^{-2}(d\xi^2+dy^2+dz^2)]\ .$$

The space $\{\tau^+(K_1); (-K_1{}^2)^{-1}ds^2\}$ is the open

Einstein universe.

K_1^a is zero on the two-surface $x=t=0$ and is null on $|t|=|x|$. The timelike trajectories of K_1, are determined by the equations $y=$constant, $z=$constant, $x^2-t^2=c^2$ $(c>0)$. The future-pointing timelike trajectories of K_1 are illustrated in Fig. 1 (the diagram possesses the symmetry of inversion in the origin; this determines the past-pointing timelike trajectories in $x<0$). Dashed lines are used to denote the horizon of K_1 which is defined by $K_1{}^2=0$. The zeros of K_1 are illustrated by a solid dot while arrows indicate the direction of the vector field on a trajectory.

The trajectories are all the familiar rectangular hyperbolas associated with a motion of constant, nonzero acceleration. This is a feature common to all flat-space, curl-free conformal Killing vectors: every trajectory has constant acceleration (this is proved in Sec. VI).

3. $K_2^a=x^a$ (The Milne Universe)

$\tau^+(K_2)$ is the region $|t|>r$, $t>0$; $\tau^-(K_2)$ is the region $|t|>r$, $t<0$. In the coordinates defined by $t=e^\tau\cosh\rho$, $r=e^\tau\sinh\rho$, $\tau^+(K_2)$ is the region $-\infty<\tau<\infty$, $0\le\rho<\infty$, $K_2{}^2=-e^{2\tau}$, and the line element becomes

$$ds^2=e^{2\tau}(-d\tau^2+d\rho^2+\sinh^2\rho\, d\Omega^2)\ ,$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2\ .$$

The further transformation $\xi^{-1}=\cosh\rho-\sinh\rho\cos\theta$, $y=\xi\sinh\rho\sin\theta\cos\phi$, $z=\xi\sinh\rho\sin\theta\sin\phi$ takes the line element to

$$ds^2=e^{2\tau}[-d\tau^2+\xi^{-2}(d\xi^2+dy^2+dz^2)]\ .$$

In these coordinates $\tau^+(K_2)$ is the region $-\infty<\tau,y,z<\infty$, $0<\xi<\infty$.

The space $\{\tau^+(K_2);\ (-K_2{}^2)^{-1}ds^2\}$ is the open Einstein universe. K_2^a is zero at the origin and null on the light cone of the origin. The timelike trajectories are given by the equations $\theta=$constant, $\phi=$constant, $t=Ar$ $(|A|<1)$, and are illustrated in Fig. 2.

4. $K_3^a=2x^a(m\cdot x)-m^ax^2;\ m^2<0$

Take $m^a=(-1,0,0,0)$, then $K_3{}^2=-(x^2)^2$ and $\tau(K_3)=\tau^+(K_3)$ is Minkowski space without the light cone of the origin.

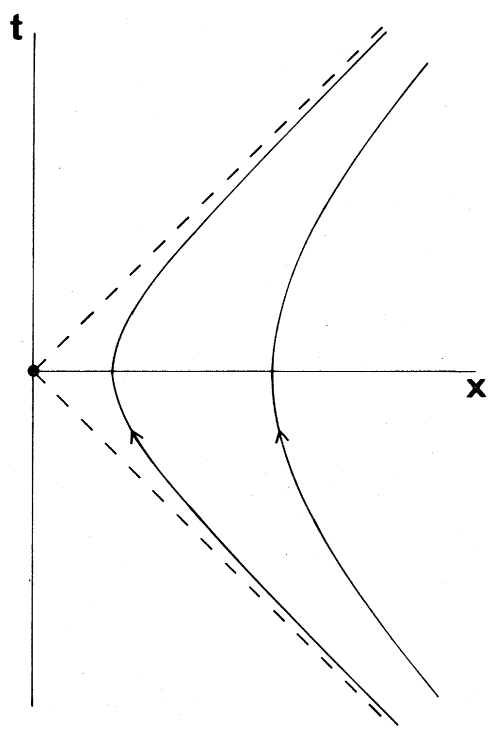

FIG. 1. The future-pointing timelike trajectories of the Rindler Killing vector field K_1. The zero of the field is illustrated by a solid dot while the horizon (defined by $K_1{}^2=0$) is denoted by a dashed line. The arrows on trajectories indicate the direction of the vector field on a trajectory.

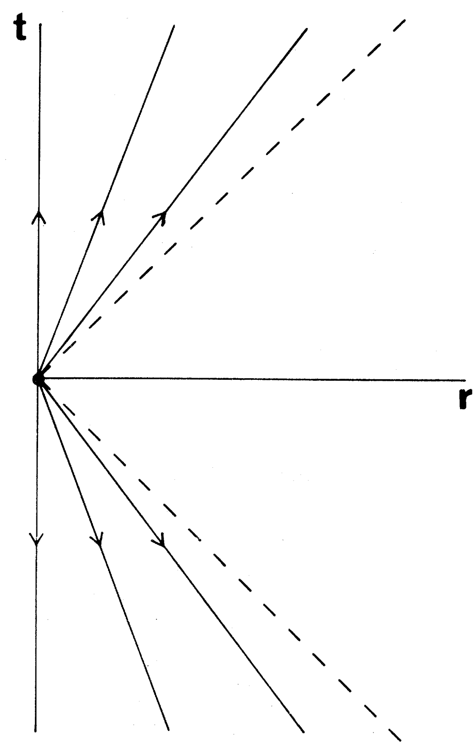

FIG. 2. The timelike trajectories of the Milne vector field K_2. The horizon is just the light cone of the origin.

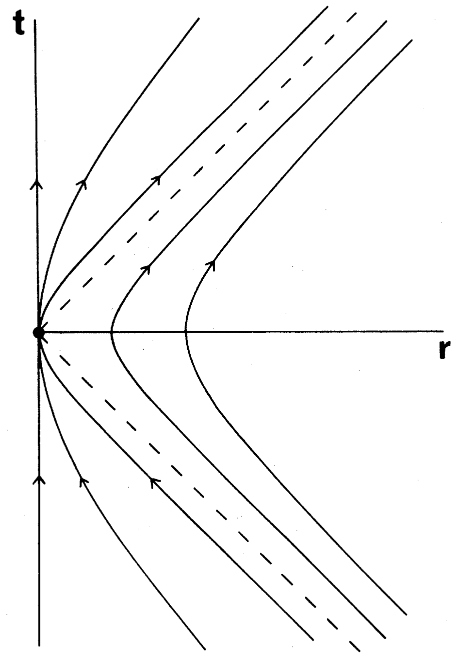

FIG. 3. The timelike trajectories of K_3.

With the coordinate transformation $t'-r' =(t+r)^{-1}$, $t'+r'=(t-r)^{-1}$, $\tau^+(K_3)$ is the region $0<|t'+r'|<\infty$, $0<|t'-r'|<\infty$ and the line element becomes

$$ds^2=(t'^2-r'^2)^{-2}(-dt'^2+dr'^2+r'^2d\Omega^2) .$$

The space $\{\tau^+(K_3);\ (-K_3{}^2)^{-1}ds^2\}$ can be analytically extended to the whole of Minkowski space.

K_3^a is zero at the origin and null on the light cone of the origin. The timelike trajectories are given by the equations $\theta=$constant, $\phi=$constant, $(r-c)^2-t^2=c^2$ $(c\neq 0)$, and are illustrated in Fig. 3.

5. $K_4^a=nm^a+2x^a(m\cdot x)-m^ax^2;\ m^2=0,\ n<0$

Take $m^a=(-1,1,0,0)$, $n=-1$ and use cylindrical polar coordinates in the y and z directions: $y=\rho\cos\psi$, $z=\rho\sin\psi$. Then $K_4{}^0=1+(t+x)^2+\rho^2$ and $K_4{}^2=-4(t+x)^2$ so $\tau^+(K_4)$ is Minkowski space minus the plane $t+x=0$.

With the coordinate transformation $t+x =(t'+x')^{-1}$, $t-x=(t'+x')^2\rho^2-(t'-x')$, $\rho=\rho'|t+x|$, $\psi=\psi'$, $\tau^+(K_4)$ is the region $0<|t'+x'|<\infty$ and the line element becomes

$$ds^2=(t'+x')^{-2}(-dt'^2+dx'^2+dy'^2+dz'^2) ;$$

$\{\tau^+(K_4);\ (-K_4{}^2)^{-1}ds^2\}$ can be analytically extended to the whole of Minkowski space. K_4^a is nowhere zero but is null on the plane $t+x=0$ as illustrated in Fig. 4(a). The timelike trajectories are given by the equations $\psi=$constant, $\rho=A(t+x)$,

$$t^2-x^2-\rho^2=c(t+x)-1 ,$$

and are illustrated in Figs. 4(b) and 4(c).

6. $K_5^a=nm^a+2x^a(m\cdot x)-m^ax^2;\ m^2<0,\ n>0$

Take $m^a=(1,0,0,0)$ and $n=1$. Then $K_5{}^2=-[(t+r)^2-1]\,[(t-r)^2-1]$. $\tau^+(K_5)$ is the region $|t+r|<1$, $|t-r|<1$ and $\tau^-(K_5)$ is the region $|t+r|>1$, $|t-r|>1$.

With the coordinate transformation $t+r =\tanh\frac{1}{2}(t'+r')$, $t-r=\tanh\frac{1}{2}(t'-r')$, $\tau^+(K_5)$ is the region $-\infty<t'<\infty$, $0\leq r'<\infty$ and the line element becomes

$$ds^2=\tfrac{1}{4}\text{sech}^2\tfrac{1}{2}(t'+r')\text{sech}^2\tfrac{1}{2}(t'-r') \times(-dt'^2+dr'^2+\sinh^2 r'd\Omega^2) .$$

With the coordinate transformation $(t+r) =\coth\frac{1}{2}(t'-r')$, $t-r=\coth\frac{1}{2}(t'+r')$, $\tau^-(K_5)$ is the region $-\infty<t'<\infty$, $0\leq r'<\infty$ and the line element becomes

$$ds^2=\tfrac{1}{4}\text{csch}^2\tfrac{1}{2}(t'+r')\text{csch}^2\tfrac{1}{2}(t'-r') \times(-dt'^2+dr'^2+\sinh^2 r'd\Omega^2) .$$

By comparison with case 3 we see that (up to a constant scale factor) the space $\{\tau^+(K_5);\ (-K_5{}^2)^{-1}ds^2\}$ is the open Einstein universe.

K_5^a has zeros at the points $r=0$, $t=\pm 1$ and on the two-surface $r=1$, $t=0$. It is null on the cones $r=|t\pm 1|$. The timelike trajectories are given by the equations $\theta=$constant, $\phi=$constant, $(r-c)^2-t^2=c^2-1$ $(|c|>1)$ and are illustrated in Fig. 5.

7. $K_6^a=nm^a+2x^a(m\cdot x)-m^ax^2;\ m^2>0,\ n<0$

Take $m^a=(0,1,0,0)$, $n=-1$ and use cylindrical polar coordinates in the y, z directions as for case 5. Then $K_6^0=2xt$ and

$$K_6{}^2=[t^2-\rho^2-(x-1)^2] \times[t^2-\rho^2-(x+1)^2] .$$

The region $\tau(K_6)$ is the union of the interiors of the cones $t^2=\rho^2+(x\pm 1)^2$ minus their intersection. These cones are illustrated in Fig. 6(a), together with the region $\tau^+(K_6)$.

In the coordinates defined by $t=\tau\cosh\chi$, $\rho=\tau\sinh\chi$, $K_6{}^2=[(\tau-x)^2-1][(\tau+x)^2-1]$, $\tau^+(K_6)$ is the region $|\tau+x|>1$, $|\tau-x|<1$, and $\tau^-(K_6)$ is the region $|\tau+x|<1$, $|\tau-x|>1$. In

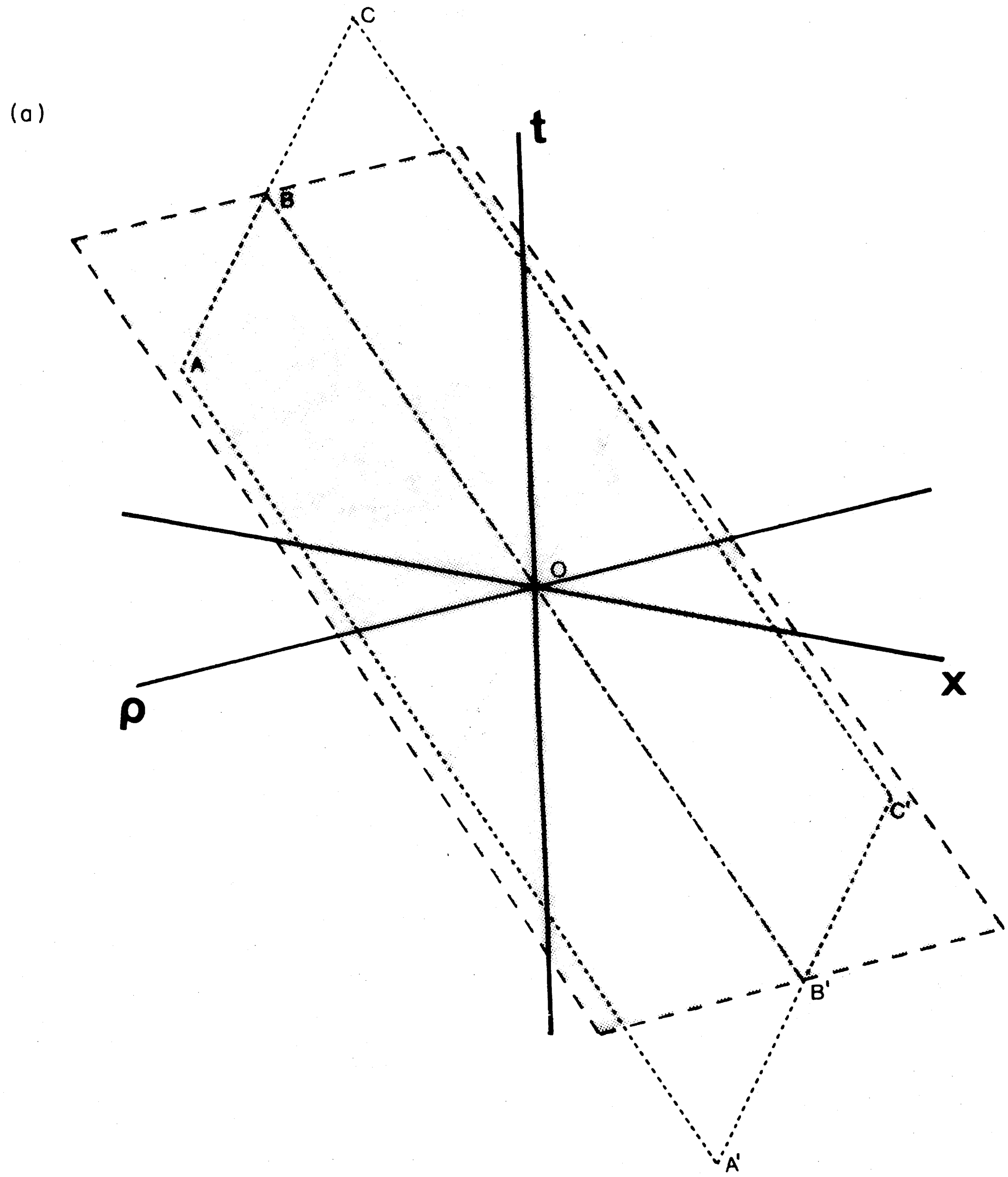

FIG. 4. (a) The shaded plane denotes the horizon of the vector field K_4, while ABCA′B′C′ is a typical plane through $\rho=0$, $x+t=0$. (b) The timelike trajectories of K_4 in the t-x plane. (c) The timelike trajectories of K_4 in the plane ABCA′B′C′.

these coordinates the line element is

$$ds^2=-d\tau^2+dx^2+\tau^2(d\chi^2+\sinh^2\chi\, d\psi^2)\ .$$

With the further coordinate transformation $\tau-x=\tanh\frac{1}{2}(t'+x')$, $\tau+x=\coth\frac{1}{2}(t'-x')$, $\tau^+(K_6)$ is the region $-\infty<t', x'<\infty$, $0\leq\chi<\infty$, $0\leq\psi<2\pi$ and the line element is

$$ds^2=\tfrac{1}{4}\,\mathrm{sech}^2\tfrac{1}{2}(x'+t')\mathrm{csch}^2\tfrac{1}{2}(x'-t')ds'^2\ ,$$

where

$$ds'^2=-dt'^2+dx'^2+\cosh^2x'(d\chi^2+\sinh^2\chi d\psi^2)\ .$$

With the coordinate transformation $\xi^{-1}=\cosh x'(\cosh\chi-\tanh x')$, $\rho=\xi\cosh x'\sinh\chi$,

$$ds'^2=-dt'^2+\xi^{-2}(d\xi^2+d\rho^2+\rho^2d\psi^2)$$

and $\tau^+(K_6)$ is $-\infty<t'<\infty$, $0<\xi, \rho<\infty$,

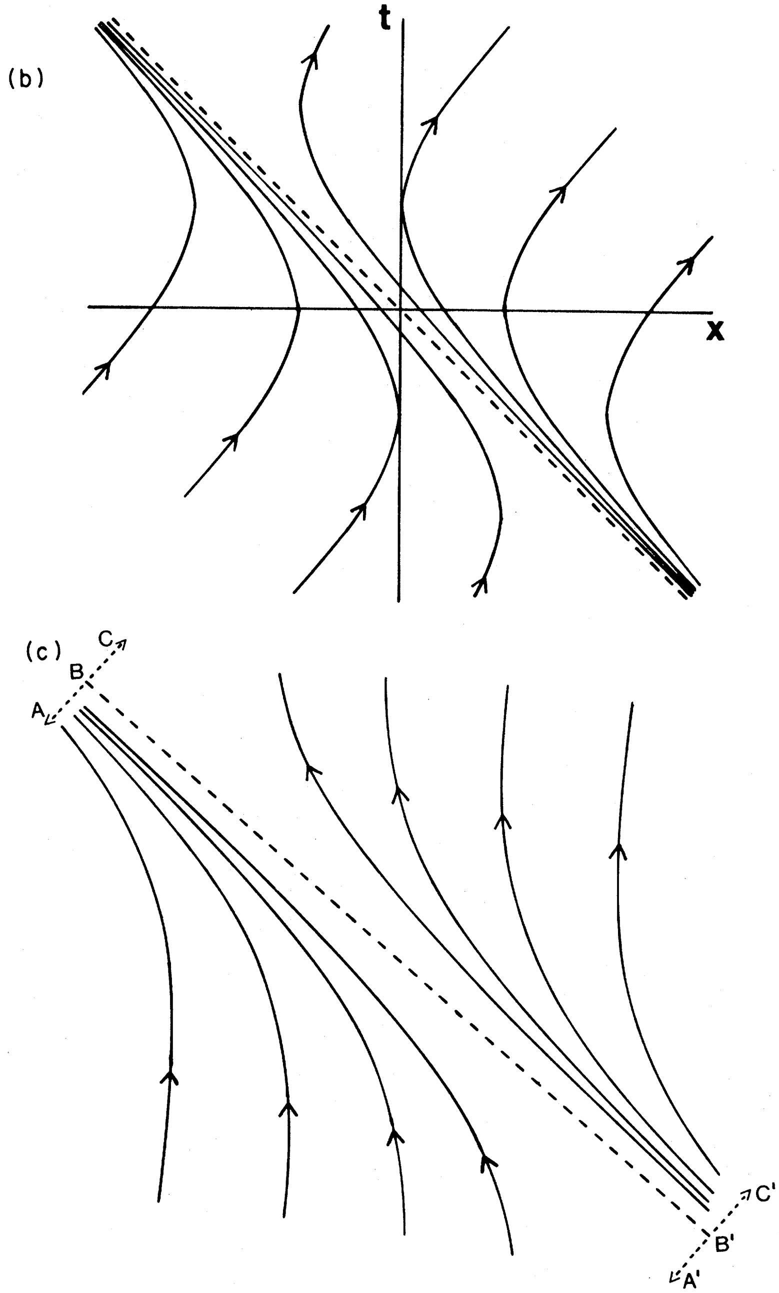

FIG. 4. (*Continued.*)

$0 \le \psi < 2\pi$. Hence $\{\tau^+(K_6); (-K_6{}^2)^{-1}ds^2\}$ is the open Einstein universe. $\tau^-(K_6)$ may be treated similarly.

K_6^a is zero at the points $t=\rho=0$, $x=\pm 1$ and on the two-surface $x=0$, $t^2-\rho^2=1$, which is the intersection of the two cones. It is null on the cones $(x\pm 1)^2+\rho^2=t^2$.

The timelike trajectories are defined by the equations $\psi=$constant, $\rho=At$,

$$x^2-[(1-A^2)^{1/2}t+c]^2 = 1-c^2 \qquad (|A|<1, |c|<1)$$

and are illustrated in Figs. 6(b) and 6(c).

8. $K_7=nm^a+2x^a(m\cdot x)-m^a x^2$; $m^2<0$, $n<0$

Take $m^a=(-1,0,0,0)$ and $n=-1$. Then $\tau(K_7)=\tau^+(K_7)$ is the whole of Minkowski space, and $K_7{}^2=-[(t-r)^2+1]\,[(t+r)^2+1]$. With the coordinate transformation $t-r=\tan\frac{1}{2}(t'-r')$,

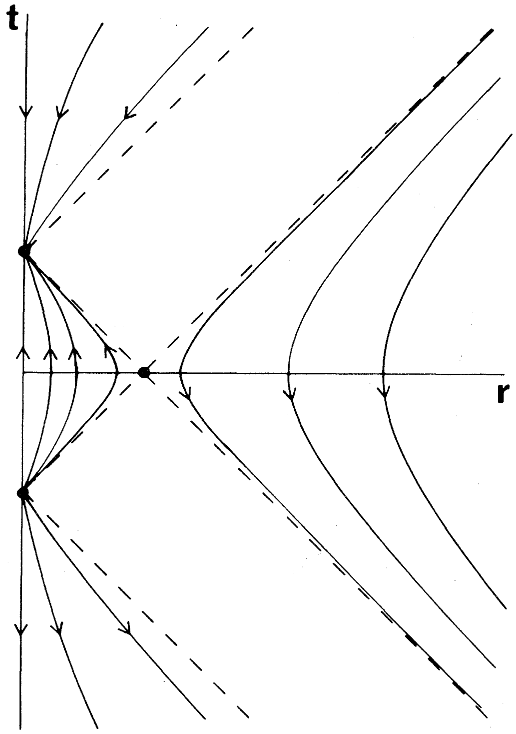

FIG. 5. The timelike trajectories of K_5.

$t+r=\tan\frac{1}{2}(t'+r')$, $\tau(K_7)$ is the region $|t'+r'|<\pi$, $|t'-r'|<\pi$, $r'\geq 0$. The line element becomes

$$ds^2=\frac{1}{4}\sec^2\frac{1}{2}(t'+r')\sec^2\frac{1}{2}(t'-r')\times(-dt'^2+dr'^2+\sin^2 r' d\Omega^2) .$$

The space $\{\tau(K_7); (-K_7{}^2)^{-1}ds^2\}$ can be analytically extended to the Einstein static universe.

K_7 has no zeros or horizons. All trajectories are timelike and are defined by the equations $\theta=$constant, $\phi=$constant, $(r-c)^2-t^2=1+c^2$ $(c\leq\infty)$ and are illustrated in Fig. 7. The trajectory corresponding to $c=0$ is of special interest as it marks the trajectory of greatest acceleration (the acceleration a is given by the formula $a^{-1}=1+c^2$). This trajectory is distinguished by a triple arrow in Fig. 7.

V. STATES AND STRESS TENSORS

We first describe the natural vacuums and Green's functions for the ultrastatic spaces. We continue to denote by a tilde quantities in the curved ultrastatic spaces. Thus $2\tilde{\sigma}$ is the square of the geodesic distance, $\tilde{K}$ the constant, timelike Killing field, and so on. The curvature scalar is unambiguously denoted by R. Covariant derivatives of quantities bearing a tilde will be with respect to the metric $\tilde{g}_{ab}$. The metric g_{ab} will denote a flat metric and K a conformal Killing vector field on flat space. As in Sec. III:

$$\tilde{g}_{ab}=(-K^2)^{-1}g_{ab} , \tag{5.1}$$

$$\tilde{K}^a=K^a,\ \ \tilde{K}_a=(-K^2)^{-1}K_a , \tag{5.2}$$

and

$$\tilde{K}_{a;b}=0 . \tag{5.3}$$

Many of the formulas we use in this section are valid for all three ultrastatic spaces; one need only substitute the appropriate value for R: $+|R|$, $-|R|$, or 0. Where we need to distinguish the spaces we append a subscript (or superscript), $+$, $-$, or 0, to denote the positive-, negative-, or zero-curvature versions of the relevant or equation.

The elementary (single geodesic[12]) Feynman Green's functions corresponding to Eq. (2.26) with $b_0=0$ can be given in closed form[13]:

$$\tilde{G}(x,x')=[8\pi^2(\tilde{\sigma}+i\epsilon)]^{-1}\tilde{\Delta}^{1/2} , \tag{5.4}$$

where

$$\tilde{\Delta}^{1/2}(x,x')=(R/6)^{1/2}\tilde{\gamma}\csc(R/6)^{1/2}\tilde{\gamma} , \tag{5.5}$$

and $\tilde{\gamma}(x,x')$ is the geodesic distance on the spatial section:

$$\tilde{\gamma}^2=(\tilde{K}^a\tilde{\sigma}_{;a})^2+2\tilde{\sigma} . \tag{5.6}$$

In terms of the coordinates defined in Sec. IV [Eqs. (4.18) and (4.19)]

$$\cosh\tilde{\gamma}_-(\chi,\chi';\theta,\theta';\phi,\phi')=\cosh\chi\cosh\chi'-\sinh\chi\sinh\chi'\cos\Gamma$$

and

$$\cos\tilde{\gamma}_+(\chi,\chi';\theta,\theta';\phi,\phi')=\cos\chi\cos\chi'+\sin\chi\sin\chi'\cos\Gamma ,$$

where

$$\cos\Gamma\equiv\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi') .$$

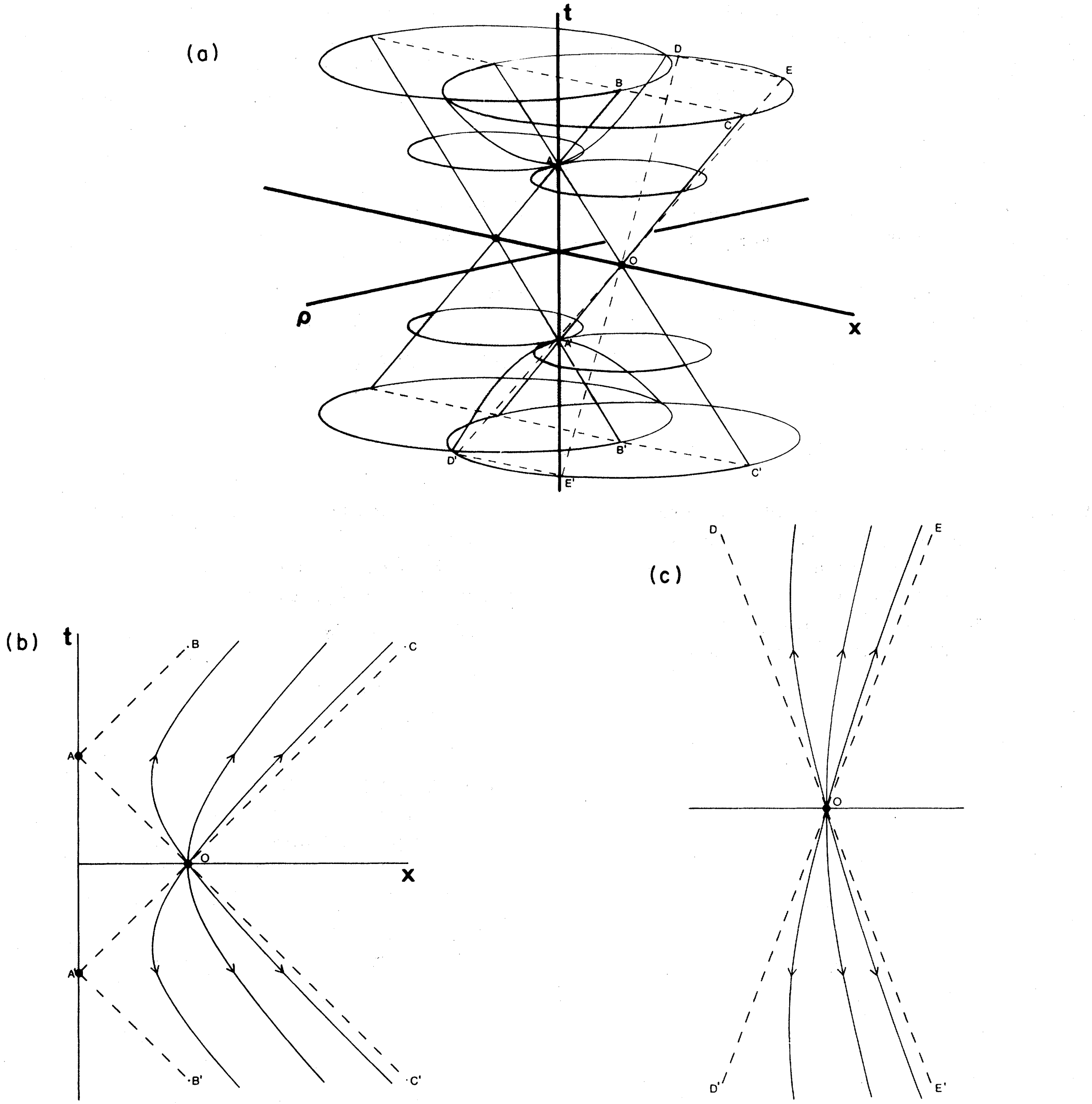

FIG. 6. (a) The shaded region denotes $\tau^+(K_6)$ the region in which K_6 is timelike and future pointing. DEOD′E′ is a typical plane through $t=\rho=0$. (b) The timelike trajectories of K_6 in the $\rho=0$ plane. (c) The timelike trajectories of K_6 in the plane DEOD′E′.

When R is negative $\widetilde{\Delta}_-$ has no singularities and the vacuum defined by $\widetilde{G}_-$ is usually taken to be the global vacuum for the open Einstein universe.[14] When R is zero Δ_0 is constant and Eq. (5.4) reduces to the usual Minkowski Green's function

$$G_0(x,x')=[8\pi^2(\sigma+i\epsilon)]^{-1} . \tag{5.7}$$

When R is positive (5.5) gives

$$\widetilde{\Delta}_+^{1/2}(x,x')=|R/6|^{1/2}\widetilde{\gamma}_+\csc|R/6|^{1/2}\widetilde{\gamma}_+ \tag{5.8}$$

and is singular when $\tilde{\gamma}_+(x,x')\,|R/6|^{1/2}=n\pi$. This singularity survives in the Green's function (5.4) and for this reason it is not taken to define the global vacuum state for the Einstein static universe. One defines instead the Green's function that can be obtained from (5.4) by making it periodic in $\tilde{\gamma}_+$.[12] The analogous function when R is negative is obtained from (5.4) by periodizing in $i\tilde{\gamma}_-$. This process results in the functions

$$\tilde{G}_\pi(x,x') = -(R/6)\{8\pi^2[\cos(R/6)^{1/2}\tilde{K}_a\tilde{\sigma}^{;a} - \cos(R/6)^{1/2}\tilde{\gamma} + i\epsilon]\}^{-1} . \qquad (5.9)$$

The function $\tilde{G}_\pi^+$ defines the natural vacuum state for the Einstein static universe; it is the function that is arrived at by summing over complete sets of modes that are defined to be positive frequency with respect to $\tilde{K}_+$.[15]

It is a consequence of the formulas given in Sec. III that the squares of the geodesic distances in flat space and the curved ultrastatic spaces, 2σ and $2\tilde{\sigma}$, respectively, are related by the equation

$$[K^2(x)K^2(x')]^{1/2}\sigma(x,x') = -(R/6)^{-1}\{\cos[(R/6)^{1/2}\tilde{K}_a\tilde{\sigma}^{;a}] - \cos[(R/6)^{1/2}\tilde{\gamma}]\} . \qquad (5.10)$$

Thus we can reexpress Eq. (5.9) as

$$\tilde{G}_\pi(x,x') = (8\pi^2)^{-1}[K^2(x)K^2(x')]^{1/2}[\sigma(x,x') + i\epsilon]^{-1} . \qquad (5.11)$$

Written in this form it is immediately apparent that the functions $\tilde{G}_\pi$ are Green's functions on the ultrastatic spaces; they are merely conformal to the Minkowski Green's functions G_0. Equation (5.11) also emphasizes the point made earlier that elementary Feynman Green's functions do not necessarily map into one another under conformal transformations.

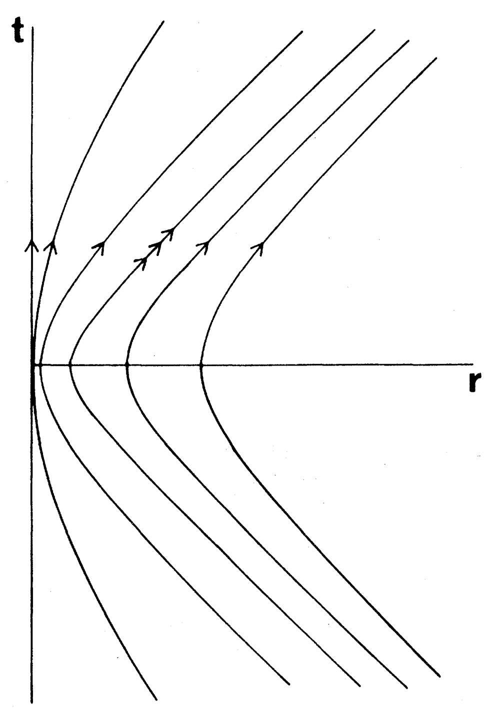

FIG. 7. The timelike trajectories of K_7. The trajectory distinguished by a triple arrow is the trajectory of greatest acceleration.

The ultrastatic vacuums corresponding to the functions $\tilde{G}$ and $\tilde{G}_\pi$ are defined by the equations

$$\langle 0,\tilde{\Delta}\,|\,T[\hat{\Phi}(x)\hat{\Phi}(x')]\,|\,0,\tilde{\Delta}\rangle = \tilde{G}(x,x') , \qquad (5.12)$$

$$\langle 0,\tilde{\pi}\,|\,T[\hat{\Phi}(x)\hat{\Phi}(x')]\,|\,0,\tilde{\pi}\rangle = \tilde{G}_\pi(x,x') , \qquad (5.13)$$

where $\langle 0,\tilde{\Delta}\,|\,0,\tilde{\Delta}\rangle = \langle 0,\tilde{\pi}\,|\,0,\tilde{\pi}\rangle = 1$.

$|0,\tilde{\pi}_-\rangle$ is a thermal state with respect to $|0,\tilde{\Delta}_-\rangle$ with temperature[16] $i(2\pi)^{-1}|R/6|^{1/2}$. $|0,\tilde{\pi}_+\rangle$ is a thermal state with respect to $|0,\tilde{\Delta}_+\rangle$ with temperature $(2\pi)^{-1}|R/6|^{1/2}$. The usual Minkowski vacuum is defined by

$$\langle 0\,|\,T[\hat{\Phi}(x)\hat{\Phi}(x')]\,|\,0\rangle = G_0(x,x') , \qquad (5.14)$$

where

$$\langle 0\,|\,0\rangle = 1 .$$

The vacuums in flat space corresponding to the eight conformal Killing fields K_i are obtained from the ultrastatic vacuums by performing the relevant conformal transformations. They are defined as follows:

$$\langle 0,K_i\,|\,T[\hat{\Phi}(x)\hat{\Phi}(x')]\,|\,0,K_i\rangle \equiv G_i(x,x') , \qquad (5.15)$$

where $\langle 0,K_i\,|\,0,K_i\rangle = 1, \{x,x'\}\subset\tau^+(K_i)$ and

$$G_i(x,x') \equiv [K_i^{\,2}(x)K_i^{\,2}(x')]^{-1/2}\tilde{G}_-(x,x') \qquad (5.16)$$

for $i\in\{1,2,5,6\}$,

$$G_i(x,x')\equiv[K_i{}^2(x)K_i{}^2(x')]^{-1/2}\tilde{G}_0(x,x') , \tag{5.17}$$

for $i\in\{0,3,4\}$;

$$G_i(x,x')\equiv[K_i{}^2(x)K_i{}^2(x')]^{-1/2}\tilde{G}_+(x,x') , \tag{5.18}$$

for $i=7$.

The states $|0,K_i\rangle$ for $i=0$, 1, 2, are the usual Minkowski, Rindler, and Milne vacuums, respectively. There is a slight redundancy in notation occasioned by the degeneracy of the Minkowski vacuum in that

$$|0\rangle\equiv|0,K_0\rangle\equiv|0,\Delta_0\rangle\equiv|0,\pi_0\rangle .$$

Notice that the flat vacuums are defined in terms of the elementary Feynman functions G. No new vacuums for flat space are to be found by transforming ultrastatic periodic functions $\tilde{G}_\pi$; by Eq. (5.11) they all get mapped to $G_0(x,x')$ with x and x' restricted to the appropriate region $\tau(K_i)$.

We can describe the relations between the various vacuums in terms of the notation of Sec. II:

$$|0,K_i\rangle\simeq|0,\tilde{\Delta}_-\rangle,\ \ i\in\{1,2,5,6\} , \tag{5.19}$$

$$|0,K_i\rangle\simeq|0,\tilde{\Delta}_0\rangle,\ \ i\in\{0,3,4\} , \tag{5.20}$$

$$|0,K_i\rangle\simeq|0,\tilde{\Delta}_+\rangle,\ \ i=7 . \tag{5.21}$$

Equation (5.11) implies the additional relations

$$|0,\tilde{\pi}_-\rangle\simeq|0\rangle , \tag{5.22}$$

$$|0,\tilde{\pi}_+\rangle\simeq|0\rangle . \tag{5.23}$$

The expectation values of stress tensor operators are most easily calculated in the ultrastatic spaces and then transformed to the conformal spaces as required. All expectation values that we shall need can be written as the coincidence limit of the differential operator $\tilde{D}^{ab}$ acting on the difference of two Green's functions. In any given ultrastatic space there are, at most, two Green's functions of (current) interest, $\tilde{G}_\pi$ and $\tilde{G}$, and hence only one nonvanishing difference $\tilde{G}_\pi-\tilde{G}$. Thus we begin our discussion of stress tensors with a calculation of the quantity

$$\tilde{T}_N^{ab}(\pi)=\lim_{x'\to x}\tilde{D}^{ab}(x,x')\times[\tilde{G}_\pi(x,x')-\tilde{G}(x,x')] , \tag{5.24}$$

where

$$\tilde{D}^{ab}(x,x')\equiv\tfrac{1}{6}\{\tilde{g}^{ab}(x)[-\tfrac{1}{6}R(x)-\tilde{g}^{c}_{c'}\tilde{\nabla}_c\tilde{\nabla}^{c'}]-2\tilde{\nabla}^a\tilde{\nabla}^b+4\tilde{g}^{b}_{b'}\tilde{\nabla}^a\tilde{\nabla}^{b'}+R^{ab}(x)\} , \tag{5.25}$$

and $\tilde{G}_\pi$ and $\tilde{G}$ are given by Eqs. (5.9) and (5.4), respectively.

The calculation is given in the Appendix. It is worth noting that one must calculate $\tilde{T}_N^{ab}(\pi)$ in a way that is required by its more fundamental representation, namely,

$$\tilde{T}_N^{ab}(\pi)=\langle 0,\tilde{\pi}\,|\,{}^*_*\hat{T}^{ab}{}^*_*\,|\,0,\tilde{\pi}\rangle-\langle 0,\tilde{\Delta}\,|\,{}^*_*\hat{T}^{ab}{}^*_*\,|\,0,\tilde{\Delta}\rangle . \tag{5.26}$$

By writing Eqs. (5.24) and (5.25) we have chosen an ordering $({}^*_*)$. This is quite legitimate, but once chosen it must be adhered to and must be the same for both functions $\tilde{G}_\pi$ and $\tilde{G}$. The result is

$$\tilde{T}_N^{ab}(\pi)=\alpha(\tfrac{1}{6}R)^2(\tilde{g}^{ab}+4\tilde{K}^a\tilde{K}^b) , \tag{5.27}$$

where

$$\alpha^{-1}=1440\pi^2 . \tag{5.28}$$

The flat-space expectation values are obtainable from Eq. (5.27) and either formula (2.40) or (2.38). Equations (5.19)–(5.23) identify the relevant conformal vacuums. The results are summarized in the formulas

$$T_N^{ab}(K_i)\equiv\langle 0,K_i\,|\,{}^*_*\hat{T}^{ab}(x){}^*_*\,|\,0,K_i\rangle-\langle 0\,|\,{}^*_*\hat{T}^{ab}(x){}^*_*\,|\,0\rangle , \tag{5.29}$$

whence

$$T_N^{ab}(K_i)=\alpha\frac{(\Box K_i{}^2-3K_i{}^2\Box\ln K_i{}^2)^2}{16(K_i{}^2)^2}\times\left[\frac{4K_i^aK_i^b}{K_i{}^2}-g^{ab}\right] \tag{5.30}$$

for all $i\in\{0,1,2,\ldots,7\}$. All quantities in Eqs. (5.30) refer to flat space: We have expressed the constant curvature R of Eq. (5.27) in terms of the conformal Killing field by using Eq. (4.16). Equation (5.30) is valid in the region $\tau(K_i)$ of Minkowski space.

We can make contact with renormalized stress tensors by substituting in Eq. (2.42),

$$R_{ab}=\tfrac{1}{3}R(\tilde{g}_{ab}+\tilde{K}_a\tilde{K}_b) , \tag{5.31}$$

the expression for the Ricci tensor in the ultrastatic spaces to obtain

$$\tilde{T}_R^{ab}(\tilde{u}) = \Omega^{-6} T_R^{ab}(u) + \alpha(\tfrac{1}{6}R)^2(\tilde{g}^{ab} + 4\tilde{K}^a\tilde{K}^b) . \tag{5.32}$$

Again using Eqs. (5.19)−(5.23) we find

$$\tilde{T}_R^{ab}(\pi) = \alpha(\tfrac{1}{6}R)^2(\tilde{g}^{ab} + 4\tilde{K}^a\tilde{K}^b) \tag{5.33}$$

and

$$\tilde{T}_R^{ab}(\Delta) = (-K_i{}^2)^3 T_R^{ab}(K_i) + \frac{\alpha}{16}(\Box K_i{}^2 - 3K_i{}^2 \Box \ln K_i{}^2) \times [(-K_i{}^2)g^{ab} + 4K_i^a K_i^b] . \tag{5.34}$$

Moreover, for conformally flat ultrastatic spaces, $\tilde{G}(x,x') \equiv \tilde{G}_s(x,x')$ [see Eq. (2.34)] and so $\tilde{T}_R^{ab}(\Delta) \equiv 0$.

These results are in agreement with previous expressions for renormalized stress tensors[16] for these spaces. It is important to remember that "the" renormalized stress tensor for the Einstein static universe is $\tilde{T}_R^{ab}(\pi)$, whereas "the" renormalized stress tensor for the open Einstein universe is $\tilde{T}_R^{ab}(\Delta)$.

If, in Eq. (5.29), we choose $(^*_*)$ to be normal ordering with respect to the Minkowski vacuum (:) we have

$$T_N^{ab}(K_i) = \langle 0,K_i \,|\, :\hat{T}^{ab}: \,|\, 0,K_i \rangle , \tag{5.35}$$

and the interpretation that the tensor $T_N^{ab}(K_i)$ describes the energy and momentum density of the state $|0,K_i\rangle$ with respect to the Minkowski vacuum. The energy density ρ_i is defined to be

$$\rho_i \equiv T_N^{ab}(K_i) K_{ia} K_{ib} (-K_i{}^2)^{-1} . \tag{5.36}$$

By Eq. (5.30) we have

$$\rho_i = -\frac{3\alpha}{16} \frac{(\Box K_i{}^2 - 3K_i{}^2 \Box \ln K_i{}^2)^2}{(K_i{}^2)^2} . \tag{5.37}$$

Thus the states $|0,K_i\rangle$ all have energy densities less than or equal to zero. (ρ_i is zero for $i \in \{0,3,4,\}$.)

One can choose other orderings for Eq. (5.29). If we denote by $(:_i)$ normal ordering with respect to $|0,K_i\rangle$ we can derive the equation

$$\langle 0,K_i \,|\, :_j \hat{T}^{ab}(x) :_j \,|\, 0,K_i \rangle = T_N^{ab}(K_i) - T_N^{ab}(K_j) , \tag{5.38}$$

where $x \in \tau(K_i) \cap \tau(K_j)$. We shall reserve discussion of the physical relevance of these results until Sec. VI.

We conclude this section with a brief discussion of the state $|0,K_7\rangle$ and its conformal Killing vector $K_7{}^a$. By Eqs. (5.37) and (4.41) we see that $\rho_7 < 0$ everywhere. Moreover $T_N^{ab}(K_7)$ has no singularities in Minkowski space and tends to zero at infinity. There is nothing particularly odd in this; $|0,K_7\rangle$ is a regular vacuum state for Minkowski space with energy less than the Minkowski vacuum $|0\rangle$, but it is not a translation-invariant state.

Although, as mentioned earlier, the function $\tilde{G}_+$ has pathologies as a Green's function for the Einstein static universe, the conformal function G_7 is well behaved on the whole of Minkowski space. In spherical polar Minkowski coordinates

$$G_7(t,r;t',r') \sim |R/6|^{1/2}(4\pi^3 |r - r'|)^{-1} \tag{5.39}$$

for large spacelike separations.

Finally we note that G_7 may be represented as a mode sum: One performs an analytic continuation $(R \to -R)$ of the natural modes for the open Einstein universe. This gives a mode sum representation for $\tilde{G}_+$ which is related by a conformal transformation to that for G_7.

VI. MEASUREMENTS BY NONINERTIAL OBSERVERS

We set out a few comments here concerning the relevance of the tensors $T_N^{ab}(K_i)$ to the physics of observers whose world lines in flat space are the trajectories of the conformal Killing vector fields K_i. These trajectories are defined by Eq. (4.20) and are drawn in Figs. 1−7 of Sec. IV. Their common properties may be established with the aid of the following equations:

$$fK_{a;b} = K_a f_{;b} - K_b f_{;a} + \lambda f g_{ab} , \tag{6.1}$$

$$2f^3 f_{;ab} = f^2(f\Box f - f_{;c}f^{;c})g_{ab} + 2(f\Box f - 2f_{;c}f^{;c})K_a K_b , \tag{6.2}$$

$$2f(f\lambda_{;a} - \lambda f_{;a}) + (f\Box f - f_{;c}f^{;c})K_a = 0 , \tag{6.3}$$

where, as before, $f \equiv |-K_a K^a|^{1/2}$, $\lambda \equiv \frac{1}{4} K^c_{;c}$, and K is any curl-free, conformal Killing vector field in flat space. Equation (6.1) follows from Eqs. (4.6) and (3.2). Equation (6.2) is Eq. (3.6) written in terms of K_a, and Eq. (6.3) is merely a useful consequence of the previous two. Equation (4.20) implies that

$$\ddot{x}^a = f^{-2}\lambda K^a + f^{-1} f^{;a} \tag{6.4}$$

and hence

$$\dddot{x}^a = f^{-2}(\lambda^2 + f_{;c} f^{;c}) x^a \ . \tag{6.5}$$

The acceleration $a \equiv f^{-1}(\lambda^2 + f^{;a} f_{;a})^{1/2}$, is constant along the world lines: $\dot{a} = 0$. Equation (6.5) is in an integrable form; any individual trajectory is part of a rectangular hyperbola corresponding to constant acceleration.

Let us now consider a point "observer" O who moves on one of these trajectories through flat space which we take to be in the Minkowski vacuum state. We wish to discuss the relevance to O of the fields K_i and their vacuums.

Any open interval of a given trajectory having nonzero acceleration can be regarded as belonging to any of the vector fields K_i, $i \in \{1,3,4,5,6,7\}$. Thus, if O is strictly confined to his world line he cannot determine to which field he belongs. Although, of course, he may be able to obtain some information if he reaches the boundary of his space in a finite proper time. However if O wishes to make quantum-mechanical observations his apparatus must extend into some neighborhood of his world line. How this extension is made is crucial to the nature of subsequent observations. Thus, although O may have a pointlike existence, his apparatus, in order to be useful, necessarily defines a vector field. Which vector field is defined in this way is something that only O can decide. We shall consider only those configurations where the constituent atoms of O's apparatus or laboratory have world lines that are described by the trajectories of one of the vector fields K_i.

For example, suppose O takes with him a rigid, nonrotating, Unruh box[17] with which to measure the Minkowski vacuum. Let the material of the box be such that the scalar field vanishes on the sides. It is the concept of rigidity that describes how the box behaves in relation to O and provides the extension from O's world line to a particular vector field. In this case it is K_1, the Rindler Killing vector field, that is implicitly defined.

The largest possible rigid box that O can have will fill a Rindler space. Moreover it is easily shown that

$$G_1(x,x') = 0$$

when x (or x') is on the horizon $K_1{}^2(x) = 0$. The Green's function for a box of finite size differs from G_1 only in its spatial boundary conditions. Thus if the box has classical dimensions its physics is essentially described by G_1 and we can say that the natural ground state for O's rigid box is the Rindler vacuum. The stress energy seen by the box will be the stress energy of the Minkowski vacuum counted in terms of Rindler particles, viz.,

$$\langle 0,K_0 | :_1\hat{T}^{ab}:_1 | 0,K_0 \rangle \ . \tag{6.6}$$

By Eq. (5.38) this expectation value equals $-T_N^{ab}(K_1)$. Hence O's rigid box will register a positive-energy thermal state with temperature proportional to $(\rho_1)^{1/4}$.

The property that the fields K_0 and K_1 can be said to describe rigid motions is characterized geometrically by the property that their expansion tensor $\theta_{ab}(K)$ vanishes,[18]

$$\theta_{ab}(K) \equiv (g_{ac} + V_a V_c)(g_{bd} + V_b V_d) V^{c;d} \ , \tag{6.7}$$

where

$$V_a = f^{-1} K_a \ .$$

If O elects to take a nonrigid (expanding) box it will record different information. In general, if the box is such that its constituent world lines are the trajectories of K_i (for some i), then its natural ground state will be the $|0,K_i\rangle$ vacuum. When immersed in the Minkowski vacuum it will behave as if in the presence of the stress-energy density $-T_N^{ab}(K_i)$. When immersed in the $|0,K_j\rangle$ vacuum it will behave as if in the presence of the stress-energy density

$$\langle 0,K_j | :_i\hat{T}^{ab}:_i | 0,K_j \rangle = T_N^{ab}(K_j) - T_N^{ab}(K_i) \ . \tag{6.8}$$

In Eq. (6.8) the fields K_i and K_j can be any pair of timelike, curl-free, conformal Killing vector fields. For example, they might both be the same up to a translation. The point being that only $|0,K_0\rangle$ is translation invariant. The derivation of Eq. (6.8) in Sec. V makes no reference to any particular coordinate system or origin.

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APPENDIX

In this appendix we use the method of point separation introduced in Sec. V to calculate the normalized stress tensor $\tilde{T}_N^{ab}(\pi)$ in the three standard conformally flat ultrastatic spaces. Since $\widetilde{G}_\pi(x,x')-\widetilde{G}(x,x')$ is a twice continuously differentiable function of its arguments the limit of (5.24) is well defined: it does not depend on the direction of separation. We are therefore at liberty to take the simplest possible separation, viz., along the $\widetilde{K}^a$ direction: $(x-x')^a=\epsilon\widetilde{K}^a$. For all three spaces we have

$$\widetilde{G}_\pi(x,x')=\frac{1}{8\pi^2}\frac{(R/6)}{\cos(R/6)^{1/2}(t-t')-\cos(R/6)^{1/2}\widetilde{\gamma}+i\epsilon}\ , \tag{A1}$$

$$\widetilde{G}(x,x')=\frac{1}{4\pi^2}\frac{(R/6)^{1/2}\widetilde{\gamma}}{\sin(R/6)^{1/2}\widetilde{\gamma}}\frac{1}{-(t-t')^2+\widetilde{\gamma}^2+i\epsilon}\ , \tag{A2}$$

where $\widetilde{\gamma}$ is defined by (5.6) and we have written $(t-t')=-\widetilde{K}^a\widetilde{\sigma}_{;a}$.

We can now calculate the derivatives of $\widetilde{G}_\pi-\widetilde{G}$ we require for (5.24) and (5.25) with $(x-x')^a=\epsilon\widetilde{K}^a$ by expanding (A1) and (A2) about $\widetilde{\gamma}=0$ and using the formulas

$$[(\tfrac{1}{2}\widetilde{\gamma}^2)_{;a}]_{\widetilde{\gamma}=0}=[(\tfrac{1}{2}\widetilde{\gamma}^2)_{||a}]_{\widetilde{\gamma}=0}=0\ ,$$

$$[(\tfrac{1}{2}\widetilde{\gamma}^2)_{;ab}]_{\widetilde{\gamma}=0}=[(\tfrac{1}{2}\widetilde{\gamma}^2)_{||ab}]_{\widetilde{\gamma}=0}=-[(\tfrac{1}{2}\widetilde{\gamma}^2)_{;ab'}]_{\widetilde{\gamma}=0}=\widetilde{g}_{ab}+\widetilde{K}_a\widetilde{K}_b\ ,$$

where $||a$ denotes differentiation in the hypersurface $t=$constant. These follow from the fact that $\widetilde{\gamma}$ is the geodesic distance in the three-section and is independent of t.

For small ϵ the required formulas are

$$[\nabla_a\nabla_b(\widetilde{G}_\pi-\widetilde{G})]_{\widetilde{\gamma}=0}=-[\nabla_a\nabla_{b'}(\widetilde{G}_\pi-\widetilde{G})]_{\widetilde{\gamma}=0}$$
$$=(1440\pi^2)^{-1}[(11\widetilde{g}_{ab}+14\widetilde{K}_a\widetilde{K}_b)(R/6)^2+O(\epsilon^2)]\ ,$$

$$[\widetilde{G}_\pi-\widetilde{G}]_{\widetilde{\gamma}=0}=-(288\pi^2)^{-1}[R+O(\epsilon^2)]\ .$$

Moreover our simple choice of separation means that $g_{ab'}=\eta_{ab}\delta^b_{b'}$. Inserting these results into (5.24), together with the formula $R_{ab}=(R/3)(\widetilde{g}_{ab}+\widetilde{K}_a\widetilde{K}_b)$ and letting the separation ϵ tend to zero we find

$$\widetilde{T}^{ab}_N(\pi)=(1440\pi^2)^{-1}(R/6)^2\,(\widetilde{g}^{ab}+4\widetilde{K}^a\widetilde{K}^b)\ .$$

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