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Quantum limits on noise in linear amplifiers

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How much noise does quantum mechanics require a linear amplifier to add to a signal it processes? An analysis of narrow-band amplifiers (single-mode input and output) yields a fundamental theorem for phase-insensitive linear amplifiers; it requires such an amplifier, in the limit of high gain, to add noise which, referred to the input, is at least as large as the half-quantum of zero-point fluctuations. For phase-sensitive linear amplifiers, which can respond differently to the two quadrature phases ("cos ωt " and "sin ωt "), the single-mode analysis yields an amplifier uncertainty principle—a lower limit on the product of the noises added to the two phases. A multimode treatment of linear amplifiers generalizes the single-mode analysis to amplifiers with nonzero bandwidth. The results for phase-insensitive amplifiers remain the same, but for phase-sensitive amplifiers there emerge bandwidth-dependent corrections to the single-mode results. Specifically, there is a bandwidth-dependent lower limit on the noise carried by one quadrature phase of a signal and a corresponding lower limit on the noise a high-gain linear amplifier must add to one quadrature phase. Particular attention is focused on developing a multimode description of signals with unequal noise in the two quadrature phases.

I. INTRODUCTION AND SUMMARY

The development of masers in the 1950's made possible amplifiers that were much quieter than other contemporary amplifiers. In particular, there emerged the possibility of constructing amplifiers with a signal-to-noise ratio of unity for a single incident photon. This possibility stirred a flurry of interest in quantum-mechanical limitations on the performance of maser amplifiers,¹⁻⁴ parametric amplifiers,^{5,6} and, more generally, all "linear amplifiers."⁷⁻¹⁰ The resulting limit is often expressed as a minimum value for the noise temperature T_n of a high-gain "linear amplifier"^{1,2,4,8:}

$$T_n \gtrsim \hbar\omega/k, \quad (1.1)$$

where $\omega/2\pi$ is the amplifier's input operating frequency. The limit (1.1) means that a "linear amplifier" must add noise to any signal it processes; the added noise must be at least the equivalent of doubling the zero-point noise associated with the input signal.

Interest in the limit (1.1) flagged in the 1960's, mainly because the issue was seen as completely resolved. (The existence of quantum limits on the

performance of "linear amplifiers" is now discussed in standard textbooks on noise¹¹ and quantum electronics.¹²) Contributing to a dwindling of interest were the difficulty of designing amplifiers that even approached quantum-limited performance and the dearth of applications that demanded such performance. Recently interest has revived¹³⁻¹⁹ because of a fortunate coincidence. The development of new amplifiers based on the dc SQUID, which are close to achieving quantum-limited sensitivity,¹⁹⁻²¹ has coincided with the realization that the detection of gravitational radiation using mechanically resonant detectors might require quantum-limited amplifiers.²² Indeed, mechanically resonant detectors might well require "amplifiers" that somehow circumvent the limit (1.1).²²

This paper returns to the question of quantum limits on noise in linear amplifiers. For the purposes of this paper an amplifier is any device that takes an input signal, carried by a collection of bosonic modes, and processes the input to produce an output signal, also carried by a (possibly different) collection of bosonic modes. A linear amplifier is an amplifier whose output signal is linearly related

to its input signal. This definition of a linear amplifier is very broad; it includes, for example, both frequency-converting amplifiers, whose output is at a frequency different from the input frequency, and phase-sensitive amplifiers, whose response depends in an essential way on the phase of the input. In contrast, previous investigations¹⁻¹⁰ have reserved the term “linear amplifier” for what are here called phase-insensitive linear amplifiers—linear devices for which a phase shift of the input produces the same or the opposite phase shift of the output. The limit (1.1) applies only to phase-insensitive linear amplifiers.

The reason for introducing this broad definition of a linear amplifier is the ability to give a unified account of the quantum limits for all such devices. An approach introduced by Haus and Mullen⁹ allows one to investigate the quantum limits on the performance of any linear amplifier, no matter how complex, without specifying any details of its operation; sufficient are the linearity assumption and the demand that the amplifier’s operation be consistent with quantum mechanics. Thus this paper seeks to review previous work, to clarify the extent of its validity, and to extend it to an analysis of all linear amplifiers.

The quantum limits obtained here have a very simple form for a narrow-band linear amplifier fed with a narrow-band input signal (bandwidth $\Delta f \ll \omega/2\pi$). In this case the input signal can be decomposed into its “ $\cos\omega t$ ” and “ $\sin\omega t$ ” phases—i.e., the input signal is proportional to $X_1 \cos\omega t + X_2 \sin\omega t = \text{Re}[(X_1 + iX_2)e^{-i\omega t}]$, where X_1 and X_2 are the amplitudes of the two quadrature phases (components of the complex amplitude). The signal information is carried by slow changes of X_1 and X_2 on a timescale $\tau \approx 1/\Delta f$. Phase-insensitive linear amplifiers treat both quadrature phases the same; the quantum limit can be stated in terms of a single added noise number A , defined as the noise the amplifier adds to the signal, the noise being referred to the input and given in units of number quanta. The precise statement of Eq. (1.1) is the *fundamental theorem for phase-insensitive linear amplifiers*:

$$A \geq \frac{1}{2} |1 - G^{-1}|, \quad (1.2)$$

where G is the amplifier’s gain in units of number of quanta. Phase-sensitive linear amplifiers can respond differently to the two quadrature phases; one must introduce gains G_1 and G_2 and added noise numbers A_1 and A_2 for both phases. The quantum limit is expressed as an *amplifier uncer-*

tainty principle:

$$A_1 A_2 \geq \frac{1}{16} |1 - (G_1 G_2)^{-1/2}|^2. \quad (1.3)$$

Specialized to the case $G_1 = G_2 = G$ and $A_1 = A_2 = \frac{1}{2}A$, Eq. (1.3) reduces to Eq. (1.2). The content of Eq. (1.3) should be clear: as a general rule, a reduction of the noise added to one quadrature phase requires an increase of the noise added to the other phase; for the special case $G_1 G_2 = 1$, however, the amplifier need not add noise to either phase.

The question this paper addresses can be put in a more general context. Motivation comes from considering the examples usually given to justify the position-momentum uncertainty relation for a particle.²³ These examples fall into two classes: either the particle’s motion is treated quantum mechanically (e.g., a single-slit experiment), in which case the uncertainty principle follows from the properties of the wave function, or some measuring apparatus is treated quantum mechanically (e.g., a Heisenberg microscope), in which case the uncertainty principle follows from the disturbance of the particle’s motion by the measuring apparatus. Are both these quantum-mechanical uncertainties present? When must a measuring apparatus add uncertainty to the quantum-mechanical uncertainties already present in the system being measured? More specifically, when must a measuring apparatus enforce an uncertainty principle that is already built into a quantum-mechanical description of the system being measured?

The analysis given here provides a complete answer for a measuring apparatus that incorporates a linear amplifier as an essential element. The answer is intimately connected to the amplifier’s gain. Confronted by a signal contaminated only by quantum noise, one uses an amplifier to increase the size of the signal without seriously degrading the signal-to-noise ratio. The noise after amplification being much larger than the minimum permitted by quantum mechanics, the signal can then be examined by crude, “classical” devices without addition of significant additional noise. Thus quantum-number gain is a crucial feature of a measurement. Indeed, the last essential quantum-mechanical stage of a measuring apparatus is a high-gain amplifier; it produces an output that we can lay our grubby, classical hands on.

The quantum mechanics of a narrow-band input signal implies an uncertainty principle for X_1 and X_2 which, if X_1 and X_2 are given in units of num-

ber of quanta, takes the form

$$(\Delta X_1)^2(\Delta X_2)^2 \geq \frac{1}{16}. \quad (1.4)$$

The quantum mechanics of a narrow-band linear amplifier implies the amplifier uncertainty principle (1.3). If there is high gain for both quadrature phases, Eq. (1.3) guarantees that the noise added by the amplifier *independently* enforces the uncertainty principle (1.4). In a real measurement that yields information about both X_1 and X_2 ($G_1, G_2 \gg 1$), the amplifier must add noise, and one cannot achieve measurement accuracies that have the minimum uncertainty product allowed by Eq. (1.4).

The paper is organized as follows. Section II introduces a general abstract formalism for describing and analyzing linear amplifiers quantum mechanically. Sections III and IV brutally seize this formalism and mercilessly beat it to death to extract from it quantum limits on the performance of linear amplifiers. Section III focuses on narrow-band linear amplifiers, analyzed using a single mode for both the input and the output. Proven are the fundamental theorem for phase-insensitive linear amplifiers and the amplifier uncertainty principle. Section IV gives a multimode description of linear amplifiers; it generalizes the single-mode analysis of Sec. III and provides bandwidth-dependent corrections to the results of Sec. III. Particular attention is paid to developing a multimode description of signals that have unequal fluctuations in the two quadrature phases; this multimode description yields a bandwidth-dependent limit on the reduction of noise in either quadrature phase. Some interesting, but peripheral, results are relegated to appendices. Section V lays the exhausted formalism to rest with a eulogy to its contributions to the quantum theory of measurement.

II. QUANTUM-MECHANICAL DESCRIPTION OF AMPLIFIERS

A. General description

An amplifier is a device that takes an input signal and produces an output signal by allowing the input signal to interact with the amplifier's internal degrees of freedom. I assume that the input and output signals are carried by sets of bosonic modes (usually modes of the electromagnetic field); the remaining degrees of freedom can be either bosonic or fermionic modes. Thus an amplifier can be thought of as a collection of interacting modes,

each labeled by a parameter α and characterized by a frequency ω_α .

I denote the set of (bosonic) input modes by \mathcal{I} and the set of (bosonic) output modes by \mathcal{O} . There is no necessary relationship between \mathcal{I} and \mathcal{O} : they can be the same set, they can have some modes in common, or they can be completely disjoint. The set consisting of all modes that are not input modes is denoted by $\overline{\mathcal{I}}$. The modes in $\overline{\mathcal{I}}$ can properly be called internal modes, because their interaction with the input signal produces the output signal. Notice that $\overline{\mathcal{I}}$ can contain some or all of the output modes.

Quantum mechanically the population of each mode is conveniently specified using its creation and annihilation operators. In the Heisenberg picture (used throughout the following), the creation and annihilation operators for mode α evolve from "in" operators $a_\alpha^\dagger, a_\alpha$ before the interaction to "out" operators $b_\alpha^\dagger, b_\alpha$ after the interaction. I assume that the "in" and "out" operators have the trivial $e^{\pm i\omega_\alpha t}$ time dependence removed.

The two sets of creation and annihilation operators obey the standard commutation and anticommutation relations for bosonic and fermionic modes. Notice that each annihilation operator can be independently multiplied by a phase factor, i.e.,

$$\tilde{a}_\alpha = e^{-i\varphi_\alpha} a_\alpha, \quad \tilde{b}_\alpha = e^{-i\theta_\alpha} b_\alpha, \quad (2.1)$$

without changing the commutation and anticommutation relations. The freedom to make the phase transformation (2.1) reflects the arbitrariness in removing the $e^{-i\omega_\alpha t}$ time dependence.

The operators $a_\alpha^\dagger a_\alpha$ and $b_\alpha^\dagger b_\alpha$ are the number operators for mode α before and after the interaction. For a bosonic mode, a_α and b_α are "before" and "after" operators for the mode's complex amplitude, measured in units of number quanta. Information about the input signal is contained in the expectation values and moments of the $a_\alpha, a_\alpha^\dagger$ for $\alpha \in \mathcal{I}$, and information about the output signal is contained in the expectation values and moments of the $b_\alpha, b_\alpha^\dagger$ for $\alpha \in \mathcal{O}$.

In general the out operators can be written as functions of the in operators:

$$\begin{aligned} b_\alpha &= \mathcal{G}_\alpha(a_\beta, a_\beta^\dagger), \\ b_\alpha^\dagger &= \mathcal{G}_\alpha^\dagger(a_\beta, a_\beta^\dagger). \end{aligned} \quad (2.2)$$

These *evolution equations* are not completely arbitrary. To be consistent with quantum mechanics, they must be derivable from a unitary transformation—i.e., they must preserve the ap-

appropriate commutation and anticommutation relations.

Completing the description of an amplifier requires specifying an “initial” state for the system—i.e., a density operator defined with respect to the in operators. One restriction is placed on the form of the initial state: the amplifier’s internal modes are assumed to be prepared in an *operating state* which is independent of the input signal. Formally, this means that the “initial” density operator ρ for the entire system is the product of a density operator $\rho_{\mathcal{I}}$ for the input modes and a density operator ρ_{op} for the internal modes (modes in $\overline{\mathcal{I}}$):

$$\rho = \rho_{\mathcal{I}} \rho_{\text{op}} . \quad (2.3)$$

The input signal determines $\rho_{\mathcal{I}}$, whereas the amplifier’s actual operating conditions determine the operating state ρ_{op} . Since I work in the Heisenberg picture, the system density operator (2.3) does not change with time.

Worth emphasizing here is the fact the evolution equations and the operating state are both essential ingredients for a complete description of an amplifier.

B. Linear amplifiers

The preceding description of an amplifier is exceedingly general and virtually useless. One way to render it useful is to specialize to the case of linear amplifiers. This drastic simplification yields a formalism that is analytically tractable but still applicable to a wide class of amplifiers.

A *linear amplifier* is one whose output signal is linearly related to its input signal, where it is now understood that the signal information is carried by the complex amplitudes of the relevant modes, rather than, for example, by the number of quanta. Thus the evolution equations for the out operators of the output modes take the form

$$b_{\alpha} = \sum_{\beta \in \mathcal{I}} (\mathcal{M}_{\alpha\beta} a_{\beta} + \mathcal{L}_{\alpha\beta} a_{\beta}^{\dagger}) + \mathcal{F}_{\alpha}, \quad \alpha \in \mathcal{O}, \quad (2.4)$$

which is a linearized version of Eqs. (2.2) for $\alpha \in \mathcal{O}$. The operators $\mathcal{M}_{\alpha\beta}$, $\mathcal{L}_{\alpha\beta}$, and \mathcal{F}_{α} depend only on the in operators of the internal modes; therefore, they commute with the a_{β} , a_{β}^{\dagger} for $\beta \in \mathcal{I}$.

Equations (2.4) embody a minimal assumption of linearity; the only relations required to be linear are the relations between the out output-mode

complex amplitudes and the in input-mode complex amplitudes. Even so, linear amplifiers are usually not strictly linear. The linearization procedure that yields Eqs. (2.4) usually requires assumptions about the size of the input signal and the nature of the operating state.

The operators \mathcal{F}_{α} are clearly responsible for the amplifier’s additive noise—i.e., noise the amplifier adds to the output signal regardless of the level of the input signal. It is the fluctuations in the \mathcal{F}_{α} —not the mean values—that are of interest, so nothing is lost by assuming $\langle \mathcal{F}_{\alpha} \rangle_{\text{op}} = 0$, where here and hereafter the subscript “op” designates an expectation value in the operating state.

Equally clear is a connection between the amplifier’s gain and the operators $\mathcal{M}_{\alpha\beta}$ and $\mathcal{L}_{\alpha\beta}$. These operators’ expectation values determine the gain, and their fluctuations produce fluctuations in the gain. Gain fluctuations introduce multiplicative noise into the output signal—i.e., noise which depends on the level of the input signal; such multiplicative noise inevitably degrades an amplifier’s performance. I am interested in limits on the performance of the best possible amplifiers, so I assume throughout the following that in the operating state fluctuations in the operators $\mathcal{M}_{\alpha\beta}$ and $\mathcal{L}_{\alpha\beta}$ are negligible. These operators can then be replaced by their expectation values $M_{\alpha\beta} \equiv \langle \mathcal{M}_{\alpha\beta} \rangle_{\text{op}}$ and $L_{\alpha\beta} \equiv \langle \mathcal{L}_{\alpha\beta} \rangle_{\text{op}}$, and Eqs. (2.4) become the basic *linear evolution equations*

$$b_{\alpha} = \sum_{\beta \in \mathcal{I}} (M_{\alpha\beta} a_{\beta} + L_{\alpha\beta} a_{\beta}^{\dagger}) + \mathcal{F}_{\alpha}, \quad \alpha \in \mathcal{O}. \quad (2.5)$$

It is now easy to apply the requirement that the in input-mode operators and the out output-mode operators obey the bosonic commutation relations ($[a_{\alpha}, a_{\beta}] = 0$ and $[a_{\alpha}, a_{\beta}^{\dagger}] = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{I}$; similarly for the out output-mode operators). The resulting *unitarity conditions* are

$$0 = \sum_{\mu \in \mathcal{I}} (M_{\alpha\mu} L_{\beta\mu} - L_{\alpha\mu} M_{\beta\mu}) + [\mathcal{F}_{\alpha}, \mathcal{F}_{\beta}], \quad (2.6a)$$

$$\delta_{\alpha\beta} = \sum_{\mu \in \mathcal{I}} (M_{\alpha\mu} M_{\beta\mu}^* - L_{\alpha\mu} L_{\beta\mu}^*) + [\mathcal{F}_{\alpha}, \mathcal{F}_{\beta}^{\dagger}], \quad (2.6b)$$

for all $\alpha, \beta \in \mathcal{O}$.

The fundamental equations needed for the analysis of quantum limits are the linear evolution equations (2.5) and the unitarity conditions (2.6). It is worth emphasizing that Eqs. (2.5) are not, in general, a complete set of evolution equations, because they give the out operators only for the output modes. Nor, for the same reasons, are Eqs.

(2.6) a complete set of unitarity conditions. Equations (2.5) and (2.6) are, however, necessary conditions for a linear amplifier, and they are sufficient for the task at hand—an investigation of quantum limits.

III. QUANTUM LIMITS FOR NARROW-BAND LINEAR AMPLIFIERS

The restriction to narrow-band amplifiers allows one to treat the input and output signals as being carried by single modes; useful, therefore, is a review of some properties of a single bosonic mode.

A. Formal considerations for a single mode

Consider an arbitrary operator R , which can be split into Hermitian real and imaginary parts:

$$R = R_1 + iR_2, \quad (3.1)$$

$$R_1 \equiv \frac{1}{2}(R + R^\dagger), \quad R_2 \equiv -\frac{1}{2}i(R - R^\dagger).$$

Some useful relations are

$$R^2 = R_1^2 - R_2^2 + i(R_1R_2 + R_2R_1), \quad (3.2a)$$

$$\frac{1}{2}(RR^\dagger + R^\dagger R) = R_1^2 + R_2^2, \quad (3.2b)$$

$$[R, R^\dagger] = -2i[R_1, R_2]. \quad (3.2c)$$

One can define the mean-square fluctuation of R by

$$\begin{aligned} |\Delta R|^2 &\equiv \frac{1}{2}\langle RR^\dagger + R^\dagger R \rangle - \langle R \rangle \langle R^\dagger \rangle \\ &= (\Delta R_1)^2 + (\Delta R_2)^2. \end{aligned} \quad (3.3)$$

Then the generalized uncertainty principle $\Delta R_1 \Delta R_2 \geq \frac{1}{2} |\langle [R_1, R_2] \rangle|$ puts a lower limit on $|\Delta R|$:

$$|\Delta R|^2 \geq \frac{1}{2} |\langle [R, R^\dagger] \rangle|. \quad (3.4)$$

The equality sign in Eq. (3.4) holds if and only if $\Delta R_1 = \Delta R_2 = \frac{1}{2} |\langle [R, R^\dagger] \rangle|^{1/2}$, which implies $\langle R^2 \rangle - \langle R \rangle^2 = 0$.

Consider now a single bosonic mode of frequency ω , with creation and annihilation operators a^\dagger, a ($[a, a^\dagger] = 1$) and number operator $a^\dagger a$. I assume that a and a^\dagger have the $e^{\mp i\omega t}$ time dependence removed. The annihilation operator is a dimensionless complex-amplitude operator for the mode. It can be split into its Hermitian real and imaginary parts:

$$a = X_1 + iX_2, \quad (3.5)$$

where

$$[X_1, X_2] = i/2. \quad (3.6)$$

The (constant) operators X_1 and X_2 are the amplitudes of the mode's quadrature phases—i.e., they give the amplitudes of the mode's “ $\cos\omega t$ ” and “ $\sin\omega t$ ” oscillations (see Fig. 1). Notice that the freedom to multiply a by a phase factor,

$$\tilde{a} = ae^{-i\varphi} \quad (3.7)$$

[a phase transformation; cf. Eq. (2.1)], is simply the freedom to make rotations in the complex-amplitude plane ($\tilde{X}_1 = X_1 \cos\varphi + X_2 \sin\varphi$; $\tilde{X}_2 = -X_1 \sin\varphi + X_2 \cos\varphi$).

The commutation relation (3.6) implies an uncertainty principle

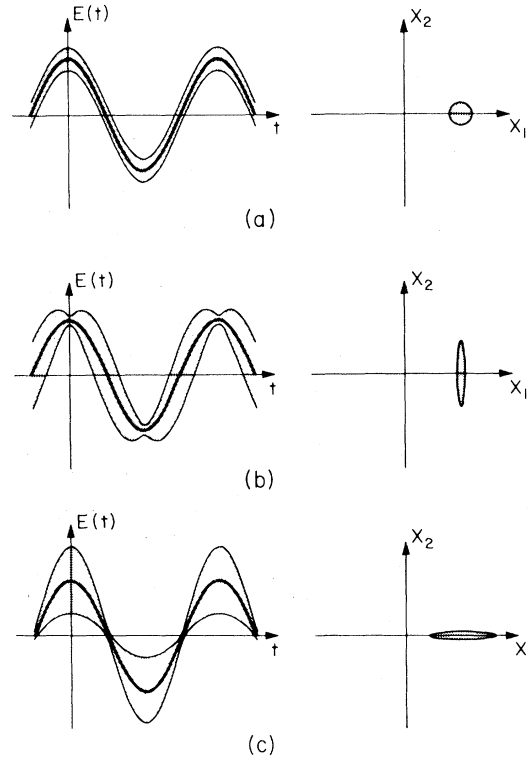


FIG. 1. Graphs of $E(t) \propto X_1 \cos\omega t + X_2 \sin\omega t$ versus time for three states of a single mode with frequency ω . To the right of each graph is a complex-amplitude plane showing the “error ellipse” of the state. All three states have $\langle X_1 \rangle \neq 0$, $\langle X_2 \rangle = 0$; ΔX_1 corresponds to amplitude fluctuations and ΔX_2 to phase fluctuations. The quantity $E(t)$ has the harmonic time dependence characteristic of the mode; for a mode of the electromagnetic field, $E(t)$ could be the electric field. In each graph the dark line gives $\langle E(t) \rangle$, and the shaded region represents the uncertainty in E . (a) A state with phase-insensitive noise; equal phase and fractional amplitude fluctuations. (b) A state with reduced amplitude fluctuations and increased phase fluctuations. (c) A state with reduced phase fluctuations and increased amplitude fluctuations.

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}, \quad (3.8)$$

and a minimum for the fluctuation in the complex amplitude

$$|\Delta a|^2 = (\Delta X_1)^2 + (\Delta X_2)^2 \geq \frac{1}{2}. \quad (3.9)$$

The minimum value of $|\Delta a|^2$ is the half-quantum of zero-point fluctuations.

A useful characterization²⁴ of the quadrature-phase fluctuations is provided by the *moment matrix*, defined by

$$\sigma_{pq} \equiv \frac{1}{2} \langle X_p X_q + X_q X_p \rangle - \langle X_p \rangle \langle X_q \rangle, \quad (3.10)$$

where here and hereafter $p, q = 1, 2$. Equations (3.2) imply

$$\langle a^2 \rangle - \langle a \rangle^2 = \sigma_{11} - \sigma_{22} + 2i\sigma_{12}, \quad (3.11a)$$

$$|\Delta a|^2 = \sigma_{11} + \sigma_{22}. \quad (3.11b)$$

A given single-mode state can be represented by an “error ellipse” in the complex-amplitude plane (see Fig. 1). The center of the ellipse lies at the expectation value $\langle X_1 + iX_2 \rangle$ of the complex amplitude, the principle axes are along the eigenvectors of σ_{pq} , and the principle radii are the square roots of the eigenvalues of σ_{pq} .

Crucial to the subsequent analysis is the notion of a single-mode state that has *phase-insensitive noise*—i.e., a state whose associated noise is distributed randomly in phase. The mode is in such a state if σ_{pq} is invariant under arbitrary phase transformations (3.7) (arbitrary rotations in the complex-amplitude plane).²⁴ This means that the fluctuations in X_1 and X_2 are equal and uncorrelated (circular error ellipse; see Fig. 1):

$$\sigma_{pq} = \frac{1}{2} |\Delta a|^2 \delta_{pq}, \quad (3.12)$$

or, equivalently, that

$$\langle a^2 \rangle - \langle a \rangle^2 = 0. \quad (3.13)$$

Examples of states with phase-insensitive noise include the coherent states²⁵ $|\mu\rangle \equiv \exp(\mu a^\dagger - \mu^* a)|0\rangle$ (μ is a complex number; $|0\rangle$ is the ground state), for which $\langle a \rangle = \mu$ and $|\Delta a|^2 = \frac{1}{2}$, and the thermal-equilibrium states, for which $\langle a \rangle = 0$ and $|\Delta a|^2 = \frac{1}{2} + (e^{\hbar\omega/kT} - 1)^{-1} = \frac{1}{2} \coth(\hbar\omega/2kT)$, where T is the temperature.

Probably the best way to think about fluctuations in X_1 and X_2 is in terms of amplitude and phase fluctuations. Suppose the mode is in a state

with $\langle X_2 \rangle = 0$ and $\langle X_1 \rangle \gg \Delta X_1, \Delta X_2$. Then the uncertainty in X_1 corresponds to amplitude fluctuations of fractional size $\Delta X_1 / \langle X_1 \rangle$, and the uncertainty in X_2 corresponds to phase fluctuations of size $\Delta X_2 / \langle X_1 \rangle$ (see Fig. 1). For a state with phase-insensitive noise, the phase fluctuations and the fractional amplitude fluctuations are equal and uncorrelated.

B. Narrow-band linear amplifiers

1. General description

Focus attention now on a narrow-band linear amplifier fed with a narrow-band input. The input and output signals are nearly sinusoidal oscillations at frequencies ω_I and ω_O , both with bandwidth $\Delta f = \Delta\omega/2\pi \ll \omega_I/2\pi, \omega_O/2\pi$. The signal information is encoded in slow changes of the complex amplitude on a timescale $\tau \approx 1/\Delta f$ (e.g., amplitude or phase modulation).

In this situation one can specialize to a single input mode ($\alpha = I$) and a single output mode ($\alpha = O$). These “modes” should be interpreted as having duration $\tau \approx 1/\Delta f$, the maximum sampling time consistent with the bandwidth. Various quantities below are characterized as having units of “number of quanta.” The number of quanta in the input mode, for example, is related to the input power per unit bandwidth \mathcal{P}_I by $\langle a_I^\dagger a_I \rangle = \mathcal{P}_I \Delta f \tau / \hbar\omega \approx \mathcal{P}_I / \hbar\omega$. Thus the phrase “measured in units of number of quanta” should be interpreted as meaning “measured in units of an equivalent flux of quanta per unit bandwidth.”

For single-mode input and output, the linear evolution equations (2.5) become

$$b_O = M a_I + L a_I^\dagger + \mathcal{F} \quad (3.14)$$

(unnecessary subscripts are omitted), and the unitarity conditions (2.6) collapse to a single equation:

$$1 = |M|^2 - |L|^2 + [\mathcal{F}, \mathcal{F}^\dagger]. \quad (3.15)$$

The complexities of a particular narrow-band linear amplifier are now buried in the single operator \mathcal{F} , which is responsible for the added noise. Fortunately, for an investigation of quantum limits, the complexities buried in \mathcal{F} need not be exhumed; the only important property of \mathcal{F} is the unitarity condition (3.15), which places a lower limit on its fluctuation

$$[|\Delta \mathcal{F}|_{\text{op}}^2 \geq \frac{1}{2} |1 - |M|^2 + |L|^2|; \text{ cf. Eq. (3.4)}].$$

The rest of this section pursues the consequences of this observation.

It is convenient to introduce complex-amplitude components for a_I and b_O :

$$a_I = X_1 + iX_2, \quad (3.16a)$$

$$b_O = Y_1 + iY_2. \quad (3.16b)$$

Associated with $X_1 + iX_2$ and $Y_1 + iY_2$ are the input and output moment matrices, denoted by σ_{pq}^I and σ_{pq}^O .

I now want to introduce the fundamental notion of phase sensitivity, and to do so, I first define what is meant by a phase-insensitive amplifier. The fundamental property of a phase-insensitive linear amplifier is that when the input signal has phase-insensitive noise, the output, both in terms of the signal and the noise, shows no phase preference; the only effect of a phase shift of the input is an equivalent phase shift of the output. This idea is formalized by defining a *phase-insensitive linear amplifier* as one that satisfies the following two conditions.

Condition (i). The expression for $\langle b_O \rangle$ is invariant under arbitrary phase transformations $\varphi = \varphi_I = \theta_O$ (*phase-preserving amplifier*) or $\varphi = \varphi_I = -\theta_O$ (*phase-conjugating amplifier*).

Condition (ii). If the input signal has phase-insensitive noise, then the output signal also has phase-insensitive noise,²⁴ i.e.,

$$\langle b_O^2 \rangle = \langle b_O \rangle^2 \text{ if } \langle a_I^2 \rangle = \langle a_I \rangle^2.$$

Condition (i) means that a phase shift of the input signal produces the same (phase-preserving) or the opposite (phase-conjugating) phase shift of the output signal [see Eq. (2.1) and recall $\langle \mathcal{F} \rangle_{\text{op}} = 0$], and condition (ii) means that the noise added by the amplifier is distributed randomly in phase [see Eq. (3.13)]. An amplifier that fails to meet conditions (i) and (ii) is called a *phase-sensitive linear amplifier*.

The consequences of these two conditions are easy to work out. Condition (i) implies that

$$\text{phase preserving: } L = 0, \quad (3.17a)$$

$$\text{phase conjugating: } M = 0;$$

and condition (ii) implies that²⁴

$$\langle \mathcal{F}^2 \rangle_{\text{op}} = 0. \quad (3.17b)$$

Keep in mind that Eqs. (3.17) are constraints on both the evolution equations and the operating state. An amplifier that is phase insensitive when

prepared in a particular operating state might be phase sensitive when prepared differently.

The output of a phase-sensitive linear amplifier depends in an essential way on the phase of the input. In particular, its response picks out a preferred set of input quadrature phases. A rotation in the input complex-amplitude plane can choose the preferred X_1 and X_2 , and a rotation in the output complex-amplitude plane can choose $Y_1 + iY_2$ so that Y_1 responds to the preferred X_1 and Y_2 to the preferred X_2 . Specifically, appropriate phase transformations φ_I and θ_O [Eq. (2.1)] can always bring Eq. (3.14) into a *preferred form* where M and L are real and positive. The evolution equation (3.14) then splits into the following equations:

$$Y_1 = (M + L)X_1 + \mathcal{F}_1, \quad (3.18a)$$

$$Y_2 = (M - L)X_2 + \mathcal{F}_2,$$

where

$$\mathcal{F}_1 \equiv \frac{1}{2}(\mathcal{F} + \mathcal{F}^\dagger), \quad \mathcal{F}_2 \equiv -\frac{1}{2}i(\mathcal{F} - \mathcal{F}^\dagger). \quad (3.18b)$$

One can now define gains for the preferred quadrature phases,

$$G_1 \equiv (M + L)^2, \quad G_2 \equiv (M - L)^2, \quad (3.19)$$

and a mean gain

$$G \equiv \frac{1}{2}(G_1 + G_2) = |M|^2 + |L|^2, \quad (3.20)$$

all gains being measured in units of number of quanta [(power gain) $_p = (\omega_O/\omega_I)G_p$]. The gain of a phase-insensitive amplifier is independent of phase ($G = G_1 = G_2$).

2. Characterization of noise

When the equations are written in preferred form, the uncertainties in the output quadrature phases have the simple form

$$(\Delta Y_p)^2 = G_p(\Delta X_p)^2 + (\Delta \mathcal{F}_p)_{\text{op}}^2, \quad (3.21)$$

the first term on the right being the amplified input noise and the second term the noise added by the amplifier.

Only one number is needed to characterize the noise added by a phase-insensitive amplifier, because Eq. (3.17b) implies $(\Delta \mathcal{F}_1)_{\text{op}} = (\Delta \mathcal{F}_2)_{\text{op}}$. For an arbitrary input signal, the total mean-square fluctuation at the output of a phase-insensitive amplifier is given by

$$|\Delta b_O|^2 = G |\Delta a_I|^2 + |\Delta \mathcal{F}|_{\text{op}}^2. \quad (3.22)$$

The added noise is conveniently characterized by an *added noise number*

$$A \equiv |\Delta \mathcal{F}|_{\text{op}}^2 / G, \quad (3.23)$$

which gives the added noise referred to the input and measured in units of number of quanta.

For a phase-sensitive amplifier one defines added noise numbers for both preferred phases:

$$A_p \equiv (\Delta \mathcal{F}_p)_{\text{op}}^2 / G_p \quad (3.24)$$

[Eqs. (3.21)]. More generally, one defines an *added noise moment matrix* σ_{pq}^A by

$$\frac{1}{2} \langle \mathcal{F}_p \mathcal{F}_q + \mathcal{F}_q \mathcal{F}_p \rangle_{\text{op}} = (G_p G_q)^{1/2} e^{i(\gamma_p - \gamma_q)} \sigma_{pq}^A, \quad (3.25)$$

where $\gamma_1=0$, and $\gamma_2=0$ if $|M| > |L|$ or $\gamma_2=\pi$ if $|M| < |L|$ ($\sigma_{pp}^A = A_p$). When the equations are written in preferred form, the input and output moment matrices are related by

$$\sigma_{pq}^O = (G_p G_q)^{1/2} e^{i(\gamma_p - \gamma_q)} (\sigma_{pq}^I + \sigma_{pq}^A) \quad (3.26)$$

[Eqs. (3.18)]. A phase-insensitive amplifier can be compactly defined by the requirements $G = G_1 = G_2$ (phase-insensitive gain) and $\sigma_{pq}^A = \frac{1}{2} A \delta_{pq}$ [random-phase added noise; cf. Eqs. (3.17)].

3. Examples

The term “linear amplifier” is usually reserved for what are here called phase-preserving linear amplifiers. Maser amplifiers¹¹ and dc SQUID amplifiers^{20,21} are examples of phase-preserving amplifiers that can be made to operate near the quantum limit. A phase-preserving amplifier produces an amplified replica of a narrow-band input signal, which preserves precisely the input phase information.

The types of linear amplifiers distinguished in this subsection can perhaps best be illustrated by the simple, but protean, example of a parametric amplifier. Stripped to its essentials, a paramp consists of two modes, conventionally called the “signal” ($\alpha=s$, frequency ω_s) and the “idler” ($\alpha=i$, frequency ω_i), which are coupled (by some non-linearity) via a “pump” at frequency $\omega_p = \omega_s + \omega_i$. The pump is actually a quantum-mechanical mode, but it is assumed to be excited in a large-amplitude coherent state, so that it can be regarded as classical and so that it remains unaffected by its cou-

pling to the signal and the idler. The pump then produces a classically modulated interaction at frequency ω_p between the signal and the idler. The resulting complete set of evolution equations can be put in the form^{5,6}

$$b_s = a_s \cosh r + a_i^\dagger \sinh r, \quad (3.27a)$$

$$b_i = a_s^\dagger \sinh r + a_i \cosh r, \quad (3.27b)$$

where r is a real constant determined by the strength and duration of the interaction.

When a paramp is operated in the standard way, the signal mode carries both the input and output signals. Then Eq. (3.27a) is the relevant evolution equation, and the paramp is a phase-preserving amplifier with $G = \cosh^2 r$ and $\mathcal{F} = a_i^\dagger \sinh r$. The paramp could, however, be operated with the signal mode carrying the input signal and the idler mode carrying the output signal. Then Eq. (3.27b) would be the relevant evolution equation, and the paramp would be a phase-conjugating amplifier with $G = \sinh^2 r$ and $\mathcal{F} = a_i \cosh r$. In both cases the idler is the one internal mode, and the added noise can be traced to the idler’s initial mean-square fluctuations $|\Delta a_i|_{\text{op}}^2$.

A parametric amplifier is not usually operated as a phase conjugator; a formally equivalent device that is so operated is a degenerate four-wave mixer.²⁶ In a four-wave mixer the modes of interest are two counterpropagating electromagnetic waves, an “incident” wave (input mode) and a “reflected” wave (output mode), both with frequency ω . These two waves are coupled in a nonlinear medium to two counterpropagating “pump” waves of frequency ω , assumed to be classical. The evolution equations for the mixer have the same form as Eqs. (3.27) (Ref. 27); one simply identifies the signal mode as the incident wave and the idler mode as the reflected wave. Four-wave mixers have recently attracted a great deal of attention precisely because of their ability to produce a phase-conjugated reflected wave.

An instructive special case of a parametric amplifier is a degenerate parametric amplifier, which results when the signal and idler coincide ($\omega_s = \omega_i = \frac{1}{2} \omega_p$). The one mode of a degenerate paramp can be regarded as a simple harmonic oscillator, whose frequency is modulated at twice its fiducial frequency. The one evolution equation is

$$b = a \cosh r + a^\dagger \sinh r \quad (3.28)$$

[cf. Eqs. (3.27)]. Written in terms of the components of the complex amplitude, Eq. (3.28) be-

comes

$$Y_1 = e^r X_1, \quad Y_2 = e^{-r} X_2, \quad (3.29)$$

revealing that a degenerate paramp is a phase-sensitive amplifier with $G_1 = G_2^{-1} = e^{2r}$ and $\mathcal{F} = 0$. Because $\mathcal{F} = 0$, an ideal degenerate paramp is noiseless.

C. Fundamental theorem

1. The theorem

In hand now are the tools necessary to state and prove the *fundamental theorem for phase-insensitive linear amplifiers*: The added noise number A for such an amplifier satisfies the inequality

$$A \geq \frac{1}{2} |1 \mp G^{-1}|, \quad (3.30)$$

where the upper (lower) sign holds for phase-preserving (phase-conjugating) amplifiers. The fundamental theorem is a trivial consequence of Eqs. (3.23), (3.4), (3.15), (3.20), and (3.17a), the crucial equation being the unitarity condition (3.15). Rewritten in terms of the output noise, the fundamental theorem becomes

$$|\Delta b_O|^2 = G(|\Delta a_I|^2 + A) \geq \frac{1}{2}G + \frac{1}{2}|G \mp 1|, \quad (3.31)$$

where the first and second terms are due to the input and added noises, respectively.

$$\frac{1}{2} \coth[\hbar\omega_I/2k(T + T_n)] \equiv |\Delta b_O|^2/G = A + \frac{1}{2} \coth(\hbar\omega_I/2kT). \quad (3.33)$$

Both F and T_n depend on the amount of input noise. To get quantities that characterize the amplifier's noise only, one can specialize to the case of minimum input noise ($|\Delta a_I|^2 = \frac{1}{2}$; $T=0$). The fundamental theorem (3.30), when written in terms of the resulting noise figure $F_0 = 1 + 2A$ and noise temperature $T_{n0} = \hbar\omega_I/k \ln(1 + A^{-1})$, becomes (for $G \geq 1$)

$$F_0 \geq 2 \mp G^{-1} \rightarrow 2, \quad (3.34a)$$

$$T_{n0} \geq \frac{\hbar\omega_I}{k} \left[\ln \left[\frac{3 \mp G^{-1}}{1 \mp G^{-1}} \right] \right]^{-1} \rightarrow \frac{\hbar\omega_I}{k \ln 3}, \quad (3.34b)$$

where the figures on the right are the limits as $G \rightarrow \infty$. Notice that if $kT \gg \hbar\omega_I$, $T_n = (\hbar\omega_I/k)A$

The fundamental theorem implies that a high-gain phase-insensitive amplifier must add noise to any signal it processes, the added noise being at least the equivalent of an additional half-quantum of noise at the input. In contrast, a passive ($G = 1$) phase-preserving device need not add any noise. For a phase-preserving attenuator ($G < 1$), the fundamental theorem guarantees that the added noise is large enough to ensure $|\Delta b_O|^2 \geq \frac{1}{2}$.

The added noise number is a particularly convenient way of characterizing the noise added by a phase-insensitive amplifier that operates near the quantum limit. It is independent of how much noise is carried by the input signal; indeed, it is independent of whether the input signal has phase-insensitive noise; it is not, however, conventional. Instead, one usually characterizes an amplifier's performance by a noise figure or a noise temperature. To define these quantities, one assumes phase-insensitive input noise, and one associates with the input noise a temperature T defined by $|\Delta a_I|^2 = \frac{1}{2} \coth(\hbar\omega_I/2kT)$. The *noise figure* F is the ratio of the input (power) signal-to-noise ratio to the output (power) signal-to-noise ratio²⁸:

$$F \equiv |\Delta b_O|^2/G |\Delta a_I|^2 = 1 + (A / |\Delta a_I|^2); \quad (3.32)$$

and the *noise temperature* T_n is the increase in input temperature required to account for all the output noise referred to the input²⁸:

satisfies the inequality $T_n \geq (\hbar\omega_I/2k)(1 \mp G^{-1})$ for $G \geq 1$.

Previous investigations^{2,5,8} that give an exact limit on T_{n0} have gotten the result $T_{n0} \geq \hbar\omega_I/k \ln 2$ for $G \gg 1$, instead of Eq. (3.34b). In Weber's analysis² of masers and in the analysis of parametric amplifiers by Louisell, Yariv, and Siegmann,⁵ this discrepancy is really a matter of convention; it arises from defining T_{n0} by $(e^{\hbar\omega_I/kT_{n0}} - 1)^{-1} = A + \frac{1}{2}$, instead of $(e^{\hbar\omega_I/kT_{n0}} - 1)^{-1} = A$ —i.e., the half-quantum of zero-point noise is left out of the left-hand side of Eq. (3.33). The discrepancy is ultimately due to the fact that noise temperature is not a very useful quantity when $kT/\hbar\omega_I \lesssim 1$. These difficulties can be avoided by sticking to A as the way of characterizing the performance of amplifiers that operate

near the quantum limit.

In the case of Heffner's general analysis⁸ of (phase-preserving) linear amplifiers, the discrepancy is more serious. His use of the relation $(\Delta f)\tau = \frac{1}{2}$, instead of $(\Delta f)\tau = 1$, leads to the incorrect conclusion that, for $G \gg 1$, the added noise must be at least the equivalent of a full quantum at the input, instead of just a half-quantum. Translated into a noise temperature, this error accounts for the discrepancy between $\ln 3$ and $\ln 2$.

2. Review of past work and discussion

Previous analyses of quantum noise in phase-insensitive linear amplifiers fall into two classes. In the first class are analyses^{9,10,16} that, like the one here, derive the fundamental theorem from a set of linear evolution equations and the corresponding unitarity conditions. Indeed, the analysis here is patterned after the pioneering work of Haus and Mullen.⁹ Takahasi's later analysis¹⁰ is similar to, but more restrictive than that of Haus and Mullen.

The approach taken here owes much to Haus and Mullen, but there are improvements in delineating the assumptions required to prove the fundamental theorem. Haus and Mullen assume a linear relation between the in and out creation and annihilation operators for all modes, a condition more restrictive than that embodied in Eqs. (2.5) or Eq. (3.14). In addition, Haus and Mullen suggest that the fundamental theorem relies on an assumption of time independence—i.e., that the Hamiltonian for the amplifier must have no explicit time dependence. This assumption, which is violated by all phase-insensitive amplifiers except those that are both phase and frequency preserving, is replaced here by the less stringent requirement of phase insensitivity. Finally, a distinction is drawn here between phase-preserving and phase-conjugating amplifiers, with the result that one obtains different limits for the two cases.

The second class consists of analyses^{7,8} that attempt to obtain a quantum limit using only the X_1 - X_2 uncertainty principle (3.8). The most detailed of these analyses, due to Heffner,⁸ is widely referred to as a general proof of the fundamental theorem. Heffner's argument, rewritten in X_1 - X_2 language, runs as follows: assume the input signal is noiseless; show that if the amplifier adds no noise, then an ideal measurement of the output

complex amplitude allows one to infer X_1 and X_2 with (equal) accuracies that violate the uncertainty principle; conclude that a high-gain phase-insensitive linear amplifier must add noise that is at least equivalent to a half-quantum at the input. This argument works precisely and only because it neglects the input noise required by quantum mechanics, thereby forcing the amplifier to supply noise that is equivalent to the neglected input noise. If X_1 and X_2 are allowed to have uncertainties that satisfy the uncertainty principle (3.8), then this argument yields *no* information about the amplifier noise.²⁹ Heffner's argument begs the question: why must a phase-insensitive linear amplifier add noise to an input signal, when the noise associated with the input signal is already sufficient to satisfy the uncertainty principle?

How does the analysis here succeed where Heffner's argument fails? Indeed, there is, at first sight, a paradox. The fundamental theorem is a consequence of the unitarity condition (3.15), which follows from the commutation relation $[b_O, b_O^\dagger] = 1$. How does this commutation relation, which by itself implies only $|\Delta b_O|^2 \geq \frac{1}{2}$, manage to imply in the analysis here the much stronger constraint $|\Delta b_O|^2 \geq G \mp \frac{1}{2}$ for $G \geq 1$? The answer is hidden in the innocuous, but fundamental, assumption (2.3) that the initial state of the input mode and the operating state of the internal modes are independent and uncorrelated. There *are* states of the entire system for which $|\Delta b_O|^2 = \frac{1}{2}$, but these states are forbidden because they require that the input noise and the initial internal-mode noise be correlated.

This observation penetrates to the heart of the question of quantum noise in linear amplifiers. In addition to the input mode, a high-gain phase-insensitive linear amplifier must have one or more internal modes [Eqs. (3.15) and (3.17a) forbid $\mathcal{F} = 0$ when $G > 1$], whose interaction with the input signal produces the amplified output in the output mode. The internal modes must have at least the quantum-mechanical zero-point fluctuations, and these fluctuations are amplified along with the input signal to produce a noise $\frac{1}{2}(G \mp 1)$ ($G \geq 1$) at the output. Since the amplified internal-mode fluctuations and the amplified input fluctuations are uncorrelated, they add in quadrature to produce the total output noise (3.31).

This situation is particularly clear for a parametric amplifier [Eqs. (3.27)], whose only internal mode is the idler. The idler's irreducible zero-point fluctuations, which appear amplified at the

output, are responsible for the lower limit on the added noise.

D. Amplifier uncertainty principle

For phase-sensitive amplifiers the fundamental theorem (3.30) is replaced by a more general *amplifier uncertainty principle*, which limits the product of the added noise numbers for the preferred quadrature phases:

$$(A_1 A_2)^{1/2} \geq \frac{1}{4} |1 \mp (G_1 G_2)^{-1/2}|, \quad (3.35)$$

where the upper (lower) sign holds if $|M| \geq |L|$ ($|M| \leq |L|$). The amplifier uncertainty principle follows trivially from Eqs. (3.24), (3.2c), (3.15), and (3.19), the crucial relation again being the unitarity condition (3.15). Notice the similarity between the amplifier uncertainty principle and the ordinary uncertainty principle (3.8); notice also that for phase-insensitive amplifiers Eq. (3.35) reduces to Eq. (3.30).

The amplifier uncertainty principle implies that, as a general rule, a reduction in the noise added to one quadrature phase requires an increase in the noise added to the other phase. That this can be useful should be fairly clear. Consider, for example, an amplifier such that $(G_1 G_2)^{1/2} \gg 1$. One can tailor the input so that it has reduced noise in one quadrature phase ($\Delta X_1 \ll \frac{1}{2}$) and so that the signal information is carried by changes in the amplitude of that phase (e.g., amplitude or phase modulation). One can then design the amplifier so that it amplifies the phase of interest ($G_1 \gg 1$) and so that it has reduced noise for that phase ($A_1 \ll \frac{1}{4}$). Using phase-sensitive detection, one can then read out the amplified signal in the chosen phase with accuracy far better than is possible using phase-insensitive techniques. Quantum mechanics does not hand out this improvement for nothing; the price paid is increased noise in the other phase ($\Delta X_2 \gg \frac{1}{2}$, $A_2 \gg \frac{1}{4}$).

A version of this idea, referred to as a "back-action-evading" measurement technique (or a "quantum nondemolition" technique), has been suggested to improve the potential sensitivity of resonant-mass gravitational-wave detectors.^{15,17,18} Back-action evasion can be described as follows: the gravitational-wave detector is a mechanical oscillator; a transducer is coupled to the oscillator so that it responds strongly to the oscillator's X_1 and weakly to its X_2 ; the transducer's output is delivered to an ordinary high-gain phase-preserving

linear amplifier. If the mechanical oscillator is regarded as the input mode, this entire system becomes a phase-sensitive linear amplifier with $G_1 \gg G_2$ and $A_1 \ll \frac{1}{4}$. Use of the back-action-evading technique permits a measurement of X_1 with accuracy better than could be obtained were the transducer a phase-preserving device.

For the special case $|M|^2 - |L|^2 = 1$, which implies $G_1 G_2 = 1$, an amplifier need not add noise to either phase. In this case the price paid is that only one phase is amplified. An example of such an amplifier is a degenerate parametric amplifier [Eqs. (3.28) and (3.29)].

Although the precise statement of the amplifier uncertainty principle apparently has not been obtained previously, it has been realized for some time that it is possible to construct phase-sensitive linear amplifiers that add no noise to one quadrature phase. Haus and Townes³⁰ and Oliver³¹ pointed out the possibility of building such amplifiers, and Takahasi¹⁰ considered the specific example of a degenerate paramp. In each of these cases, however, the input signal was considered to have phase-insensitive noise, so only part of the potential noise reduction was realized. Yuen³² has suggested that an ideal two-photon laser would be a noiseless phase-sensitive amplifier (formally identical to a degenerate paramp). He seems to conclude, however, that one could amplify both quadrature phases, without adding noise to either, by first splitting the input signal into its two quadrature phases and then amplifying the two phases separately with noiseless amplifiers. This possibility is ruled out by the amplifier uncertainty principle; physically, the reason is that splitting the input signal into its two phases introduces noise into both, which is then amplified by the two amplifiers.

IV. MULTIMODE DESCRIPTION OF LINEAR AMPLIFIERS

This section generalizes the preceding analysis by giving a multimode treatment of linear amplifiers. Allowing the input and output signals to have many modes opens a Pandora's box—an enormous range of possibilities, even when one considers only linear relationships between the input and output signals. To get a handle on this situation, I restrict attention here to the multimode generalizations of the amplifiers considered in Sec. III. The idea is to find the corrections to the single-mode

analysis which result from the nonzero bandwidth of real signals and real amplifiers. In this section the exposition is slashed to the bone, except where new results emerge from the multimode analysis.

A. General description

Throughout this section each input mode and each output mode is denoted by its frequency. Thus the input- and output-mode sets \mathcal{I} and \mathcal{O} are sets of positive frequencies; for simplicity, I assume that for each frequency in \mathcal{I} (\mathcal{O}) there is precisely one input (output) mode. The input-mode creation and annihilation operators are assumed to obey continuum commutation relations:

$$[a(\omega), a(\omega')] = 0, \quad (4.1a)$$

$$[a(\omega), a^\dagger(\omega')] = 2\pi\delta(\omega - \omega'), \quad (4.1b)$$

where $\omega, \omega' \in \mathcal{I}$; the same commutation relations are assumed to hold for the output-mode operators $b(\omega), b^\dagger(\omega), \omega \in \mathcal{O}$. The commutation relations (4.1b) imply that $a^\dagger(\omega)a(\omega)d\omega/2\pi$ is the number of quanta in the input signal within the bandwidth $d\omega/2\pi$ —i.e., $a^\dagger(\omega)a(\omega)$ is the number of quanta per hertz.

From the operators $a(\omega), a^\dagger(\omega), \omega \in \mathcal{I}$, one constructs a Hermitian *input signal operator*

$$\begin{aligned} \phi(t) \equiv & \int_{\mathcal{I}} d\omega (\hbar\omega/8\pi^2)^{1/2} \\ & \times [a(\omega)e^{-i\omega t} + a^\dagger(\omega)e^{i\omega t}], \end{aligned} \quad (4.2)$$

the \mathcal{I} indicating that the integration runs over the input-mode frequencies. Similarly, from the operators $b(\omega), b^\dagger(\omega), \omega \in \mathcal{O}$, one constructs an *output signal operator*

$$\begin{aligned} \psi(t) \equiv & \int_{\mathcal{O}} d\omega (\hbar\omega/8\pi^2)^{1/2} \\ & \times [b(\omega)e^{-i\omega t} + b^\dagger(\omega)e^{i\omega t}]. \end{aligned} \quad (4.3)$$

The signal operators obey the commutation relations

$$[\phi(t), \phi(t')] = -\frac{i}{2\pi} \int_{\mathcal{I}} d\omega \hbar\omega \sin\omega(t-t'), \quad (4.4a)$$

$$[\psi(t), \psi(t')] = -\frac{i}{2\pi} \int_{\mathcal{O}} d\omega \hbar\omega \sin\omega(t-t'). \quad (4.4b)$$

It should be understood that even if an input mode and an output mode are denoted by the same frequency, they need not be the same mode.

The linear evolution equations (2.5) have an ob-

vious analog for this case of continuous modes:

$$\begin{aligned} b(\omega) = & \int_{\mathcal{I}} d\omega' [M(\omega, \omega')a(\omega') \\ & + L(\omega, \omega')a^\dagger(\omega')] \\ & + \mathcal{F}(\omega), \quad \omega \in \mathcal{O}. \end{aligned} \quad (4.5)$$

The unitarity conditions (2.6) have an equally obvious analog, not written here because the general form is not needed in the subsequent analysis.

In a real situation the input signal operator (similar considerations apply to the output signal operator) is derived from the operator of some field (e.g., the electric field operator). The input signal operator is constructed only from those modes of the field that actually contribute to the input signal. The other modes of the field are not neglected; they are included among the internal modes, and their effects, if any, appear in the operators $\mathcal{F}(\omega)$. On the other hand, the input signal operator (4.2) certainly does not have the most general possible form. One can easily imagine signal operators with more than one mode for each input frequency. This generalization, however, adds nothing to an understanding of quantum limits, so I stick to the simpler form (4.2) here.

B. Multimode description of phase-sensitive amplifiers

1. Signals with time-stationary noise

I now review the concept of a signal with time-stationary noise. The input signal operator is used as an example; the same considerations apply to the output signal operator.

It is convenient to introduce the positive- and negative-frequency parts of the signal operator:

$$\begin{aligned} \phi^{(+)}(t) \equiv & \int_{\mathcal{I}} d\omega (\hbar\omega/8\pi^2)^{1/2} a(\omega)e^{-i\omega t}, \\ & \end{aligned} \quad (4.6a)$$

$$\begin{aligned} \phi^{(-)}(t) \equiv & \int_{\mathcal{I}} d\omega (\hbar\omega/8\pi^2)^{1/2} a^\dagger(\omega)e^{+i\omega t}, \\ & \end{aligned} \quad (4.6b)$$

$$\phi = \phi^{(+)} + \phi^{(-)}, \quad \phi^{(+)\dagger} = \phi^{(-)}, \quad (4.7)$$

which have the commutators

$$\begin{aligned} [\phi^{(+)}(t), \phi^{(+)}(t')] = [\phi^{(-)}(t), \phi^{(-)}(t')] = 0, \\ & \end{aligned} \quad (4.8a)$$

$$\begin{aligned} [\phi^{(+)}(t), \phi^{(-)}(t')] = \int_{\mathcal{I}} (d\omega/2\pi) \frac{1}{2} \hbar\omega \\ \times e^{-i\omega(t-t')}. \end{aligned} \quad (4.8b)$$

The factor $(\hbar\omega/8\pi^2)^{1/2}$ in Eq. (4.2) is chosen so that the instantaneous power carried by the signal is $:\phi^2:=2\phi^{(-)}\phi^{(+)}+\phi^{(-)2}+\phi^{(+)2}$, where the double colons signify normal ordering. If \mathcal{S} is bounded away from zero frequency, then the last two terms in the instantaneous power average to zero over a sufficiently long time, so that one obtains an operator for the mean power,

$$P(t)=2\phi^{(-)}(t)\phi^{(+)}(t). \quad (4.9)$$

The total signal energy is $\int_{-\infty}^{\infty} P(t)dt = \int_{\mathcal{S}} (d\omega/2\pi)\hbar\omega a^\dagger(\omega)a(\omega)$.

The signal is in a state with *time-stationary noise* if the following conditions are met:

$$\langle a(\omega)a(\omega') \rangle - \langle a(\omega) \rangle \langle a(\omega') \rangle = 0, \quad (4.10a)$$

$$\begin{aligned} \frac{1}{2} \langle a(\omega)a^\dagger(\omega') + a^\dagger(\omega')a(\omega) \rangle \\ - \langle a(\omega) \rangle \langle a^\dagger(\omega') \rangle = 2\pi S(\omega)\delta(\omega - \omega'), \end{aligned} \quad (4.10b)$$

for all $\omega, \omega' \in \mathcal{S}$. These conditions constitute an obvious generalization of the concept of phase-insensitive noise for a single mode [cf. Eq. (3.13)]; each mode has random-phase noise, and the noise in different modes is uncorrelated. The (real) quantity $S(\omega)$, defined by Eq. (4.10b), is a dimensionless *spectral density* for the noise in ϕ (analog of $|\Delta a|^2$ for a single mode). One easily shows that

$$S(\omega) \geq \frac{1}{2} \quad (4.11)$$

[Eqs. (4.10b), (4.1b), (3.3), and (3.4)], the lower limit being the contribution of zero-point fluctuations [cf. Eq. (3.9)].

The significance of $S(\omega)$ is revealed by its relation to the symmetrized two-point correlation function, which is defined by

$$\begin{aligned} K(u) \equiv \frac{1}{2} \langle \phi(t)\phi(t+u) + \phi(t+u)\phi(t) \rangle \\ - \langle \phi(t) \rangle \langle \phi(t+u) \rangle \end{aligned} \quad (4.12)$$

Condition (i) implies that [recall $\langle \mathcal{F}(\omega) \rangle_{\text{op}} = 0$]

$$\text{phase preserving: } M(\omega, \omega') = |f'(\omega)|^{1/2} M(\omega) \delta(\omega' - \bar{\omega}) \text{ and } L(\omega, \omega') = 0, \quad (4.16a)$$

$$b(\omega) = |f'(\omega)|^{1/2} M(\omega) a(\bar{\omega}) + \mathcal{F}(\omega),$$

$$\text{phase conjugating: } L(\omega, \omega') = |f'(\omega)|^{1/2} L(\omega) \delta(\omega' - \bar{\omega}) \text{ and } M(\omega, \omega') = 0, \quad (4.16b)$$

$$b(\omega) = |f'(\omega)|^{1/2} L(\omega) a^\dagger(\bar{\omega}) + \mathcal{F}(\omega),$$

[$K(u)$ is real and $K(-u) = K(u)$] and which satisfies

$$K(u) = \int_{\mathcal{S}} (d\omega/2\pi)\hbar\omega S(\omega) \cos\omega u, \quad (4.13)$$

$$K(0) = (\Delta\phi)^2 = \int_{\mathcal{S}} (d\omega/2\pi)\hbar\omega S(\omega). \quad (4.14)$$

Since $:\phi^2:$ is the instantaneous power, Eq. (4.14) shows that $S(\omega)d\omega/2\pi$ is the fluctuation in ϕ within the bandwidth $d\omega/2\pi$, given as an equivalent flux of quanta. Thus $S(\omega)$ characterizes the fluctuations in ϕ by giving an equivalent flux of quanta per hertz (units of number of quanta).

For a signal with time-stationary noise, the expected power is

$$\begin{aligned} \langle P \rangle = 2 |\langle \phi^{(+)} \rangle|^2 \\ + \int_{\mathcal{S}} (d\omega/2\pi)\hbar\omega [S(\omega) - \frac{1}{2}]. \end{aligned} \quad (4.15)$$

The zero-point fluctuations, which are a real source of noise if one is interested in measurements of ϕ , do not contribute to the expected power.

2. Phase-insensitive linear amplifiers

A *phase-insensitive linear amplifier* satisfies the following two conditions, which are an obvious generalization of the conditions for the narrow-band case.

Condition (i). Each output mode with frequency $\omega \in \mathcal{O}$ is coupled to precisely one input mode with frequency $\bar{\omega} = f(\omega) \in \mathcal{S}$ [$f(\omega)$ is a one-to-one map of \mathcal{O} onto \mathcal{S}], and the expression for $\langle b(\omega) \rangle$ is invariant under arbitrary phase transformations $\varphi = \varphi(\bar{\omega}) = \theta(\omega)$ (*phase-preserving amplifier*) or $\varphi = \varphi(\bar{\omega}) = -\theta(\omega)$ (*phase-conjugating amplifier*).

Condition (ii). If the input signal has time-stationary noise, then the output signal also has time-stationary noise.

where $\bar{\omega} = f(\omega)$ [cf. Eq. (3.17a)]. The gain, in units of number of quanta, at input frequency $\bar{\omega} = f(\omega)$ is

$$G(\bar{\omega}) \equiv \left| \frac{|\langle b(\omega) \rangle|^2 d\omega / 2\pi}{|\langle a(\bar{\omega}) \rangle|^2 d\bar{\omega} / 2\pi} \right| = \begin{cases} |M(\omega)|^2, & \text{phase preserving,} \\ |L(\omega)|^2, & \text{phase conjugating.} \end{cases} \quad (4.17)$$

The unitarity conditions corresponding to the evolution equations (4.16) are

$$[\mathcal{F}(\omega), \mathcal{F}(\omega')] = 0, \quad (4.18a)$$

$$[\mathcal{F}(\omega), \mathcal{F}^\dagger(\omega')] = 2\pi\delta(\omega - \omega')[1 \mp G(\bar{\omega})], \quad (4.18b)$$

for all $\omega, \omega' \in \mathcal{O}$, where the upper (lower) sign holds for phase-preserving (phase-conjugating) amplifiers [cf. Eqs. (3.15), (3.17a), and (3.20); also cf. Eqs. (4.1)].

Condition (ii) places constraints on the moments of the operators $\mathcal{F}(\omega)$, $\mathcal{F}^\dagger(\omega)$ in the operating state:

$$\langle \mathcal{F}(\omega)\mathcal{F}(\omega') \rangle_{\text{op}} = 0, \quad (4.19a)$$

$$\frac{1}{2} \langle \mathcal{F}(\omega)\mathcal{F}^\dagger(\omega') + \mathcal{F}^\dagger(\omega')\mathcal{F}(\omega) \rangle_{\text{op}} = 2\pi G(\bar{\omega}) S^A(\bar{\omega}) \delta(\omega - \omega'), \quad (4.19b)$$

for all $\omega, \omega' \in \mathcal{O}$ [cf. Eq. (3.17b)]. Equations (4.19) guarantee that the noise added by the amplifier is time stationary [cf. Eqs. (4.10)].

The quantity $S^A(\bar{\omega})$, defined by Eq. (4.19b) as a function of the input frequency $\bar{\omega} = f(\omega)$, is the *added noise spectral density* [analog of added noise number; cf. Eq. (3.23)]; it is the spectral density of the noise added by the amplifier, referred to the input and given in units of number of quanta. If the input signal has time-stationary noise, then the output spectral density is given by

$$S^O(\omega) = G(\bar{\omega}) [S^I(\bar{\omega}) + S^A(\bar{\omega})], \quad \bar{\omega} = f(\omega) \quad (4.20)$$

[cf. Eqs. (3.22) and (3.23)], where the superscripts I and O designate the spectral densities of the input and output signals.

The multimode description of a phase-insensitive linear amplifier amounts to saying that each pairing of an output mode with an input mode is phase insensitive in the narrow-band sense and that all such pairings are independent. Therefore, it should not be surprising that the unitarity conditions (4.18) imply the following *fundamental theorem for phase-insensitive linear amplifiers*:

$$S^A(\bar{\omega}) \geq \frac{1}{2} \left| 1 \mp \frac{1}{G(\bar{\omega})} \right|, \quad (4.21)$$

where the upper (lower) sign holds for phase-preserving (phase-conjugating) amplifiers [cf. Eq. (3.30)].

The phase transformation $\varphi(\omega) = \omega\tau$ and $\theta(\omega) = \omega\tau$ [Eqs. (2.1)] corresponds to a time translation τ of the input and output signals. The conditions that the general expressions for $\langle b(\omega) \rangle$ [Eqs. (4.5)] be invariant under this time translation are $M(\omega, \omega') = M(\omega)\delta(\omega - \omega')$ and $L(\omega, \omega') = 0$. Such amplifiers are called *time stationary* (phase and frequency preserving). All other linear amplifiers must have some sort of internal clock, because the expression for $\langle b(\omega) \rangle$ is aware of one's choice for the zero of time.

C. Multimode description of phase-sensitive amplifiers

Having warmed up on phase-insensitive amplifiers, one is now ready to tackle the problem of giving a multimode description for phase-sensitive amplifiers. The first task is to develop a multimode description of a signal in terms of its quadrature phases.

1. Quadrature-phase description of signals

Once again the input signal operator (4.2) is used as an example. I assume that associated with the signal is a carrier frequency Ω ; the quadrature phases are to be defined relative to this frequency. Furthermore, I assume that \mathcal{S} is symmetric about Ω —i.e., $\Omega + \epsilon \in \mathcal{S}$ if and only if $\Omega - \epsilon \in \mathcal{S}$.

Introduce now Hermitian operators $\phi_1(t)$ and $\phi_2(t)$ defined by

$$\phi^{(\pm)} = (\hbar\Omega/2)^{1/2} (\phi_1 \pm i\phi_2) e^{\mp i\Omega t} \quad (4.22)$$

[cf. Eq. (3.5)], which definition implies the commutation relations

$$\begin{aligned} [\phi_1(t), \phi_1(t')] &= [\phi_2(t), \phi_2(t')] \\ &= -\frac{i}{2\pi} \int_{\mathcal{S}} d\epsilon \frac{\epsilon}{\Omega} \text{sinc}(t - t'), \end{aligned} \quad (4.23a)$$

$$[\phi_1(t), \phi_2(t')] = \frac{i}{2\pi} \int_{\mathcal{S}} d\epsilon \text{cose}(t - t') \quad (4.23b)$$

[Eqs. (4.8)], where the integrals run over the set $\mathcal{R} \equiv \{ \epsilon > 0 \mid \Omega + \epsilon \in \mathcal{S} \}$. Written in terms of ϕ_1 and ϕ_2 , the signal operator (4.2) is given by

$$\phi(t) = (2\hbar\Omega)^{1/2} [\phi_1(t)\cos\Omega t + \phi_2(t)\sin\Omega t], \quad (4.24)$$

and the power operator (4.9) becomes

$$P = \hbar\Omega(\phi_1^2 + \phi_2^2) - \int_{\mathcal{R}} (d\epsilon/\pi) \frac{1}{2} \hbar\Omega \quad (4.25)$$

(cf. $a^\dagger a = X_1^2 + X_2^2 - \frac{1}{2}$ in the single-mode case). Notice that $d\epsilon/\pi$ —not $d\epsilon/2\pi$ —is the appropriate integration interval for bandwidths expressed in hertz, because, ϵ being always positive, $d\epsilon/\pi$ corresponds to sidebands above and below the carrier frequency.

The operators ϕ_1 and ϕ_2 are the amplitudes of the “cos Ωt ” and “sin Ωt ” quadrature phases—i.e., they are the multimode analogs of X_1 and X_2 for a single mode. The advantage of the multimode description is its explicit display of the time dependence of the quadrature-phase amplitudes.

The reason for interest in ϕ_1 and ϕ_2 can be loosely described as follows. The commutators $[\phi_1(t), \phi_1(t')]$ and $[\phi_2(t), \phi_2(t')]$ are much smaller than $(\hbar\Omega)^{-1}[\phi(t), \phi(t')]$, provided \mathcal{S} covers a range of frequencies small compared to Ω ; as a result, the fluctuations in ϕ_1 or ϕ_2 can be much smaller than the minimum fluctuations in ϕ . This way of looking at ϕ_1 and ϕ_2 —and their relation to so-called quantum nondemolition observables—is explored in Appendix A.

Although ϕ_1 or ϕ_2 has the potential for reduced fluctuations, the nonvanishing of $[\phi_1(t), \phi_1(t')]$ and $[\phi_2(t), \phi_2(t')]$ means that those fluctuations, unlike the uncertainty in X_1 or X_2 , cannot be reduced to zero. There are limits to the reduction of noise in ϕ_1 or ϕ_2 and to the reduction of the noise that a linear amplifier adds to ϕ_1 or ϕ_2 . These limits, which are bandwidth-dependent corrections to the single-mode results, emerge naturally from the multimode analysis.

Each of the quadrature-phase amplitudes can be decomposed into its Fourier components:

$$\phi_p(t) = \int_{\mathcal{R}} (d\epsilon/2\pi) [\alpha_p(\epsilon)e^{-i\epsilon t} + \alpha_p^\dagger(\epsilon)e^{i\epsilon t}] \quad (4.26)$$

($p = 1, 2$). The Fourier components are related to the creation and annihilation operators by

$$\alpha_1(\epsilon) = \frac{1}{2} \left[\left[\frac{\Omega + \epsilon}{\Omega} \right]^{1/2} a(\Omega + \epsilon) + \left[\frac{\Omega - \epsilon}{\Omega} \right]^{1/2} a^\dagger(\Omega - \epsilon) \right], \quad (4.27a)$$

$$\alpha_2(\epsilon) = -\frac{1}{2} i \left[\left[\frac{\Omega + \epsilon}{\Omega} \right]^{1/2} a(\Omega + \epsilon) - \left[\frac{\Omega - \epsilon}{\Omega} \right]^{1/2} a^\dagger(\Omega - \epsilon) \right], \quad (4.27b)$$

$\epsilon \in \mathcal{R}$ [Eqs. (4.22) and (4.6)], from which one, using Eqs. (4.1), derives the commutation relations

$$[\alpha_1(\epsilon), \alpha_1(\epsilon')] = [\alpha_1(\epsilon), \alpha_2(\epsilon')] = [\alpha_2(\epsilon), \alpha_2(\epsilon')] = 0, \quad (4.28a)$$

$$[\alpha_1(\epsilon), \alpha_1^\dagger(\epsilon')] = [\alpha_2(\epsilon), \alpha_2^\dagger(\epsilon')] = \pi(\epsilon/\Omega)\delta(\epsilon - \epsilon'), \quad (4.28b)$$

$$[\alpha_1(\epsilon), \alpha_2^\dagger(\epsilon')] = -[\alpha_2(\epsilon), \alpha_1^\dagger(\epsilon')] = i\pi\delta(\epsilon - \epsilon'), \quad (4.28c)$$

for all $\epsilon, \epsilon' \in \mathcal{R}$. The operators $\alpha_1(\epsilon)$ and $\alpha_2(\epsilon)$ are linear combinations of the Fourier components of ϕ at the frequencies $\Omega + \epsilon$, the linear combinations being precisely those that describe amplitude and phase modulation at frequency ϵ of a carrier signal with time dependence $\cos \Omega t$.

I now introduce the notion of time-stationary quadrature-phase noise, by which I mean that the fluctuations in ϕ_1 and ϕ_2 separately are time stationary but that these fluctuations might be correlated. This notion is formalized by defining a state to have *time-stationary quadrature-phase noise* if

$$\langle \alpha_p(\epsilon)\alpha_q(\epsilon') \rangle - \langle \alpha_p(\epsilon) \rangle \langle \alpha_q(\epsilon') \rangle = 0, \quad (4.29a)$$

$$\frac{1}{2} \langle \alpha_p(\epsilon)\alpha_q^\dagger(\epsilon') + \alpha_q^\dagger(\epsilon')\alpha_p(\epsilon) \rangle - \langle \alpha_p(\epsilon) \rangle \langle \alpha_q^\dagger(\epsilon') \rangle = 2\pi S_{pq}(\epsilon)\delta(\epsilon - \epsilon'), \quad (4.29b)$$

for all $\epsilon, \epsilon' \in \mathcal{R}$ ($p, q = 1, 2$). Equation (4.29b) defines a dimensionless *spectral-density matrix* $S_{pq}(\epsilon)$ [analog of single-mode moment matrix (3.10)]; the diagonal elements of S_{pq} characterize the fluctua-

tions in ϕ_1 and ϕ_2 , and the off-diagonal elements characterize their correlation. Using Eq. (4.29b), one can easily show that S_{pq} is Hermitian:

$$S_{pq}^*(\epsilon) = S_{qp}(\epsilon). \quad (4.30)$$

The time-domain equivalent of S_{pq} is the two-point correlation matrix

$$K_{pq}(u) \equiv \frac{1}{2} \langle \phi_p(t+u)\phi_q(t) + \phi_q(t)\phi_p(t+u) \rangle - \langle \phi_p(t+u) \rangle \langle \phi_q(t) \rangle \quad (4.31)$$

[$K_{pq}(u)$ is real and $K_{pq}(-u) = K_{qp}(u)$], which is related to $S_{pq}(\epsilon)$ by

$$K_{pq}(u) = \int_{\mathcal{R}} (d\epsilon/\pi)^{\frac{1}{2}} [S_{pq}(\epsilon)e^{-i\epsilon u} + S_{pq}^*(\epsilon)e^{i\epsilon u}]. \quad (4.32)$$

Equation (4.32), evaluated at $u = 0$, implies

$$(\Delta\phi_p)^2 = \int_{\mathcal{R}} (d\epsilon/\pi) S_{pp}(\epsilon), \quad (4.33a)$$

$$\frac{1}{2} \langle \phi_1\phi_2 + \phi_2\phi_1 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle = \int_{\mathcal{R}} (d\epsilon/\pi)^{\frac{1}{2}} [S_{12}(\epsilon) + S_{12}^*(\epsilon)]. \quad (4.33b)$$

These results allow one to obtain easily the variance of ϕ

$$(\Delta\phi)^2(t) = \hbar\Omega \int_{\mathcal{R}} (d\epsilon/\pi) [S_{11} + S_{22} + (S_{11} - S_{22})\cos 2\Omega t + (S_{12} + S_{12}^*)\sin 2\Omega t] \quad (4.34)$$

[Eqs. (4.24), (4.33), and (4.30); cf. Eq. (4.14)], and the expected power

$$\langle P \rangle = \hbar\Omega (\langle \phi_1 \rangle^2 + \langle \phi_2 \rangle^2) + \hbar\Omega \int_{\mathcal{R}} (d\epsilon/\pi) (S_{11} + S_{22} - \frac{1}{2}) \quad (4.35)$$

[Eqs. (4.25) and (4.33); cf. Eq. (4.15)]. The spectral-density matrix gives the fluctuations in ϕ_1 and ϕ_2 as equivalent fluxes of quanta (at the carrier frequency) per hertz.

It is often convenient to have the conditions (4.29) for time-stationary quadrature-phase noise written in terms of the creation and annihilation operators:

$$\langle a(\Omega \pm \epsilon)a(\Omega \pm \epsilon') \rangle - \langle a(\Omega \pm \epsilon) \rangle \langle a(\Omega \pm \epsilon') \rangle = 0, \quad (4.36a)$$

$$\frac{1}{2} \langle a(\Omega + \epsilon)a^\dagger(\Omega - \epsilon') + a^\dagger(\Omega - \epsilon')a(\Omega + \epsilon) \rangle - \langle a(\Omega + \epsilon) \rangle \langle a^\dagger(\Omega - \epsilon') \rangle = 0, \quad (4.36b)$$

$$\begin{aligned} \langle a(\Omega + \epsilon)a(\Omega - \epsilon') \rangle - \langle a(\Omega + \epsilon) \rangle \langle a(\Omega - \epsilon') \rangle \\ = 2\pi\delta(\epsilon - \epsilon') \left[\frac{\Omega}{\Omega + \epsilon} \right]^{1/2} \left[\frac{\Omega}{\Omega - \epsilon'} \right]^{1/2} \{ S_{11}(\epsilon) - S_{22}(\epsilon) + i[S_{12}(\epsilon) + S_{21}(\epsilon)] \}, \end{aligned} \quad (4.36c)$$

$$\begin{aligned} \frac{1}{2} \langle a(\Omega \pm \epsilon)a^\dagger(\Omega \pm \epsilon') + a^\dagger(\Omega \pm \epsilon')a(\Omega \pm \epsilon) \rangle - \langle a(\Omega \pm \epsilon) \rangle \langle a^\dagger(\Omega \pm \epsilon') \rangle \\ = 2\pi\delta(\epsilon - \epsilon') \frac{\Omega}{\Omega \pm \epsilon} \{ S_{11}(\epsilon) + S_{22}(\epsilon) \mp i[S_{12}(\epsilon) - S_{21}(\epsilon)] \} \end{aligned} \quad (4.36d)$$

[cf. Eqs. (3.11)]. Comparison of Eqs. (4.10) and (4.36) reveals that a state with time-stationary quadrature-phase noise has time-stationary noise if and only if

$$S_{11}(\epsilon) = S_{22}(\epsilon) = \frac{1}{4} \left[\left[\frac{\Omega + \epsilon}{\Omega} \right] S(\Omega + \epsilon) + \left[\frac{\Omega - \epsilon}{\Omega} \right] S(\Omega - \epsilon) \right], \quad (4.37a)$$

$$S_{12}(\epsilon) = -S_{21}(\epsilon) = \frac{i}{4} \left[\left[\frac{\Omega + \epsilon}{\Omega} \right] S(\Omega + \epsilon) - \left[\frac{\Omega - \epsilon}{\Omega} \right] S(\Omega - \epsilon) \right] \quad (4.37b)$$

[cf. Eq. (3.12)]. The factors $(\Omega \pm \epsilon)/\Omega$ are essentially a units conversion: $S(\Omega \pm \epsilon)$ is in units of number of quanta at frequency $\Omega \pm \epsilon$, whereas

$S_{pq}(\epsilon)$ is in units of number of quanta at Ω .

The crucial properties of S_{pq} follow from the commutation relations (4.28). Equations (4.28b)

imply directly that

$$S_{11}(\epsilon) \geq \frac{1}{4}(\epsilon/\Omega), \quad S_{22}(\epsilon) \geq \frac{1}{4}(\epsilon/\Omega), \quad (4.38)$$

which are the previously advertised limits on reduction of noise in ϕ_1 or ϕ_2 . For a state with time-stationary noise, the minimum value of $S_{pp}(\epsilon)$ corresponds to a quarter-quantum at the carrier frequency [Eq. (4.37a)]. In contrast, for a state with time-stationary quadrature-phase noise, $S_{pp}(\epsilon)$ can be reduced to correspond to a quarter-quantum at frequency ϵ —precisely the reduction in noise (in terms of noise power per hertz) which could be achieved if the signal in one quadrature phase were transformed from frequencies near Ω to frequencies near zero. Indeed, one can regard as the main result of this subsection the demonstration that a signal in one quadrature phase of a high carrier frequency can have as small an amount of quantum noise as a comparable signal at frequencies near zero. From this point of view, an ordinary signal like Eq. (4.2) should be thought of as being the one “quadrature phase” of a signal with zero carrier frequency.

By writing Eqs. (4.29), (4.28a), and (4.28c) in terms of the Hermitian real and imaginary parts of α_1 and α_2 and by using the generalized uncertainty principle $\Delta B \Delta C \geq \frac{1}{2} |\langle [B, C] \rangle|$, one can prove an uncertainty principle for the spectral-density matrix,

$$S_{11}(\epsilon)S_{22}(\epsilon) \geq \frac{1}{16}, \quad (4.39)$$

which is the multimode generalization of the ordinary uncertainty principle (3.8) for X_1 and X_2 . Equality in the uncertainty principle (4.39) implies $S_{12}(\epsilon) + S_{21}(\epsilon) = 0$ (S_{12} pure imaginary).

Appendix B considers a class of states with time-stationary quadrature-phase noise, the multimode “squeezed states.”

2. Phase-sensitive linear amplifiers

The objective now is to give a multimode description of the sort of phase-sensitive amplifier considered in Sec. III. I assume that the input and output signals [Eqs. (4.2) and (4.3)] have carrier frequencies Ω_I and Ω_O and that \mathcal{S} and \mathcal{O} are symmetric about Ω_I and Ω_O , respectively; furthermore, I assume that \mathcal{S} and \mathcal{O} map onto the same low-frequency set $\mathcal{R} \equiv \{ \epsilon > 0 \mid \Omega_I + \epsilon \in \mathcal{S} \} = \{ \epsilon > 0 \mid \Omega_O + \epsilon \in \mathcal{O} \}$. The operators $\phi^{(\pm)}$, ϕ_1 , ϕ_2 , α_1 , and α_2 associated with the input signal are defined as before, except that Ω_I replaces Ω ; the analogous operators for the output signal are denoted by $\psi^{(\pm)}$, ψ_1 , ψ_2 , β_1 , and β_2 . The input and output spectral-density matrices are denoted by S_{pq}^I and S_{pq}^O .

Focus attention now on phase-sensitive amplifiers whose evolution equations (4.5) can be put in the form

$$\begin{aligned} \left[\frac{\Omega_O + \epsilon}{\Omega_O} \right]^{1/2} b(\Omega_O + \epsilon) &= \left[\frac{\Omega_I + \epsilon}{\Omega_I} \right]^{1/2} M(\epsilon) a(\Omega_I + \epsilon) + \left[\frac{\Omega_I - \epsilon}{\Omega_I} \right]^{1/2} L(\epsilon) a^\dagger(\Omega_I - \epsilon) \\ &+ \left[\frac{\Omega_O + \epsilon}{\Omega_O} \right]^{1/2} \mathcal{F}(\Omega_O + \epsilon), \end{aligned} \quad (4.40a)$$

$$\begin{aligned} \left[\frac{\Omega_O - \epsilon}{\Omega_O} \right]^{1/2} b(\Omega_O - \epsilon) &= \left[\frac{\Omega_I - \epsilon}{\Omega_I} \right]^{1/2} M^*(\epsilon) a(\Omega_I - \epsilon) + \left[\frac{\Omega_I + \epsilon}{\Omega_I} \right]^{1/2} L^*(\epsilon) a^\dagger(\Omega_I + \epsilon) \\ &+ \left[\frac{\Omega_O - \epsilon}{\Omega_O} \right]^{1/2} \mathcal{F}(\Omega_O - \epsilon), \end{aligned} \quad (4.40b)$$

$\epsilon \in \mathcal{R}$, where, if \mathcal{R} contains zero frequency, one must require $M(0)$ and $L(0)$ to be real. A given output frequency near Ω_O is coupled to the corresponding input frequency near Ω_I and to the image-sideband frequency. Notice that if the output signal were carried only by frequencies above Ω_O and the input signal only the corresponding frequencies above (below) Ω_I , then Eq. (4.40a) would be the evolution equation for a phase-preserving (phase-conjugating) amplifier. The input frequencies below (above) Ω_I would be included in the internal modes, and they would contribute to the amplifier's added noise.

Equations (4.40) can be translated into the much simpler form

$$\beta_1(\epsilon) = [G_1(\epsilon)]^{1/2} e^{i\gamma_1(\epsilon)} \alpha_1(\epsilon) + \mathcal{F}_1(\epsilon), \quad (4.41a)$$

$$\beta_2(\epsilon) = [G_2(\epsilon)]^{1/2} e^{i\gamma_2(\epsilon)} \alpha_2(\epsilon) + \mathcal{F}_2(\epsilon) \quad (4.41b)$$

[Eqs. (4.27)], where

$$[G_1(\epsilon)]^{1/2} e^{i\gamma_1(\epsilon)} \equiv M(\epsilon) + L(\epsilon), \quad (4.42)$$

$$[G_2(\epsilon)]^{1/2} e^{i\gamma_2(\epsilon)} \equiv M(\epsilon) - L(\epsilon)$$

(notice that $e^{i\gamma_p(0)}$ is real), and where

$$\mathcal{F}_1(\epsilon) \equiv \frac{1}{2} \left[\left[\frac{\Omega_0 + \epsilon}{\Omega_0} \right]^{1/2} \mathcal{F}(\Omega_0 + \epsilon) + \left[\frac{\Omega_0 - \epsilon}{\Omega_0} \right]^{1/2} \mathcal{F}^\dagger(\Omega_0 - \epsilon) \right], \quad (4.43a)$$

$$\mathcal{F}_2(\epsilon) \equiv -\frac{1}{2} i \left[\left[\frac{\Omega_0 + \epsilon}{\Omega_0} \right]^{1/2} \mathcal{F}(\Omega_0 + \epsilon) - \left[\frac{\Omega_0 - \epsilon}{\Omega_0} \right]^{1/2} \mathcal{F}^\dagger(\Omega_0 - \epsilon) \right]. \quad (4.43b)$$

Equations (4.41) imply that ψ_1 is coupled to ϕ_1 and ψ_2 is coupled to ϕ_2 [cf. Eqs. (3.18)]. The frequency-dependent gains for the two quadrature phases, in units of number of quanta, are $G_1(\epsilon)$ and $G_2(\epsilon)$ [cf. Eq. (3.19)].

The unitarity conditions corresponding to Eqs. (4.40) are most easily obtained by applying the appropriate commutation relations to Eqs. (4.41):

$$[\mathcal{F}_1(\epsilon), \mathcal{F}_1(\epsilon')] = [\mathcal{F}_1(\epsilon), \mathcal{F}_2(\epsilon')] = [\mathcal{F}_2(\epsilon), \mathcal{F}_2(\epsilon')] = 0, \quad (4.44a)$$

$$[\mathcal{F}_1(\epsilon), \mathcal{F}_1^\dagger(\epsilon')] = \pi(\epsilon/\Omega_I) \delta(\epsilon - \epsilon') [(\Omega_I/\Omega_0) - G_1(\epsilon)], \quad (4.44b)$$

$$[\mathcal{F}_2(\epsilon), \mathcal{F}_2^\dagger(\epsilon')] = \pi(\epsilon/\Omega_I) \delta(\epsilon - \epsilon') [(\Omega_I/\Omega_0) - G_2(\epsilon)], \quad (4.44c)$$

$$[\mathcal{F}_1(\epsilon), \mathcal{F}_2^\dagger(\epsilon')] = i\pi \delta(\epsilon - \epsilon') [1 - (G_1 G_2)^{1/2} e^{i(\gamma_1 - \gamma_2)}], \quad (4.44d)$$

for all $\epsilon, \epsilon' \in \mathcal{R}$. These unitarity conditions have the same form as Eqs. (4.28), the main difference being that $[\mathcal{F}_1(\epsilon), \mathcal{F}_2^\dagger(\epsilon')]$ is not necessarily pure imaginary.

I place one further requirement on the amplifiers of interest: if the input signal has time-stationary quadrature-phase noise, then so must the output signal. The consequences of this requirement are most easily presented in terms of \mathcal{F}_1 and \mathcal{F}_2 :

$$\langle \mathcal{F}_p(\epsilon) \mathcal{F}_q(\epsilon') \rangle_{\text{op}} = 0, \quad (4.45a)$$

$$\frac{1}{2} \langle \mathcal{F}_p(\epsilon) \mathcal{F}_q^\dagger(\epsilon') + \mathcal{F}_q^\dagger(\epsilon') \mathcal{F}_p(\epsilon) \rangle_{\text{op}} = 2\pi \delta(\epsilon - \epsilon') (G_p G_q)^{1/2} e^{i(\gamma_p - \gamma_q)} S_{pq}^A(\epsilon), \quad (4.45b)$$

for all $\epsilon, \epsilon' \in \mathcal{R}$ [cf. Eqs. (4.29)]. Equation (4.45b) defines a Hermitian *added noise spectral-density matrix* $S_{pq}^A(\epsilon)$ [analog of the added noise moment matrix; cf. Eq. (3.25)]. The input and output spectral-density matrices are related by

$$S_{pq}^O = (G_p G_q)^{1/2} e^{i(\gamma_p - \gamma_q)} (S_{pq}^I + S_{pq}^A) \quad (4.46)$$

[cf. Eq. (3.26)].

Writing the conditions (4.45) in terms of moments of $\mathcal{F}(\omega)$ and $\mathcal{F}^\dagger(\omega)$ yields a set of equations which can be obtained from Eqs. (4.36) by the replacements $a \rightarrow \mathcal{F}$, $\Omega \rightarrow \Omega_0$, and $S_{pq} \rightarrow (G_p G_q)^{1/2} \times e^{i(\gamma_p - \gamma_q)} S_{pq}^A$. This set of equations, together with Eqs. (4.40), reveals that a phase-sensitive amplifier

of the type considered here is phase insensitive if and only if (i) $L(\epsilon) = 0$ [$G_1 = G_2$ and $e^{i(\gamma_1 - \gamma_2)} = 1$; phase preserving] or $M(\epsilon) = 0$ [$G_1 = G_2$ and $e^{i(\gamma_1 - \gamma_2)} = -1$; phase conjugating] and (ii) $S_{11}^A = S_{22}^A$ and $S_{12}^A + S_{21}^A = 0$. If the amplifier is phase insensitive, then $S_{pq}^A(\epsilon)$ is related to $S^A(\bar{\omega})$ [Eq. (4.19b)] by equations of the form (4.37), where Ω is replaced by Ω_I .

The unitarity conditions (4.44) are now used to derive constraints on the added noise spectral-density matrix. Equations (4.44b) and (4.44c) imply that

$$S_{pp}^A \geq \frac{1}{4} (\epsilon/\Omega_I) |1 - (\Omega_I/G_p \Omega_0)|, \quad (4.47)$$

which limits the reduction in the noise the amplifier adds to either quadrature phase. In the case of high gain ($G_p \gg \Omega_I/\Omega_O$), the limit on the noise added to a particular phase is the same as the limit (4.38) on the input noise in that phase. At frequencies where $G_p \leq \Omega_I/\Omega_O$, Eq. (4.47), together with Eqs. (4.46) and (4.38), simply ensures that

$$S_{11}^A S_{22}^A \geq \max \left\{ \frac{1}{16} [\cos(\gamma_1 - \gamma_2) - (G_1 G_2)^{-1/2}]^2, \frac{1}{16} \sin^2(\gamma_1 - \gamma_2) \right\}. \quad (4.48)$$

Equality implies $e^{i(\gamma_1 - \gamma_2)} S_{12}^A \pm e^{-i(\gamma_1 - \gamma_2)} S_{21}^A = 0$, where the upper (lower) sign holds if the first (second) term in the maximum applies. Notice the similarity of the amplifier uncertainty principle to the corresponding multimode uncertainty principle (4.39) for the input signal. The multimode amplifier uncertainty principle is somewhat more complicated than its single-mode analog (3.35), because the phase factors γ_1 and γ_2 allow a continuous transition between the two signs in Eq. (3.35). For a phase-insensitive amplifier, the amplifier uncertainty principle (4.48), evaluated at $\epsilon=0$, reduces to Eq. (4.21).

An important special case, not covered explicitly by the preceding analysis, is an amplifier whose output signal (4.3), assumed to be at frequencies near zero, is coupled to one quadrature phase of the input signal. This sort of amplifier is phase sensitive and has all the previously obtained potential for reduction in noise; the output signal can be regarded as the one "quadrature phase" of a signal with zero carrier frequency [see discussion following Eq. (4.38)]. To convert this case into the sort of situation analyzed above, one can simply imagine multiplying the output signal by $\cos\Omega_O t$, thereby converting the output to frequencies near Ω_O .

Examples of this sort of amplifier are easy to come by. Consider an input that consists of a strong carrier signal (local-oscillator signal) at frequency Ω_I superposed on an input signal at frequencies near Ω_I . Imagine running this input through a square-law device, which responds to the input power, or a rectifier, which retains only the positive part of the input. In either case, the output signal at frequencies near zero is coupled to only one quadrature phase of the input signal—the phase that produces amplitude modulation of the carrier. This sort of device can be operated in either a homodyne mode ($\mathcal{S} = \{ \omega \mid \Omega_I - \Delta\omega \leq \omega \leq \Omega_I + \Delta\omega \}$) or a heterodyne mode (\mathcal{S} consisting of upper and lower of sidebands of Ω_I , separated

$$S_{pp}^O \geq \frac{1}{4} (\epsilon/\Omega_O).$$

By writing Eqs. (4.44a), (4.44d) and (4.45) in terms of the Hermitian real and imaginary parts of \mathcal{F}_1 and \mathcal{F}_2 and by using the generalized uncertainty principle, one can obtain the multimode version of the *amplifier uncertainty principle*:

from Ω_I). A homodyne device is always phase sensitive. If the input signal to a heterodyne device consists of both sidebands, then it, too, is phase sensitive. In its usual mode of operation, however, the input signal to a heterodyne device consists of only one of the sidebands. The modes in the unused sideband are then included in the internal modes [cf. discussion following Eq. (4.40)]. Operated in this way, a heterodyne device is phase insensitive, and the unavoidable quantum noise added by the device is due to zero-point noise in the unused sideband.

V. CONCLUSION

Perhaps the greatest impact of the results obtained here lies in illuminating the role of amplification in the quantum theory of measurement. Consider a signal that carries the minimum noise permitted by quantum mechanics (e.g., a signal with $S_{11} S_{22} = \frac{1}{16}$). One cannot examine the signal directly with standard, "classical" devices, because they would drastically degrade the signal-to-noise ratio. Instead, one first amplifies the signal with a high-gain linear amplifier; then further signal processing need not add significant amounts of noise. If one wants to get the information in both quadrature phases of the signal with equal accuracy, then the best situation is to let the signal have time-stationary noise [$S(\omega) = \frac{1}{2}$] and to use the best phase-insensitive linear amplifier, which adds noise that is equal to the noise carried by the signal. On the other hand, if one is interested in only one quadrature phase, then it is possible to generate the signal with a much reduced amount of noise in that phase ($S_{11} = \epsilon/4\Omega_I$) and to use the best phase-sensitive amplifier, which adds precisely this reduced amount of noise to the phase of interest.

What are the lessons to be learned here? *First, quantum mechanics extracts its due twice.* If it

places a limit on how well a signal can be measured, then it requires a high-gain linear amplifier that processes the signal to add enough noise to enforce independently the same limit. *Second, in doing a quantum-mechanical analysis of a measurement, one should include in the analysis the first high-gain amplifier in the chain of measuring devices.* A high-gain amplifier is required to get the measurement information up to the classical level, and it is the last stage of the measurement chain which might be required to add a significant amount of unavoidable quantum noise.

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$$[\phi(t), \phi(t')] = -(i/\pi)\hbar\Omega \left[\sin\Omega u \int_{\mathcal{R}} d\epsilon \cos\epsilon u + \cos\Omega u \int_{\mathcal{R}} d\epsilon (\epsilon/\Omega) \sin\epsilon u \right], \quad (\text{A4})$$

where $u \equiv t - t'$, it is assumed \mathcal{S} is symmetric about the carrier frequency Ω , and \mathcal{R} is defined after Eqs. (4.23). Equation (A4) reveals that if \mathcal{S} covers a range of frequencies small compared to Ω ($\epsilon \ll \Omega$), then the minimum value for $\Delta^2(t, t')$, as a function of u , oscillates nearly sinusoidally at frequency Ω . The minimum value is largest when $\cos\Omega u = 0$, and it is smallest when $\sin\Omega u = 0$.

The quadrature-phase amplitudes ϕ_1 and ϕ_2 [Eq. (4.22)] are a way of taking advantage of this sinusoidal oscillation. Their commutators $[\phi_1(t), \phi_1(t')]$ and $[\phi_2(t), \phi_2(t')]$ [Eq. (4.23a)] account for the small (second) term in Eq. (A4), and their cross-commutator $[\phi_1(t), \phi_2(t')]$ [Eq. (4.23b)] accounts for the large (first) term. Thus one of the quadrature-phase amplitudes can have at all times the reduced fluctuations available in $\phi(t)$ every half-period; the large fluctuations in $\phi(t)$ are pushed into the other quadrature phase.

The quadrature-phase amplitudes are, loosely speaking, a step in the direction of QND observables. Indeed, in the limit of zero bandwidth, they become the QND observables X_1 and X_2 for a single mode.¹⁷ It is possible to construct nonzero-bandwidth QND observables from the operators $a(\omega), a^\dagger(\omega)$, $\omega \in \mathcal{S}$, but these observables do not

APPENDIX A

For an arbitrary observable $\chi(t)$, the generalized uncertainty principle limits the product of the uncertainties at different times:

$$\Delta\chi(t)\Delta\chi(t') \geq \frac{1}{2} | \langle [\chi(t), \chi(t')] \rangle |; \quad (\text{A1})$$

therefore, it also limits the mean uncertainty $\Delta^2(t, t') \equiv \frac{1}{2} [(\Delta\chi)^2(t) + (\Delta\chi)^2(t')]$:

$$\Delta^2(t, t') \geq \frac{1}{2} | \langle [\chi(t), \chi(t')] \rangle |. \quad (\text{A2})$$

The best situation occurs when

$$[\chi(t), \chi(t')] = 0 \quad \text{for all times } t, t', \quad (\text{A3})$$

in which case $\chi(t)$ could have zero uncertainty at all times; such an observable is called a (generalized) *quantum nondemolition (QND) observable*.^{17,33}

The signal operator $\phi(t)$ [Eq. (4.2)] is not a QND observable. Nonetheless, considerable insight can be obtained by writing its commutation relation (4.4a) in the form

have the simple amplitude- and phase-modulation interpretation that applies to ϕ_1 and ϕ_2 .

APPENDIX B: MULTIMODE SQUEEZED STATES

This appendix introduces and summarizes some properties of multimode squeezed states. Throughout I use the formal apparatus developed in Secs. IV B 1 and IV C 1 to apply to the signal operator (4.2).

Single-mode squeezed states were introduced independently by Stoler^{34,35} and Lu,^{36,37} and they have been analyzed in detail by Yuen.³² Hollenhorst³⁸ has applied them to the problem of quantum nondemolition measurements of a harmonic oscillator, and their application to optical communications has been exhaustively analyzed by Yuen, Shapiro,^{39,40} and Machado Mata.⁴¹ It has recently been proposed that squeezed states can reduce the photon-counting noise in a laser interferometer designed to detect gravitational radiation.⁴² A compact summary of some properties of single-mode squeezed states can be found in Sec. II A of Ref. 42.

To define multimode squeezed states, one first introduces the unitary *displacement operator*,²⁵ de-

finied by

$$D[\mu(\omega)] \equiv \exp \left[\int_{\mathcal{J}} (d\omega/2\pi) [\mu(\omega) a^\dagger(\omega) - \mu^*(\omega) a(\omega)] \right], \quad (\text{B1})$$

where $\mu(\omega)$ is a complex function of frequency. Notice that $D^\dagger[\mu(\omega)] = D^{-1}[\mu(\omega)] = D[-\mu(\omega)]$. The most important property of the displacement operator is that

$$D^\dagger a(\omega) D = a(\omega) + \mu(\omega), \quad (\text{B2})$$

$$D^\dagger a^\dagger(\omega) D = a^\dagger(\omega) + \mu^*(\omega),$$

$\omega \in \mathcal{J}$. Applied to the vacuum state $|0\rangle$, the displacement operator generates a *coherent state*,²⁵

$$|\mu(\omega)\rangle \equiv D[\mu(\omega)] |0\rangle, \quad (\text{B3})$$

$$S[\zeta(\epsilon)] \equiv \exp \left[\int_{\mathcal{H}} (d\epsilon/2\pi) [-\zeta(\epsilon) a^\dagger(\Omega + \epsilon) a^\dagger(\Omega - \epsilon) + \zeta^*(\epsilon) a(\Omega + \epsilon) a(\Omega - \epsilon)] \right] \quad (\text{B6})$$

[cf. Eq. (2.7) of Ref. 42], where it is now assumed that \mathcal{H} is symmetric about Ω , and where

$$\zeta(\epsilon) = r(\epsilon) e^{2i\varphi(\epsilon)} \quad (\text{B7})$$

is a complex function. Notice that $S^\dagger[\zeta(\epsilon)] = S^{-1}[\zeta(\epsilon)] = S[-\zeta(\epsilon)]$. The crucial property of the squeeze operator is that

$$S^\dagger a(\Omega \pm \epsilon) S = a(\Omega \pm \epsilon) \cosh r - a^\dagger(\Omega \mp \epsilon) e^{2i\varphi} \sinh r, \quad (\text{B8})$$

$$S^\dagger a^\dagger(\Omega \pm \epsilon) S = a^\dagger(\Omega \pm \epsilon) \cosh r - a(\Omega \mp \epsilon) e^{-2i\varphi} \sinh r$$

($\epsilon \in \mathcal{H}$), where r and φ are evaluated at ϵ [cf. Eq. (2.8) of Ref. 42]. The squeeze operator couples the annihilation operator at a given frequency to itself and to the creation operator at the image-sideband frequency.

A multimode *squeezed state* is obtained by first

$$S_{11} + S_{22} \mp i(S_{12} - S_{21}) = \frac{1}{2} \frac{\Omega \pm \epsilon}{\Omega} (\cosh^2 r + \sinh^2 r), \quad (\text{B11})$$

$$S_{11} - S_{22} + i(S_{12} + S_{21}) = - \left[\frac{\Omega + \epsilon}{\Omega} \right]^{1/2} \left[\frac{\Omega - \epsilon}{\Omega} \right]^{1/2} \sinh r \cosh r e^{2i\varphi}.$$

which is an eigenstate of each annihilation operator:

$$a(\omega) |\mu(\omega')\rangle = \mu(\omega) |\mu(\omega')\rangle. \quad (\text{B4})$$

The coherent state $|\mu(\omega)\rangle$ has time-stationary noise. Among its important properties are

$$\langle \phi(t) \rangle = \int_{\mathcal{J}} d\omega (\hbar\omega/8\pi^2)^{1/2} \times [\mu(\omega) e^{-i\omega t} + \mu^*(\omega) e^{i\omega t}],$$

$$S(\omega) = \frac{1}{2}, \quad (\text{B5})$$

$$\langle P(t) \rangle = 2 |\langle \phi^{(+)}(t) \rangle|^2,$$

$$(\Delta P)^2 = 2 \langle P \rangle \int_{\mathcal{J}} (d\omega/2\pi) \frac{1}{2} \hbar\omega.$$

The next step toward multimode squeezed states is introduction of the unitary *squeeze operator*

squeezing the vacuum and then displacing it:

$$|\mu(\omega); \zeta(\epsilon)\rangle \equiv D[\mu(\omega)] S[\zeta(\epsilon)] |0\rangle. \quad (\text{B9})$$

If $\zeta(\epsilon) = 0$ the squeezed state is a coherent state. The expected signal for the squeezed state (B9) is given by [$\langle a(\omega) \rangle = \mu(\omega)$, $\langle \alpha_p(\epsilon) \rangle = \mu_p(\epsilon)$]:

$$\langle \phi(t) \rangle = \int_{\mathcal{J}} d\omega (\hbar\omega/8\pi^2)^{1/2} [\mu(\omega) e^{-i\omega t} + \mu^*(\omega) e^{i\omega t}],$$

$$\langle \phi_p(t) \rangle = \int_{\mathcal{H}} (d\epsilon/2\pi) [\mu_p(\epsilon) e^{-i\epsilon t} + \mu_p^*(\epsilon) e^{i\epsilon t}], \quad (\text{B10})$$

where $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ are related to $\mu(\Omega + \epsilon)$ and $\mu^*(\Omega - \epsilon)$ in the obvious way [see Eqs. (4.27)]. A squeezed state has time-stationary quadrature-phase noise, and its spectral-density matrix is given in general by

Notice that in general $S_{11} + S_{22} = \frac{1}{2}(\cosh^2 r + \sinh^2 r)$. A tedious calculation provides the expectation value and variance of the power:

$$\begin{aligned} \langle P \rangle &= \hbar\Omega(\langle \phi_1 \rangle^2 + \langle \phi_2 \rangle^2) + \hbar\Omega \int_{\mathcal{R}} (d\epsilon/\pi) \sinh^2 r, \\ (\Delta P)^2 &= \hbar \int_{\mathcal{R}} (d\epsilon/\pi) [|Q_+ \cosh r - Q_-^* \sinh r|^2 + |Q_- \cosh r - Q_+^* \sinh r|^2] \\ &\quad + \left[\int_{\mathcal{R}} (d\epsilon/\pi) \hbar\Omega \sinh^2 r \right] \left[\int_{\mathcal{R}} (d\epsilon/\pi) \hbar\Omega \cosh^2 r \right] + \left| \int_{\mathcal{R}} (d\epsilon/\pi) \hbar(\Omega^2 - \epsilon^2)^{1/2} e^{-2i\varphi} \sinh r \cosh r \right|^2, \\ Q_{\pm} &\equiv (\Omega \pm \epsilon)^{1/2} \langle \phi^{(+)} \rangle e^{i(\Omega t - \varphi)} \end{aligned} \quad (\text{B12})$$

[cf. Eq. (2.11) of Ref. 42].

An important special case occurs when $\varphi(\epsilon) = 0$ for all $\epsilon \in \mathcal{R}$; then the squeezing occurs with the same phase at all values of ϵ . In this case the spectral-density matrix (B11) becomes

$$S_{11} = \frac{1}{4} e^{-2r} + \frac{1}{4} \sinh 2r \left[1 - \left[1 - \frac{\epsilon^2}{\Omega^2} \right]^{1/2} \right], \quad (\text{B13a})$$

$$S_{22} = \frac{1}{4} e^{2r} - \frac{1}{4} \sinh 2r \left[1 - \left[1 - \frac{\epsilon^2}{\Omega^2} \right]^{1/2} \right], \quad (\text{B13b})$$

$$S_{12} = -S_{21} = i(\epsilon/4\Omega)(\cosh^2 r + \sinh^2 r) \quad (\text{B13c})$$

[cf. Eq. (2.11) of Ref. 42], and the uncertainty product is given by

$$S_{11} S_{22} = \frac{1}{16} \left[1 + \frac{\epsilon^2}{\Omega^2} \sinh^2 2r \right] \quad (\text{B14})$$

[cf. Eq. (4.39)]. Equations (B13a) and (B13b) demonstrate that as ϵ increases, one's ability to reduce S_{11} or S_{22} decreases. Indeed, for a given ϵ , the minimum value of S_{11} occurs when $\cosh 2r = \Omega/\epsilon$ ($r > 0$), the minimum being $S_{11} = \frac{1}{4}(\epsilon/\Omega)$ [cf. Eq. (4.38)]. At the minimum the uncertainty product is $S_{11} S_{22} = \frac{1}{16} [2 - (\epsilon^2/\Omega^2)]$.

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²⁴Throughout this paper there is an implicit assumption that the noise is Gaussian, so that it can be characterized completely by its second moments. A more general analysis would deal not with the moment matrix but with the characteristic function $\Phi(\eta, \xi) \equiv \langle \exp[2i(\langle X_1 \rangle \eta + \langle X_2 \rangle \xi)] \rangle = \langle \exp(\mu a^\dagger - \mu^* a) \rangle (\mu \equiv i\eta - \xi)$, which contains information about all the moments and cross moments of X_1 and X_2 . The requirement for a state to have phase-insensitive noise would be that $\Phi(\eta, \xi) \exp[-2i(\langle X_1 \rangle \eta + \langle X_2 \rangle \xi)]$ be invariant under arbitrary phase transformations (3.7) or, equivalently, that $\Phi(\eta, \xi) = Q(|\mu|^2) \times \exp[2i(\langle X_1 \rangle \eta + \langle X_2 \rangle \xi)]$, where Q is an analytic function with $Q(0) = 1$. The notion of a phase-insensitive linear amplifier could be similarly generalized. The wording of the two conditions for a narrow-band phase-insensitive amplifier would remain the same, but their implications [Eqs. (3.17)] would be changed. Equation (3.17b) would be replaced by the requirement that $\langle \exp(\mu \mathcal{F}^\dagger - \mu^* \mathcal{F}) \rangle_{\text{op}}$ be a function only of $|\mu|^2$ (recall that $\langle \mathcal{F} \rangle_{\text{op}} = 0$).

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