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Two-variable expansions for nucleon-nucleon scattering

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Previously proposed two-variable expansions of scattering amplitudes are adapted to the case of nucleon-nucleon scattering. The expansions can be interpreted as partial-wave expansions of singlet-triplet amplitudes, complemented by an expansion of each partial-wave amplitude as a function of energy in terms of basis functions of the group O(4).

I. INTRODUCTION

Two recent articles were devoted to two-variable expansions of scattering amplitudes for particles with arbitrary spins[1,2] (to be referred to as I and II). The scattering amplitudes for a two-body reaction were expanded in terms of known functions of the center-of-mass-system (c.m.s.) energy and scattering angle. In article I the expansions were obtained by making use of the representation theory of the homogeneous Lorentz group; they involved both integrals and sums over O(3,1) basis functions and were applicable for infinite energy ranges. In article II the expansions were made discrete, i.e., a restriction was made to a finite energy range (from threshold to some chosen energy), and the group O(3,1) was replaced by the rotation group O(4), so that the obtained expansions involved sums only. The motivation for the entire program was discussed in I and II; for references to earlier and related papers we refer to a review article.[3]

The purpose of this paper is to adapt the discrete O(4) expansions to the case of nucleon-nucleon scattering and to relate them to the usual formalism of phase-shift analysis at a fixed energy (single-variable expansions).[4] The obtained expansions can then be directly applied to analyze simultaneously all nucleon-nucleon data over a large energy region. This is particularly pertinent in view of current nucleon-nucleon experimental programs at various accelerators, providing a large body of spin-dependent data. The availability of a variety of different methods of reconstructing scattering amplitudes from the data is of utmost importance: a direct reconstruction necessitates the performance of a complete set of experiments[5] at each energy and angle. Ordinary phase-shift analysis only makes use of data at one given energy. The closest in spirit to the present approach is that of "locally energy-dependent partial-wave analysis",[6] where the considered energy region is divided into several subregions, in each of which the partial waves are parametrized separately. Our approach builds in some of the kinematics and makes use of orthogonal sets of basis functions. This should improve the convergence and stability of data fits. It is also a uniform approach for all reactions and should hence be relatively free of specific theoretical prejudices.

II. EXPANSIONS OF INITIAL-SPIN–FINAL-SPIN AMPLITUDES FOR ARBITRARY SPINS

The expansions in I and II were performed for helicity and canonical coupling amplitudes. Nucleon-nucleon phase-shift analysis is usually performed using singlet-triplet amplitudes. We first provide some general formulas for this type of amplitude, then specify to the case of two nucleons. We use the notations and results of I and II.

Consider the reaction $1+2\rightarrow 3+4$ where the particles have spins s_i and spin projections σ_i onto the third axis ($i=1,\ldots,4$). Let (s,σ) and (s',σ') be the total spin and its projection in the final and initial states, respectively. Let $T_{\sigma_3\sigma_4\sigma_1\sigma_2}(\alpha,\theta,\phi)$ be the total scattering amplitude in the canonical quantization for which a two-variable O(4) expansion is given by formula (11) of II. The scattering amplitudes in the initial-spin—final-spin formalism are

$$M_{s\sigma s'\sigma'}(\alpha,\theta,\phi)=\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}(s_3\sigma_3 s_4\sigma_4\,|\,s\sigma)(s_1\sigma_1 s_2\sigma_2\,|\,s'\sigma')T_{\sigma_3\sigma_4\sigma_1\sigma_2}(\alpha,\theta,\phi)\ , \tag{1}$$

and their partial-wave expansion is given by

$$M_{s\sigma s'\sigma'}(\alpha,\theta,\phi)=\sum_{ll'j}[4\pi(2l+1)]^{1/2}\frac{2l'+1}{2j+1}(l\sigma'-\sigma s\sigma\,|\,j\sigma')(l'0s'\sigma'\,|\,j\sigma')A^{j}_{lsl's'}(\alpha)Y_{l\sigma'-\sigma}(\theta,\phi)\ , \tag{2}$$

where $A^{j}_{lsl's'}(\alpha)$ are the partial-wave amplitudes. The Clebsch-Gordan coefficients and spherical harmonics are as in Ref. 7, θ and ϕ are the c.m.s. scattering and azimuthal angles, and α is related to the energy. For elastic scattering, when the initial- and final-state thresholds coincide, we identify

$$\sin\alpha=\frac{\tilde{s}-(m_1+m_2)^2}{\tilde{s}_M-(m_1+m_2)^2},\quad 0\le\alpha\le\frac{\pi}{2} \tag{3}$$

where $\tilde{s}_M$ is some maximal value of the invariant energy $\tilde{s}=(p_1+p_2)^2$, to be chosen as the specific application requires.

The two-variable O(4) expansions are obtained directly from the results of II as

$$M_{s\sigma s'\sigma'}(\alpha,\theta,\phi)=\sum_{ll'j}[4\pi(2l+1)]^{1/2}\frac{2l'+1}{(2j+1)^2}(l\sigma'-\sigma s\sigma\,|\,j\sigma')(l'0s'\sigma'\,|\,j\sigma')$$
$$\times\sum_{\lambda\lambda'}(l0s\lambda\,|\,j\lambda)(l'0s'\lambda'\,|\,j\lambda')\sum_{n\nu}[(n+1)^2-\nu^2]T^{n\nu}_{jss'\lambda'}d^{n\nu}_{js\lambda}(\alpha)^{*}Y_{l\sigma'-\sigma}(\theta,\phi)\ , \tag{4}$$

where $T^{n\nu}_{jss'\lambda'}$ are the O(4) amplitudes. They are constants and the entire energy dependence is contained in the O(4) transformation matrices $d^{n\nu}_{js\lambda}(\alpha)$, discussed in detail in II. We can now perform the sum over l' and further simplify by introducing new basis functions

$$F^{n\nu}_{jsl}(\alpha)=\sum_{\lambda}\left[\frac{2l+1}{2j+1}\right]^{1/2}(l0s\lambda\,|\,j\lambda)d^{n\nu}_{js\lambda}(\alpha) \tag{5}$$

satisfying the following orthogonality and completeness relations:

$$\sum_{l=|j-s|}^{j+s}\int_0^{\pi}\sin^2\alpha\,d\alpha\,F^{n\nu}_{jsl}(\alpha)F^{n'\nu'}_{jsl}(\alpha)^{*}=\frac{\pi}{2}\frac{(2j+1)(2s+1)}{(n+1)^2-\nu^2}\delta_{\nu\nu'}\delta_{nn'}\ ,$$
$$\sum_{\nu=-\min(js)}^{\min(js)}\ \sum_{n=\max(js)}^{\infty}[(n+1)^2-\nu^2]F^{n\nu}_{jsl}(\alpha)F^{n\nu}_{jsl'}(\alpha')^{*}=\frac{\pi}{2}\frac{(2j+1)(2s+1)}{\sin^2\alpha}\delta(\alpha-\alpha')\delta_{ll'} \tag{6}$$

as well as the symmetry relations

$$F^{n-\nu}_{jsl}(\alpha)=(-1)^{l+s-j}F^{n\nu}_{jsl}(\alpha)\ , \tag{7}$$

$$F^{n\nu}_{jsl}(\alpha)^{*}=F^{n\nu}_{jsl}(-\alpha)\ . \tag{8}$$

Parity conservation implies $A^{j}_{lsl's'}(\alpha)=\eta(-1)^{l+l'}A^{j}_{lsl's'}(\alpha)$ where $\eta=\eta_1\eta_2\eta_3\eta_4$ is a product of all intrinsic parities. In terms of the O(4) amplitudes parity conservation gives

$$T^{n\nu}_{jss'\lambda'}=\eta(-1)^{s-s'}T^{n-\nu}_{jss'-\lambda'}\ . \tag{9}$$

If the initial- or final-state particles are identical we have

$$A^j_{lsl's'}=(-1)^{l'+s'}A^j_{lsl's'} \text{ or } A^j_{lsl's'}=(-1)^{l+s}A^j_{lsl's'} ,$$

i.e., for the O(4) amplitudes we obtain

$$T^{n\nu}_{jss'\lambda'}=(-1)^j T^{n\nu}_{jss'-\lambda'}=(-1)^j T^{n-\nu}_{jss'\lambda'} . \qquad (10)$$

The Pauli principle (e.g., for $pp \to pp$ and $nn \to nn$) or the generalized Pauli principle (a consequence of isospin invariance, e.g., for $np \to np$) implies

$$T^{n-\nu}_{jss'-\lambda'}=T^{n\nu}_{jss'\lambda'} . \qquad (11)$$

Time-reversal invariance and unitarity lead to more complicated conditions on the O(4) amplitudes. The expansion (4) in terms of the functions (5) after summation over l' reduces to

$$M_{s\sigma s'\sigma'}(\alpha,\theta,\phi)=\sum_{l=|\sigma'-\sigma|}^{\infty}\sum_{j=|l-s|}^{l+s}\sum_{\nu=-\min(j,s)}^{\min(j,s)}\sum_{n=\max(j,s)}^{\infty}\left[\frac{4\pi}{2j+1}\right]^{1/2}(l\sigma'-\sigma s\sigma\,|\,j\sigma')Y_{l\sigma'-\sigma}(\theta,\phi)$$

$$\times[(n+1)^2-\nu^2]T^{n\nu}_{jss'\sigma'}F^{n\nu}_{jsl}(\alpha)^* . \qquad (12)$$

III. NUCLEON-NUCLEON SCATTERING IN THE SINGLET-TRIPLET FORMALISM

Two reviews[8,9] of the nucleon-nucleon scattering formalism have recently been published and we refer to them for the relation between different types of scattering amplitudes, for formulas expressing observables in terms of amplitudes, etc. We shall now specialize the expansions (12) to the NN case. We assume parity conservation, time-reversal invariance, and the validity of the (generalized) Pauli principle, so that NN scattering is characterized by five independent spin amplitudes.[8] The condition

$$A^j_{lsl's'}=(-1)^{s-s'}A^j_{lsl's'} \qquad (13)$$

following from parity conservation and the generalized Pauli principle implies the absence of singlet-triplet transitions. Singlet- and triplet-state scattering can then be considered separately and we put

$$\widetilde{M}_{SS}(\alpha,\theta,\phi)=M_{0000}(\alpha,\theta,\phi) ,$$

$$\qquad (14)$$

$$\widetilde{M}_{\sigma\sigma'}(\alpha,\theta,\phi)=M_{1\sigma1\sigma'}(\alpha,\theta,\phi) .$$

The entire scattering matrix can then be written as

$$M^{ST}=\begin{bmatrix} M_{11} & M_{10}e^{-i\phi} & M_{1-1}e^{-2i\phi} & 0 \\ M_{01}e^{i\phi} & M_{00} & -M_{01}e^{-i\phi} & 0 \\ M_{1-1}e^{2i\phi} & -M_{10}e^{i\phi} & M_{11} & 0 \\ 0 & 0 & 0 & M_{SS} \end{bmatrix} , \qquad (15)$$

where

$$\widetilde{M}_{ab}(\alpha,\theta,\phi)=M_{ab}(\alpha,\theta)e^{i(b-a)\phi} , \qquad (16)$$

and time-reversal invariance implies

$$(M_{00}-M_{11}+M_{1-1})\sin\theta + \sqrt{2}(M_{01}+M_{10})\cos\theta=0 . \qquad (17)$$

The expansion functions for singlet ($s=0$) states are

$$d^n_l(\alpha)\equiv F^{n0}_{l0l}(\alpha) = e^{-i\pi l/2}2^l l!\left[\frac{2l+1}{n+1}\frac{(n-l)!}{(n+l+1)!}\right]^{1/2} \times \sin^l\alpha\, C^{l+1}_{n-l}(\cos\alpha) , \qquad (18)$$

where $C^{l+1}_{n-l}(\cos\alpha)$ are Gegenbauer polynomials. For triplet ($s=1$) states we define

$$g^n_{jl}(\alpha)\equiv F^{n0}_{j1l}(\alpha)=\frac{1-(-1)^{j-l}}{2}g^n_{jl}(\alpha) ,$$

$$\qquad (19)$$

$$h^n_{jl}(\alpha)\equiv F^{n1}_{j1l}(\alpha)=-(-1)^{j-l}F^{n-1}_{j1l}(\alpha) .$$

For singlet states the O(4) expansion (12) reduces to

$$M_{SS}(\alpha,\theta)=\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}(n+1)^2 S^n_l P_l(\cos\theta)d^n_l(\alpha)^* , \qquad (20)$$

where $S^n_l\equiv T^{n0}_{l000}$.

For triplet states we have

$$\widetilde{M}_{\sigma\sigma'}(\alpha,\theta,\phi)=\sum_{ljn\nu}\sqrt{4\pi}(l\sigma'-\sigma 1\sigma\,|\,j\sigma')Y_{l\sigma'-\sigma}(\theta,\phi)[(n+1)^2-\nu^2]T^{n\nu}_{j\sigma'}F^{n\nu}_{j1l}(\alpha)^*\ , \tag{21}$$

where $T^{n\nu}_{j\sigma'}\equiv T^{n\nu}_{j11\sigma'}/(2j+1)^{1/2}$.

The expansions (20) and (21) can be interpreted as being the partial-wave expansion (2) in terms of the spherical harmonics $Y_{l\sigma'-\sigma}(\theta,\phi)$ supplemented by an O(4) expansion of the partial-wave amplitude $A^j_{lsl's'}(\alpha)$ as a function of energy. This expansion can be read off from formula (4). Specializing to the nucleon-nucleon case and using the definitions (19), we obtain the following expansions of the singlet and triplet partial-wave amplitudes:

$$A^j_{j0j0}(\alpha)=\frac{1}{2j+1}\sum_{n=j}^{\infty}(n+1)^2S^n_j d^n_j(\alpha)^*\ , \tag{22}$$

$$A^j_{l1l'1}(\alpha)=\frac{1}{(2l+1)^{1/2}}\sum_{n=j}^{\infty}\sum_{\sigma'}(l'01\sigma'\,|\,j\sigma')$$
$$\times\{\,(n+1)^2T^{n0}_{j\sigma'}g^n_{jl}(\alpha)^*+n(n+2)[T^{n1}_{j\sigma'}-(-1)^{j-l}T^{n1}_{j-\sigma'}]h^n_{jl}(\alpha)^*\,\}\ . \tag{23}$$

The sum over σ' in (23) can easily be performed for each special case to yield for the triplet states

$$A^l_{l1l1}(\alpha)=\left[\frac{2}{2l+1}\right]^{1/2}\sum_{n=l}^{\infty}n(n+2)[T^{n1}_{l-1}-T^{n1}_{l1}]h^n_{ll}(\alpha)^*\ ,$$

$$A^{l+1}_{l1l1}(\alpha)=\frac{1}{(2l+1)}\sum_{n=l+1}^{\infty}(\,(n+1)^2\{\,(l+1)^{1/2}T^{n0}_{l+10}+[2(l+2)]^{1/2}T^{n0}_{l+11}\,\}g^n_{l+1l}(\alpha)^*$$
$$+n(n+2)\{\,2(l+1)^{1/2}T^{n1}_{l+10}+[2(l+2)]^{1/2}(T^{n1}_{l+11}+T^{n1}_{l+1-1})\,\}h^n_{l+1l}(\alpha)^*)\ ,$$

$$A^{l-1}_{l1l1}(\alpha)=\frac{1}{(2l+1)}\sum_{n=l-1}^{\infty}(\,(n+1)^2\{\,-\sqrt{l}\,T^{n0}_{l-10}+[2(l-1)]^{1/2}T^{n0}_{l-11}\,\}g^n_{l-1l}(\alpha)^*$$
$$+n(n+2)\{\,-2\sqrt{l}\,T^{n1}_{l-10}+[2(l-1)]^{1/2}(T^{n1}_{l-1-1}+T^{n1}_{l-11})\,\}h^n_{l-1l}(\alpha)^*)\ , \tag{24}$$

$$A^{l+1}_{l1l+21}(\alpha)=\frac{1}{[(2l+1)(2l+5)]^{1/2}}$$
$$\times\sum_{n=l+1}^{\infty}(\,(n+1)^2\{\,-(l+2)^{1/2}T^{n0}_{l+10}+[2(l+1)]^{1/2}T^{n0}_{l+11}\,\}g^n_{l+1l}(\alpha)^*$$
$$+n(n+2)\{\,-2(l+2)^{1/2}T^{n1}_{l+10}+[2(l+1)]^{1/2}(T^{n1}_{l+1-1}+T^{n1}_{l+11})\,\}h^n_{l+1l}(\alpha)^*)\ ,$$

$$A^{l+1}_{l+21l1}(\alpha)=\frac{1}{[(2l+1)(2l+5)]^{1/2}}$$
$$\times\sum_{n=l+1}^{\infty}(\,(n+1)^2\{\,(l+1)^{1/2}T^{n0}_{l+10}+[2(l+2)]^{1/2}T^{n0}_{l+11}\,\}g^n_{l+1l+2}(\alpha)^*$$
$$+n(n+2)\{\,2(l+1)^{1/2}T^{n1}_{l+10}+[2(l+2)]^{1/2}(T^{n1}_{l+1-1}+T^{n1}_{l+11})\,\}h^n_{l+1l+2}(\alpha)^*)\ .$$

Time-reversal invariance implies $A^{l+1}_{l+21l1}(\alpha)=A^{l+1}_{l1l+21}(\alpha)$ and this condition should be imposed as a constraint in any data fitting.

All expansions have so far been written for *np* scattering under the assumption of isospin invariance.

Thus, relation (9) with $\eta=1$ and $s=s'$ has been used to eliminate terms with $\nu=-1$, but relations (10) were never used. For identical particles, i.e., pp or nn scattering we have the additional condition

$$T^{n\nu}_{j\sigma'}=(-1)^j T^{n\nu}_{j-\sigma'} \tag{25}$$

to be imposed in expansions (24).

Condition (25) implies that l is even for singlet states and odd for triplet ones and that we can put

$$T^{n1}_{l1}-T^{n1}_{l-1}=2T^{n1}_{l1} \,, \qquad T^{n1}_{l\pm11}+T^{n1}_{l\pm1-1}=2T^{n1}_{l\pm11} \tag{26}$$

throughout (for l odd).

This completes the description of the O(4) formalism proposed for performing energy-dependent phase-shift analysis for np and pp scattering.

In order to establish the relation between our amplitudes $A^j_{lsl's'}$ and the amplitudes $R^j_{lsl's'}\equiv R(lsjm_j;l's'j'm_j')\delta_{jj'}\delta_{m_jm_j'}$ used by Stapp *et al.*,[4] we notice that their expansion can be written as

$$M^S_{s\sigma s'\sigma'}(\alpha,\theta,\phi)=(ik)^{-1}\sum_{ll'j} e^{-i(\pi/2)(l-l')}[\pi(2l'+1)]^{1/2}(l\sigma'-\sigma s\sigma\,|\,j\sigma')$$
$$\times(l'0s'\sigma'\,|\,j\sigma')R^j_{lsl's'}(\alpha)Y_{l\sigma'-\sigma}(\theta,\phi)\,. \tag{27}$$

Comparing with (2) we have

$$A^j_{lsl's'}(\alpha)=\frac{e^{i(\pi/2)(l'-l)}}{2ik}\frac{2j+1}{[(2l+1)(2l'+1)]^{1/2}}R^j_{lsl's'}(\alpha)\,. \tag{28}$$

Using their notations[4] for the nonvanishing partial-wave amplitudes, we have

$$\begin{aligned} A^l_{l0l0}&=\frac{1}{2ik}R_l,\quad A^l_{l1l1}=\frac{1}{2ik}R_{ll}\,,\\ A^j_{l1l1}&=\frac{1}{2ik}\frac{2j+1}{2l+1}R_{lj}\quad (j=l\pm1)\,,\\ A^j_{j-11j+11}&=A^j_{j+11j-11}=-\frac{1}{2ik}\frac{2j+1}{[(2j-1)(2j+3)]^{1/2}}R^j\,. \end{aligned} \tag{29}$$

IV. CONCLUSIONS

The kinematical part of our program concerning O(4) expansions of scattering amplitudes is completed with this article. The next step is to turn to dynamical questions, in particular the incorporation of electrodynamic effects, the treatment of pion-exchange amplitudes and the analysis of various models for NN scattering. Some work in this direction for the scattering of spinless particles[10] has already been performed, in particular in application to pion-pion scattering.[11] The most important envisaged continuation of this paper is to apply it to analyze existing[12] and forthcoming experimental NN data, in hopes of obtaining the NN scattering amplitudes as functions of energy and angle from threshold to about 1 GeV. Earlier applications of O(4) expansions to data analysis concerned particle decays like $K\rightarrow3\pi$, $\eta\rightarrow3\pi$, and $\bar{p}n\rightarrow3\pi$ annihilation at rest.[13]

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