Ghost properties of generalized theories of gravitation

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We investigate theories of gravitation, in which spacetime is non-Riemannian and the metric $g_{\mu\nu}$ is nonsymmetric, for ghosts and tachyons, using a spin-projection operator formalism. Ghosts are removed not by gauge invariance but by a Lagrange multiplier W_μ, which occurs due to the breaking of projective invariance in the theory. Unified theories based on a Lagrangian containing a term $\lambda g^{\mu\nu}g_{[\mu\nu]}$ are proved to contain ghosts or tachyons.

I. INTRODUCTION

Generalized theories of gravitation continue to receive attention in the literature.[1] General relativity, which is quite successful as a classical theory of gravity (having been confirmed by all recent experimental data[2]), is not satisfactory as a quantum theory since one-loop quantum corrections lead to divergences in the S matrix as soon as couplings to matter are added.[3] Generalization, therefore, has focused mainly on attempts to quantize gravity or unify it with other forces in nature.[4] Supergravity is an attempt at quantization which is finite to at least two loops; it is also hoped that unification of gravity with other forces in nature will occur in this theory.[5] R^2-type theories are another attempt at quantization, since such theories can be renormalizable.[6] Sezgin and van Nieuwenhuizen,[7] using the technique of spin-projection operators,[8] found five classes of R^2-type theories that were ghost and tachyon free and which were general-coordinate and local Lorentz invariant.

In this paper we use the spin-projection operator formalism[8] to investigate a third type of generalization of general relativity for ghosts and tachyons. In it the geometry of spacetime is non-Riemannian and the metric $g_{\mu\nu}$ is nonsymmetric; this theory also includes torsion. It was originally developed by Einstein and Straus[9] as a unified theory of gravity and electromagnetism and still receives attention in the literature as such.[10] However, it has recently been shown that the theory can be interpreted purely as a theory of gravity. The theory explains the stability of fermion matter and has several interesting features.[11] Geometrical formulations for the theory exist in which the metric is complex[12] or, alternatively, real.[13] Furthermore, there is a projective transformation in the theory due to the generalization of the connection $\Gamma^\lambda_{\mu\nu}$ (Refs. 11 and 14); this leads to interesting ghost properties in the theory as we shall see. Although analysis of the helicity content of certain versions of this theory has been carried out before,[15] the spin-projection analysis has never been done and the helicity content for the most general theory of this type has not been considered.

II. THE LAGRANGIAN

The Lagrangian for the theory is[16]

$$L=\mathfrak{g}^{\mu\nu}R_{\mu\nu}-\tfrac{2}{3}g^{[\mu\nu]}{}_{,\nu}W_\mu+L_s\ , \tag{1}$$

where

$$R_{\mu\nu}=\Gamma^\lambda_{\mu\nu,\lambda}-\tfrac{1}{2}(\Gamma^\lambda_{(\mu\lambda),\nu}+\Gamma^\lambda_{(\nu\lambda),\mu})+\Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{(\lambda\sigma)}-\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu}\ . \tag{2}$$

Here $\mathfrak{g}_{\mu\nu}=\sqrt{-g}\,g_{\mu\nu}$ where g is the determinant of the nonsymmetric metric $g_{\mu\nu}$. $W_\mu\equiv W^\lambda_{[\mu\lambda]}$ is the torsion vector formed from $W^\lambda_{\mu\nu}$, the general connection of the non-Riemannian spacetime. The Lagrangian (1) is invariant under the U(1) gauge transformation: $W'_\mu=W_\mu+\lambda_{,\mu}$. The physical connection $\Gamma^\lambda_{\mu\nu}$ is determined from the projective transformation[11,14]

$$\Gamma^\lambda_{\mu\nu}=W^\lambda_{\mu\nu}+\tfrac{2}{3}\delta^\lambda_\mu W_\nu\ . \tag{3}$$

L_s is given by

$$L_s=\frac{8\pi}{3}W_\mu\mathfrak{S}^\mu-8\pi\mathfrak{g}_{\mu\nu}T^{\mu\nu}\ . \tag{4}$$

$\mathfrak{S}^\mu$ is interpreted as a fermion-number current density and $T^{\mu\nu}$ is the nonsymmetric energy-momentum tensor. Both S^μ and $T^{\mu\nu}$ obey conservation laws[16]:

$$(g_{\nu\rho}\mathfrak{T}^{\nu\alpha}+g_{\rho\nu}\mathfrak{T}^{\alpha\nu})_{,\alpha}-g_{\mu\nu,\rho}\mathfrak{T}^{\mu\nu}+\tfrac{2}{3}W_{[\rho,\alpha]}\mathfrak{S}^{\alpha}=0\ , \quad (5)$$

$$\mathfrak{S}^{\mu}{}_{,\mu}=0\ , \quad (6)$$

which are related to the general coordinate invariance and U(1) gauge invariance of the theory.

In the unified field theory[17,18] a term proportional to $\mathfrak{g}^{\mu\nu}g_{[\mu\nu]}$ (sometimes referred to as the Bonnor term) is added in order that the equations of motion of the theory have the neccessary Lorentz-force term[17]; such a term is *not* a cosmological term. The sources S^{μ} and $T^{\mu\nu}$ also have a different interpretation.[10]

The reality properties of the metric vary depending on the type of theory considered; in general the metric is

$$g_{\mu\nu}=g_{(\mu\nu)}+g_{[\mu\nu]}\ , \quad (7)$$

where $g_{(\mu\nu)}$ is always real and $g_{[\mu\nu]}$ is either real or pure imaginary.

The general linearized Lagrangian of the theory is[19]

$$L^{(2)}=L^{(2)}_{\rm GR}+\tfrac{1}{2}(ah_{[\mu\nu],\lambda}h_{[\mu,\nu],\lambda}+2bh_{\mu}h_{\mu})+cW_{\mu}h_{\mu} -dh_{[\mu\nu]}h_{[\mu\nu]}+W_{\mu}S_{\mu}+h_{[\mu\nu]}T_{\mu\nu}+h_{[\mu\nu]}S_{\mu,\nu}\ , \quad (8)$$

where $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, $h_{\mu}\equiv\partial^{\nu}h_{[\mu\nu]}$, and repeated indices imply summation using $\eta^{\mu\nu}$; for example $W_{\mu}S_{\mu}\equiv\eta^{\mu\nu}W_{\mu}S_{\nu}$. $L^{(2)}_{\rm GR}$ is the linearized Lagrangian of general relativity,[19] and a, b, c, and d are constants which assume values depending on the type of theory considered. The graviton spectrum is free of ghosts, so we shall only consider the skew field part of $L^{(2)}$. The term $dh_{[\mu\nu]}h_{[\mu\nu]}$ is the linearization of $\mathfrak{g}^{\mu\nu}g_{[\mu\nu]}$.

Our conventions are $G=c=\hbar=1$ and the signature of the metric is $(-,-,-,+)$.

III. GHOST PROPERTIES

We follow the analysis and conventions of Ref. 7 and write $L^{(2)}$ as

$$L^{(2)}=\tfrac{1}{2}\sum_{A,B}\phi_A O_{AB}\phi_B\ , \quad (9)$$

where $\phi_A=(h_{[\mu\nu]},W_{\mu})$ and O_{AB} is the wave operator. Using Eqs. (10) of Ref. 7, we can decompose the fields into subspaces with spin-parity J^P and invert O_{AB} to obtain the saturated propagator

$$\Pi=-\sum_{\psi_A,\phi_B}S_A O^{-1}{}_{AB}S_B\ , \quad (10)$$

where $S_A=(K_{[\mu\nu]},S_{\mu})$, with $K_{[\mu\nu]}=T_{[\mu\nu]}+S_{\mu,\nu}$. Expanding O_{AB} in terms of the projection operators yields

$$L^{(2)}=\sum_{\psi_A,\phi_B,i,j,J^P}a_{ij}^{\psi\phi}(J^P)\psi_A P_{ij}^{\psi\phi}(J^P)_{AB}\phi_B\ , \quad (11)$$

$$\Pi=-\sum_{\psi_A,\phi_B,i,j,J^P}a^{-1}{}_{ij}^{\psi\phi}(J^P)S_A P_{ij}^{\psi\phi}(J^P)_{AB}S_B\ , \quad (12)$$

where $a_{ij}^{\psi\phi}(J^P)$ are the coefficient matrices. The relevant spin-projection operators are

$$P(1^+)=\tfrac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta}-\theta_{\nu\alpha}\theta_{\mu\beta})\ , \quad (13a)$$

$$P_{ij}(1^-)=\begin{vmatrix} \tfrac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\beta}-\theta_{\mu\beta}\omega_{\nu\alpha}-\theta_{\nu\alpha}\omega_{\mu\beta}+\theta_{\nu\beta}\omega_{\mu\alpha}) & \frac{1}{\sqrt{2}}(\not k_{\nu}\theta_{\mu\alpha}-\not k_{\mu}\theta_{\nu\alpha}) \\ \frac{1}{\sqrt{2}}(\not k_{\nu}\theta_{\mu\alpha}-\not k_{\mu}\theta_{\nu\alpha}) & \theta_{\alpha\beta} \end{vmatrix}\ , \quad (13b)$$

$$P(0^+)=\omega_{\alpha\beta}\ . \quad (13c)$$

Here

$$\theta_{\mu\alpha}=\eta_{\mu\alpha}-k_{\mu}k_{\alpha}k^{-2}\ , \quad (14a)$$

$$\omega_{\mu\alpha}=k_{\mu}k_{\alpha}k^{-2}\ , \quad (14b)$$

$$\not k_{\mu}=k_{\mu}(k^2)^{-1/2}\ . \quad (14c)$$

It is easily verified that these operators are orthonormal and complete[20] over $(h_{[\mu\nu]},W_{\mu})$. The corresponding coefficient matrices are found from Eq. (8) to be

$$a(1^+)=ak^2-d\ , \quad (15a)$$

$$a_{ij}(1^-)=\begin{vmatrix} (a+b)k^2-d & \frac{ic}{\sqrt{2}}(k^2)^{1/2} \\ -\frac{ic}{\sqrt{2}}(k^2)^{1/2} & 0 \end{vmatrix}\ , \quad (15b)$$

$$a(0^+)=0\ . \tag{15c}$$

The zero in Eq. (15c) indicates that there is a gauge invariance[7] $W'_\mu = W_\mu + \lambda_{,\mu}$ and a source constraint $S_{\mu,\mu}=0$. This is simply the aforementioned U(1) gauge invariance, which is a general feature of this type of theory.[16]

Inversion of Eqs. (15a) and (15b) gives

$$a^{-1}(1^+)=\frac{1}{ak^2-d}\ , \tag{16a}$$

$$a_{ij}{}^{-1}(1^-)=\begin{bmatrix} 0 & \frac{i\sqrt{2}}{c}(k^2)^{1/2} \\ -i\frac{\sqrt{2}}{c}(k^2)^{1/2} & -\frac{2}{c^2}[(a+b)k^2-d] \end{bmatrix}\frac{1}{k^2}\ . \tag{16b}$$

Consider the case $d\neq 0$. The criteria for freedom from ghosts and tachyons[7] gives from Eq. (16a)

$$a>0,\ \ d>0\ . \tag{17}$$

But the same criteria applied to Eq. (16b) give $d<0$, since $c^2>0$ for L to be real. Thus for $d\neq 0$ we have either a massless spin-1^- *ghost particle* or else a spin-1^+ tachyon. The helicity content is in agreement with Kursunoglu's results for the unified theory; we now see from this analysis that the unified theory based on the Bonnor term $d\mathfrak{g}^{\mu\nu}g_{[\mu\nu]}$ *must contain unphysical particles.* The saturated propagator is

$$\Pi=-\left[\frac{1}{ak^2-d}\right]T_{\mu\nu}P(1^+)_{\mu\nu\alpha\beta}-T_{\alpha\beta}-\frac{i}{ck^2}S_\alpha(k_\nu\theta_{\mu\alpha}-k_\mu\theta_{\nu\alpha})K_{\mu\nu}+\frac{i}{ck^2}K_{\mu\nu}(k_\mu\theta_{\nu\alpha}-k_\nu\theta_{\mu\alpha})S_\alpha$$
$$+\left[\frac{2(a+b)}{c^2}-\frac{d}{k^2}\right](S_\alpha\theta_{\alpha\beta}S_\beta)\ . \tag{18}$$

Terms of the form $P(1^+)_{\mu\nu\alpha\beta}S_{[\alpha,\beta]}$ did not occur because $k_\alpha\theta_{\alpha\beta}=0$.

Thus we can only have a physical theory if $d=0$.[21] In order to avoid ghosts, the only constraint we have is $a>0$. The massless spin-1^- particle no longer propagates; instead only contact terms occur in this sector. The spin-1^+ sector is no longer a tachyon. This sector has been analyzed previously[22] and represents a massless *scalar* particle. The saturated propagator is given by Eq. (18) with $d=0$.

It is interesting that there are no constraints on the parameter b. In skew field Lagrangians with $c=0$, either the 1^+ or the 1^- sector is a ghost sector, and this can only be removed by constraining a and b appropriately, yielding a gauge invariance which removes the ghost (with a source constraint).[20] Here, however, the vector W_μ acts in such a way as to make the ghost sector nonpropagating. *It is important that no $W_\mu{}^2$ terms occur in $L^{(2)}$.* Such terms would in general lead to ghosts in the 1^- sector. Since there are no such terms, the ghosts are removed by the Lagrange multiplier W_μ and not by gauge invariance.

IV. DISCUSSION

The signs of a and d depend on the reality properties of the metric $g_{\mu\nu}$. The constraints of Eq. (17) are obtained only for *real* $g_{\mu\nu}$. Thus the real theory is completely ghost free.[23] The Hermitian theory with $g_{\mu\nu}=g^*_{\nu\mu}$ has a ghost in the 1^+ sector; the relative sign between the graviton and the skew field is negative. However, if the source $T_{[\mu\nu]}$ is constrained;

$$T_{[\mu\nu]}\propto J_{[\mu,\nu]} \tag{19}$$

(where J_μ is a vector current not necessarily conserved), then, as in the case of $S_{[\mu,\nu]}$, the $P(1^+)$ sector vanishes and ghosts are avoided in the Hermitian theory. It has been demonstrated already that Eq. (19) is reasonable for open string sources,[24] so the Hermitian theory need not have ghosts. Such a constraint on $T_{[\mu\nu]}$ is consistent with the linearized version of Eq. (5).[18,25] For the real theory, no constraints on $T_{[\mu\nu]}$ are needed.

Analogous results for a spin-1^- field A_μ with a scalar Lagrangian multiplier ϕ may be obtained by replacing $h_{[\mu\nu]}$ with A_μ and W_μ with ϕ. The spin-0^+ ghost particle becomes nonpropagating due to the Lagrange multiplier ϕ.

We see that Einstein-Straus-type theories with a $d\mathfrak{g}^{\mu\nu}g_{[\mu\nu]}$ term have an unphysical particle spectrum which contains either ghosts ($d<0$) or tachyons ($d>0$). Since this term is needed for the equations of motion to have a Lorentz-force term, this type of theory cannot succeed as a unified theory. Only for $d=0$ are ghosts removed, due to the presence of the Lagrange multiplier W_μ, which follows from the projective transformation (3). In general relativity such a Lagrange multiplier cannot occur due to the projective invariance of the theory.[14]

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