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Dynamical mass generation in continuum quantum chromodynamics

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We study the formation of a mass gap, or effective gluon mass (and consequent dimensionful parameters such as the string tension, glueball mass, $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$, correlation lengths) in continuum QCD, using a special set of Schwinger-Dyson equations. These equations are derived from a resummation of the Feynman graphs which represent certain *gauge-invariant* color-singlet Green's functions, and are themselves essentially gauge invariant. This resummation is essential to the multiplicative renormalizability of QCD in the light-cone gauge, which we adopt for technical reasons. We close the dynamical equations by "solving" a Ward identity, a procedure which, while exact in the infrared regime, is subject to ambiguities and corrections in the ultraviolet regime which are beyond the scope of the present work. (These ambiguities are less prominent for QCD in three dimensions, which we discuss also.) As discussed in an earlier work, quark confinement arises from a vortex condensate supported by the mass gap. Numerical calculations of the mass gap are presented, suggesting an effective gluon mass of 500±200 MeV and a 0^+ glueball mass of about twice this value.

I. INTRODUCTION

The extraction of dimensionful quantities (e.g., the string tension) in continuum QCD is a truly quantum-mechanical problem since the classical Lagrangian has no fixed scale of mass. The pioneering instanton/meron work of Callan, Dashen, and Gross[1] emphasized classical solutions which themselves have no fixed mass scale, and then attempted to introduce the renormalization-group mass through one-loop quantum corrections. However, even this difficult calculation failed to provide a definitive cutoff mechanism for infrared singularities, and it appears[2] that the proposed phase transition to a baglike state takes one uncomfortably close to the momentum scale at which the square of the one-loop running charge

$$\bar{g}^2(k)=[b\ln(-k^2/\Lambda^2)]^{-1} \tag{1.1}$$

turns negative and unphysical. [Here

$$b=\frac{11C_A}{48\pi^2} \tag{1.2}$$

is the lowest-order coefficient in the β function $\beta=-bg^3+\cdots;C_A$ is the Casimir eigenvalue of the adjoint representation if no quarks are present, as we shall assume, and $C_A=N$ for SU(N).]

Other authors have attempted to account for the presence of fluctuating color-magnetic fields in the QCD vacuum, beginning with the famous one-loop correction to the QCD Lagrangian for constant fields.[3] But this has a minimum only for unphysical values of $\bar{g}^2$; moreover, the minimum is unstable.[4] Even in three-dimensional ($d=3$) QCD (or equivalently, $d=4$ QCD at very high temperatures) which has a dimensionful parameter in the Lagrangian ($g^2\sim$mass) perturbation theory[5] is only useful at large momenta, just as for $d=4$, and the problem of infrared singularities remains unresolved.

It may well happen that continued work on merons, instantons, corrections to the Lagrangian, etc., ultimately leads to a systematic and practical picture of confinement in continuum QCD. But it would clearly be valuable to have a picture which allowed for a direct, intuitive grasp of the role of the infrared cutoff and how it is used in calculating various dimensionful quantities. Moreover, it must be shown that such a picture is systematically derivable from first principles. We offer here the first steps in such a derivation, which leads to the conclusion that the gauge fields are effectively described as massive. The gluon "mass" is not a directly measurable quantity, but must be related to other physical parameters by difficult calculations not yet done. Nevertheless the ideas behind these calculations are easily grasped, and semiquantitative estimates of, e.g., the string tension and glueball mass can be made.

We begin with a description of massive gluons at the Lagrangian level, emphasizing that this can be made locally gauge invariant. Although we speak of the mass in the Lagrangian context as if it were a constant, in fact a *dynamically generated* gluon mass depends on momentum and vanishes at large momentum. The reader will see that, at every stage of the work, this vanishing at large momentum is essential to achieve finite results.

A. QCD with massive gluons

For some time, the author has advocated[6] that the strong interactions of QCD generate a dynamical gluon mass. Obviously such a mass will regulate all infrared singularities and yield a rather direct qualitative understanding of many dimensionful parameters of QCD, including (perhaps not so obviously) the string tension. The most direct manifestation of the gluon mass is a perimeter law for adjoint-representation Wilson loops, as expected[7] from strong-coupling expansions in lattice gauge theories, and indirectly verified[2] in Monte Carlo simulations from the energy needed to materialize gluon pairs from the vacuum.

Although we argue for a dynamically generated gluon mass, much can be learned from the kinematical description of gluon mass via a locally gauge-invariant Lagrangian.[9] It has been argued[10] that this theory is not perturbatively renormalizable in $d=4$, but this is based on the persistence of a mass term at very large momentum. Dynamically generated masses must, however, vanish at large momentum,[11] which we will demonstrate explicitly in this paper for QCD. It should be noted that a locally gauge-invariant description of gluon mass demands the existence of massless scalar fields, which—just like their Goldstone counterparts in spontaneously broken gauge theories—do not appear in the S matrix. We emphasize that there is no spontaneous symmetry breaking associated with gluon mass generation.

There are classical solutions for the massive gauge-invariant Lagrangian, such as a finite-energy $J^P=0^+$ glueball (first paper of Ref. 6), vortices, and Euclidean solutions which are the counterparts of instantons (both discussed in the second paper of Ref. 9). All are of finite size, with field strengths exponentially damped at large distances. However, the vortices have a long-range pure-gauge term in their gauge potentials, which endows them with a topological quantum number corresponding to the center of the gauge group [Z_N for SU(N)] and which is responsible for quark confinement. Given that the true dynamically generated mass vanishes at short distances, the action per unit area of a vortex is finite, and it is then very likely that the QCD vacuum is a vortex condensate characterized by $\langle G_{\mu\nu}\rangle=0$, $\langle \mathrm{Tr} G_{\mu\nu}{}^2\rangle\neq 0$ (second paper of Ref. 6). (If vortices do not condense, the mass then appears in a screened phase.) It is widely believed that the QCD vacuum is a tangle of fluctuating color fields, and Shifman, Vainshtein, and Zakharov[12] have given sum rules which can be used for an experimental determination of $\langle \mathrm{Tr} G_{\mu\nu}{}^2\rangle$.

We are now in a position to resolve the apparent paradox of producing quark confinement with massive gluons and attendant short-range forces. The key is the vortex condensate with its short-range fluctuations, as a simple argument for a massive *Abelian* gauge theory shows. To begin with, assume Gaussian statistics for the gauge-field fluctuations. With the use of Stoke's theorem, the Wilson-loop expectation value is transformed:

$$\begin{aligned} W &\equiv \left\langle \exp\left[ig \oint dx_\mu A_\mu \right]\right\rangle \\ &\simeq \exp\left[-\tfrac{1}{2} g^2 \oint dx_\mu \oint dx'_\nu \langle A_\mu A_\nu \rangle \right] \\ &= \exp\left[-\tfrac{1}{2} g^2 \int d\sigma_{\mu\nu} \int d\sigma'_{\alpha\beta} \langle G_{\mu\nu} G_{\alpha\beta}\rangle \right] . \end{aligned} \tag{1.3}$$

Then, with the assumption that the gauge fields are correlated inside cells of area $\Delta\sigma$, but completely uncorrelated in different cells, (1.3) becomes an area law:

$$W=\exp\left[-\frac{N}{2}g^2(\Delta\sigma)^2\langle G^2\rangle\right] , \tag{1.4}$$

where N is the number of cells in an area spanning the loop (i.e., the actual area is $N\Delta\sigma$). This sort of argument has been given before[13] and our version of it is not quite enough for non-Abelian gauge theories where, for one thing, the correct Stokes theorem must be used.[14] Moreover, in the naive form of the argument we have given, an area law is predicted for adjoint Wilson loops as well as quarklike loops. But it must be recognized that vortex fluctuations are quantized in such a way that for adjoint loops the flux $g\oint dx_\mu A_\mu$ is $2\pi N$ where N is an integer, and integral fluctuations in N do not affect the Wilson loop at all. This is made clear in earlier derivations of the Wilson-loop expectation value based directly on a condensate of vortices.[15] Later we will use a formula like (1.4) to estimate the string tension in terms of the gluon "mass."

Incidentally, it is worth noting that a short-ranged gauge field with finite flux is necessarily associated with a long-range gauge potential, as Stokes's theorem immediately shows. The (pure-gauge) long-range part of the potential is derived from the massless scalar fields which are necessary to maintain local gauge invariance in the presence of a gluon mass.

There is a close connection between the vortex condensate formed from massive gluons and the vacuum structure advocated by the Copenhagen school,[16] which is essentially a quantum liquid of vortexlike objects formed from the Nielsen-Olesen[4] unstable modes. There is no mass as such in the Copenhagen description, but there are tantalizing hints of dynamical mass mechanisms. In the present work we will not discuss the vacuum any further than to argue that $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle \neq 0$, which means that we cannot fully describe the long-distance dynamics of gluons; this depends on details of fluctuations in the vacuum structure which we do not yet understand.

The physical picture underlying the present work is that the forces between gluons are certainly strong enough to produce bound states, notably a $J^P = 0^+$ color-singlet glueball described by a composite field $\phi(x)$ whose vacuum expectation value is directly related to the gluon mass and to $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$. We can guess which channels are likely to have bound states simply by looking for attractive forces, and it is easy to see that the most attractive forces between two gluons are in the 0^+ color-singlet state. Here the direct Coulombic forces have strength $C_A g^2/4\pi$, and if we take $g^2 \sim b^{-1}$ with logarithmic accuracy [see Eq. (1.1)] this strength is $12\pi/11 \sim 3.4$. Additional *short-range* attraction comes from gluon spin-spin forces. Since the critical coupling for overbinding Coulombic systems (i.e., producing tachyons) is likely to be $\alpha \simeq 2$, it may well be that the Coulomb approximation leads to a tachyonic 0^+ state; in this case the composite field $\phi(x)$ will be forced to have a vacuum expectation value (VEV). Of course, since no symmetry breaking is associated with the ϕ field, there is no reason in general why it should not have a VEV. We will see below that this VEV leads to a gluon mass, which one might interpret as the Coulombic force law changing to a Yukawa force to prevent overbinding.

The next most strongly bound channel is the 0^+ color octet, the channel which carries the massless fields which are necessary to maintain local gauge invariance for massive gluons (third paper of Ref. 9). These, too, are composites resulting from strong binding.

The fact that these forces are attractive is directly related to the vanishing of the momentum-dependent dynamical mass at large momentum (i.e., the mass operator has positive anomalous dimension). The rate at which the mass term vanishes, which renormalization-group-improved perturbation theory gives as $(\ln k^2)^{-12/11}$, is directly related to the strength of the forces which we gave above as $12\pi/11$. (A more likely nonperturbative alternative is the behavior $k^{-2}(\ln k^2)^{+12/11}$, as argued by Lane and by Pagels.[17])

Essentially the same physical picture underlies the current MIT bag model[18]: the 0^+ bag state is overbound into a tachyonic state, and the consequent VEV provides a vacuum structure filled with empty bags. Color fluctuations in different bags are uncorrelated, so the bag radius corresponds in some sense to the inverse gluon mass.

Other authors[19] have attempted the problem of dynamical gluon mass generation before, but these works have suffered from lack of gauge invariance and/or Lorentz invariance. Below we will take up some of the technical issues which allow these critical invariances to be preserved in the present calculation.

B. How to generate a dynamical mass

Our general approach is to study equations of Schwinger-Dyson-type for certain Green's functions. This immediately leads to a problem: the conventionally defined gluon Green's functions of QCD are gauge dependent, and in themselves contain no physics. In contrast, the photon propagator of QED has only a trivial gauge dependence in the free propagators, with a gauge-invariant and physical vacuum polarization tensor.

It turns out that the same happy state of affairs can be reached in QCD by rearranging the Feynman graphs which contribute to *gauge-invariant* amplitudes. A new gluon "propagator" emerges, whose only gauge dependence is in the free propagator; this new propagator has pieces which come from three- and higher-point functions as well as from the usual two-point functions. That such a propagator could be defined was noted some time ago[20] and used in leading-logarithm summations of perturbation theory. Here we make the further observation that the new propagator obeys a Schwinger-Dyson equation involving itself as well as a new vertex related to the new propagator by a Ward identity. The new vertex likewise satisfies a dynamical equation involving a four-point kernel, but we will not be concerned with that equation here.

While this resummation of graphs is to some extent merely a convenience in covariant gauges, it is a necessity in the gauge we use in this paper: the light-cone gauge. We will show that even at the one-loop level the conventionally defined light-cone gauge propagator cannot be multiplicatively renormalized, but that our new propagator is precisely the linear combination of Green's functions that can be renormalized. In principle it does not matter what gauge we start with; the modified propagator always has the same proper self-energy, just as in QED. But the light-cone gauge is very convenient because it has no ghosts and because the rules for integration in momentum space are very simple (third paper of Ref. 9). A peculiar integration singularity discussed in the above references is of no concern, since it does not appear in gauge-invariant quantities.

The nonlinear propagator equation and its solution are at the heart of our study, since the resolution of all infrared problems in QCD stems from them. We will show that only a massive solution exists, which leads directly to a perimeter law for adjoint Wilson loops. Unfortunately, a quantitative expression of the (finite part of the) perimeter law in terms of the propagator mass is not easy, because the full structural complications of a non-Abelian gauge theory still persist even though the infrared singularities are gone. It is no simpler to express precisely such quantities as the string tension or $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$ in terms of the propagator mass, but in all cases we can make reasonable estimates which we might hope are accurate to within less than a factor of 2, given precise knowledge of the propagator mass. These estimates can be systematically improved, with no new problems of principle arising.

We do not, in fact, have a precise value of the propagator mass, because the propagator equation can be made self-contained only by approximating the vertex which appears in it, and because of uncertainties which arise in regulating the seagull graph (or equivalently regulating $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$). Let us list and comment upon the necessary approximations and the difficulties they produce.

1. "Solving" the Ward identity

The vertex which appears in the propagator equation obeys a Ward identity of the simple type encountered in ghost-free gauges, relating the divergence of the vertex to the inverse propagator. There are essentially two different ways (depending on whether one does or does not allow massless poles in the vertex) of writing down an expression in terms of the propagator for the part of the vertex (called the longitudinal part) which exactly satisfies the Ward identity on one of the vertex legs. To this one can, of course, add an arbitrary completely conserved (transverse) vertex part. But this unknown transverse part has two properties which make usable the first approximation of dropping it altogether: (1) in a theory with a mass gap, it vanishes by at least one power of momentum compared to the longitudinal part at small momentum; (2) the transverse part is $O(g^2)$ compared to the longitudinal part. Because of the first property, using only the longitudinal part exactly reproduces the low-energy structure of the theory, and because of the second, the ultraviolet behavior of the propagator is correctly given through one-loop graphs by dropping transverse parts. If only the longitudinal part is kept in the propagator equation, it becomes a self-contained nonlinear equation for the propagator.

The idea of solving Ward identities is quite old, and was revived a few years ago[21] when it was realized that it allowed for *linearization* of the Schwinger-Dyson equation for fermion propagators in a gauge theory. It was then used by Baker, Ball, Zachariasen, *et al.*[22] in their program of studying the equations for the conventional (gauge-dependent) gluon propagator in the axial gauge. Their results are quite different from ours, which can be traced directly to the fact that they do not allow massless scalar poles in their vertex function; thus they explicitly exclude the possibility of dynamical mass generation.

Throughout this paper we will only use the longitudinal vertex, completely dropping any transverse part. (It is worth noting that this approximation can be *systematically* improved, as will be discussed elsewhere.) This approximation yields some peculiar results in the ultraviolet regime which we now discuss.

2. Renormalizing the approximate equation

It turns out that dropping transverse vertex parts leads to mishandling overlapping divergences, with consequent ambiguities in renormalizing. Multiplicative renormalization fails in $O(g^4)$, but the equation can be made finite by subtractive renormalization [the two procedures are equivalent in $O(g^2)$]. The renormalized propagator then fails to have the correct large-k behavior at $O(g^4)$; for example, a simplified version of the propagator equation yields an inverse propagator $\sim k^2[1+2bg^2\ln(-k^2/\mu^2)]^{1/2}$ instead of $k^2[1+bg^2\ln(-k^2/\mu^2)]$. This sort of difficulty is formally expressed in erroneous values for renormalization-group coefficients; thus the β function for the approximate equation is $\beta=-2bg^3+\cdots$ instead of $-bg^3$. (The same sort of error occurs in Ref. 22, where 2 is replaced by $\frac{16}{11}$.) These errors lead to a slight but annoying dependence on the renormalization mass μ of physical quantities which should be independent of μ, but this dependence is only an artifact of the approximation used and can be *systematically* removed.

Such problems are special to $d=4$ in the sense that no renormalization is needed for $d=3$, so we have also considered this case. The propagator equation leads to mass generation just as in $d=4$, with the mass[23,24] $m\simeq bg^2$, where $b=15C_A(32\pi)^{-1}$. We will discuss the $d=3$ case only briefly, deferring details to another work.

3. *Regulating the seagull graph*

In order for there to be a gluon mass, a certain gauge-invariant projection of the seagull graph must be given a finite, nonzero value: the integral

$$\int d^4k\, d(k^2)\,, \tag{1.5}$$

where $d(k^2)$ is our modified gauge-invariant propagator, must exist after suitable regularization, and its value is then proportional to m^2, where m is the gluon mass. A similar regularization procedure is necessary to define $\langle \mathrm{Tr}G_{\mu\nu}{}^2\rangle$, which has terms like $k^2d(k^2)$ as the integrand in (1.5). The dimensional regularization rules for $d=3.4$,

$$\int \frac{d^dk}{k^2}\ln^N k^2=0,\quad N=0,1,2,\ldots \tag{1.6}$$

ensure the masslessness of the gluon order by order in perturbation theory, but it is problematical to use this in the sum of perturbative contributions to $d(k^2)$ since these have a Landau ghost pole somewhere. We will show below that if there is a mass term in the propagator which vanishes at large k like a power of $\ln k^2$, then (1.5) is in fact finite, as is necessary for the self-consistency of the theory. But this turns out to be of little practical help in dealing with the approximate propagator equation, so we discuss instead physically motivated *approximations* to (1.5) which have the essential property that (1.5) is positive in Euclidean space (for positive d). One scheme relates the integral (1.5) directly to the expectation value $\langle\phi\rangle$ of the composite 0^+ state, and yields a finite value which is then used in a self-consistency equation (rather like a BCS gap equation) to determine the gluon mass in terms of Λ. This phenomenological regularization is so severe that a finite value for (1.5) obtains even for a gluon mass not decreasing at large k; as a result, the gluon mass may be underestimated $(m\simeq 1-2\Lambda)$.

A better way is to relate m to $\langle \mathrm{Tr}G_{\mu\nu}{}^2\rangle$. We give an exact formula for this expectation value in terms of the Euclidean vacuum self-energy. In turn, this latter quantity lies a finite and calculable amount below the perturbative vacuum self-energy, given a dynamical gluon mass decreasing at large k. There are two ways of estimating the vacuum energy: a graphically oriented Hartree expansion, and a direct regulation of $\langle \mathrm{Tr}G_{\mu\nu}{}^2\rangle$ based on saving only the contribution of the fluctuating vertex condensate (second paper of Ref. 6). The value of m found these ways varies from 350 to 600 MeV.

4. *Omission of three-gluon skeleton graphs*

The skeleton graphs (with dressed vertices and propagators) for the gluon propagator consist of the usual one-loop graphs, plus two-loop graphs which have three-gluon cuts. These can be omitted without interfering with gauge invariance (because they are of different order in g^2 than the graphs we save), and we will omit them because their inclusion not only complicates the technical problems enormously, but also because, to be consistent, we should then add transverse corrections to the gluon vertex (also of higher order in g^2). The same approximation has been made in Ref. 22.

C. Current status of the dynamical-mass problem

Our goal is to understand the gluon mass and all other dimensionful parameters as numerical multiples of Λ. It is a goal which cannot even in principle be met precisely, with the above approximations, since Λ itself is not well defined until two-loop terms are retained. We will therefore have to be content with going (in some approximate sense) only halfway to the goal, by using $\langle \mathrm{Tr}G_{\mu\nu}{}^2\rangle$ instead of Λ as our scale of mass, as described above in connection with regularizing the seagull graph. We repeat that the deficiencies of the present paper can be corrected *systematically,* and that this is only the beginning of a program whose end is not yet in sight.

The main accomplishments of this program so far are the following: (1) a reorganization of the Schwinger-Dyson equations of QCD into a form which allows us to deal directly with gauge-independent and renormalization-group-independent quantities; (2) an approximate version of one of these equations which respects gauge independence, which is accurate in the infrared, and which necessarily has a mass gap; (3) a demonstration that the mass vanishes at large momenta; (4) an exact relation between $\langle \mathrm{Tr}G_{\mu\nu}{}^2\rangle$ and the (Euclidean) vacuum self-energy, which yields an approximate value of 500 ± 200 MeV for the gluon "mass"; (5) a useful semiquantitative picture of the relation of this mass to the glueball mass and string tension. The two major unresolved problems are a systematic improvement (via a dressed-loop expansion) of the lowest-order results, and a demonstration that vortices actually condense in the vacuum (i.e., that their finite action is less than their entropy).

II. REARRANGEMENT OF GRAPHS IN LIGHT-CONE-GAUGE QCD

The light-cone gauge is specified by

$$n_\mu A^\mu=0,\;\; n^2=0 \tag{2.1}$$

(plus other conditions which do not enter the graphical analysis). Although we are interested at the moment in conventional massless QCD, it is convenient to give the Lagrangian for massive gauge-invariant QCD, since we will later need a few elementary Green's functions of the massive theory (second paper of Ref. 6):

$$\mathscr{L}=\tfrac{1}{2}\,\mathrm{Tr}G_{\mu\nu}{}^2-m^2\mathrm{Tr}[A_\mu-g^{-1}U(\theta)\partial_\mu U^{-1}(\theta)]^2\,, \tag{2.2}$$

where

$$A_\mu=\frac{1}{2i}\sum_a \lambda_a A_\mu^a,\;\; \mathrm{Tr}\lambda_a\lambda_b=2\delta_{ab}\,, \tag{2.3}$$

$$U(\theta)=\exp\left[i\sum \tfrac{1}{2}\lambda_a\theta^a\right]. \tag{2.4}$$

The Lagrangian (2.2) is invariant under

$$A'_\mu=VA_\mu V^{-1}-g^{-1}(\partial_\mu V)V^{-1}\,, \tag{2.5}$$

$$U'=U(\theta')=VU \tag{2.6}$$

for a group matrix V. The equations of motion for U, namely

$$D_\mu(A^\mu-g^{-1}U\partial_\mu U^{-1})=0 \tag{2.7}$$

can be solved as a power series in g (third paper of Ref. 9):

$$\theta_a=g\frac{1}{\Box}\partial_\mu A_a^\mu-\frac{g^2}{\Box}\left[\tfrac{1}{2}(\partial\cdot A)\times\frac{1}{\Box}(\partial\cdot A)+A^\mu\times\partial_\mu\frac{1}{\Box}\partial\cdot A\right]_a+\cdots \tag{2.8}$$

[where

$$(A\times B)_a=C_{abc}A_bB_c \tag{2.9}$$

in terms of the group structure constants]. The $\Box^{-1}$ terms in θ_a reveal the massless scalar fields necessary for gauge invariance. To (2.8) can be added solutions of the homogeneous version of (2.7); these are vortices and cannot be gauged away with the help of (2.6). In graphical applications we will always eliminate θ_a with (2.8).

Add the gauge-fixing term

$$\mathscr{L}_{\mathrm{GF}}=\eta^{-1}\mathrm{Tr}(n_\mu A^\mu)^2 \tag{2.10}$$

to (2.2) and find the free propagator for the massive theory:

$$\Delta_{\mu\nu}^{(m)}=\frac{P_{\mu\nu}}{q^2-m^2+i\epsilon}-\eta\frac{q_\mu q_\nu}{(n\cdot q)^2}\,, \tag{2.11}$$

$$P_{\mu\nu}=-g_{\mu\nu}+\frac{n_\mu q_\nu+n_\nu q_\mu}{n\cdot q}\,. \tag{2.12}$$

The free massless propagator will be called $\Delta_{\mu\nu}^{(0)}$. The gauge parameter η is to be set to zero at the end of a calculation; normally we will suppress

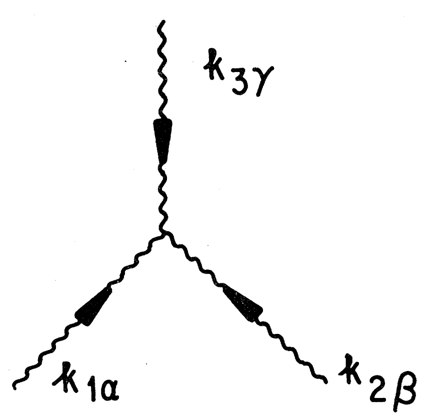

FIG. 1. Feynman graph for the massive-theory vertex of Eq. (2.13).

writing the η term.

For future use, we quote the lowest-order three-particle vertex for the massive theory (see Fig. 1):

$$\Gamma^{(m)}_{\alpha\beta\gamma}(k_1,k_2,k_3)=(k_1-k_2)_\gamma g_{\alpha\beta}+\frac{m^2}{2}\frac{k_{1\alpha}k_{2\beta}(k_1-k_2)_\gamma}{k_1{}^2k_2{}^2}+\text{c.p.}\,, \tag{2.13}$$

where c.p. means to add cyclic permutations. (An overall factor iC_{abc} is omitted.) This vertex obeys the Ward identity

$$k_1^\alpha\Gamma^{(m)}_{\alpha\beta\gamma}=\Delta^{(m)}_{\beta\gamma}(k_2)^{-1}-\Delta^{(m)}_{\beta\gamma}(k_3)^{-1} \tag{2.14}$$

which is true for the fully corrected vertex and propagators both for the massive and massless theories (third paper of Ref. 9). It reduces to the usual vertex for $m=0$.

The simple form of (2.14) is one reason for using the ghost-free light-cone gauge. There are two other reasons: (1) a scalar Green's function of one momentum cannot depend on n_μ; (2) momentum-space integrals involving n_μ are trivially reduced to n_μ-independent integrals. Neither of these statements holds for an axial gauge ($n^2\neq 0$). The first holds because Green's functions are homogeneous in n_μ of degree zero (as is the propagator) so a general scalar function of one momentum depends on q^2 and $n^2(n\cdot q)^{-2}$; the latter vanishes. The second statement is contained in the following theorem (third paper of Ref. 9). Let

$$R(p_i;q)=\frac{n\cdot(p_1-q)n\cdot(p_2-q)\cdots n\cdot(p_j-q)}{n\cdot(p_{j+1}-q)\cdots n\cdot(p_N-q)}\;; \tag{2.15}$$

then

$$\int d^dq\frac{R(p_i;q)}{[(p-q)^2-M^2]^l}=R(p_i;p)\int\frac{d^dq}{[(p-q)^2-M^2]^l}\,. \tag{2.16}$$

That is, an integration variable q is simply replaced by its shifted value in doing momentum-space integrals. This is an extraordinary simplification compared to the axial gauge.

In order to make useful gauge-invariant calculations, we introduce a set of heavy scalar test particles, described by a matrix $\Phi(x)$. These are in a representation R of the gauge group in which the center is trivially represented (e.g., the adjoint), which thus does not couple topologically to vortices. We will compute some contributions to the gauge-invariant Green's function

$$G(x,y)=\langle 0\,|\,T[\mathrm{Tr}\Phi^\dagger(x)\Phi(x)\mathrm{Tr}\Phi^\dagger(y)\Phi(y)]\,|\,0\rangle \tag{2.17}$$

omitting graphs with internal closed Φ loops. In the limit of very heavy Φ particles this Green's function is closely related to a Wilson loop with two long straight sides. Usually one thinks of G as being composed of gauge-variant pieces (e.g., propagators) which sum to a gauge-invariant total. But we will see that G can be rearranged into new propagators whose only gauge dependence is in their free parts.

A. Zero- and one-loop graphs

The relevant graphs are shown in Fig. 2 (there are other graphs, of course, but they are not needed for our purpose). We count loops as if the Φ loop were opened at x and y.

Each of these graphs has certain extraneous factors in common, such as Φ propagators and vertices, which we will leave out. As mentioned above, all our graphical calculations are done for conventional massless QCD.

With extraneous factors omitted, Figs. 2(a) and 2(b) are just the usual propagator to one-loop order. We find

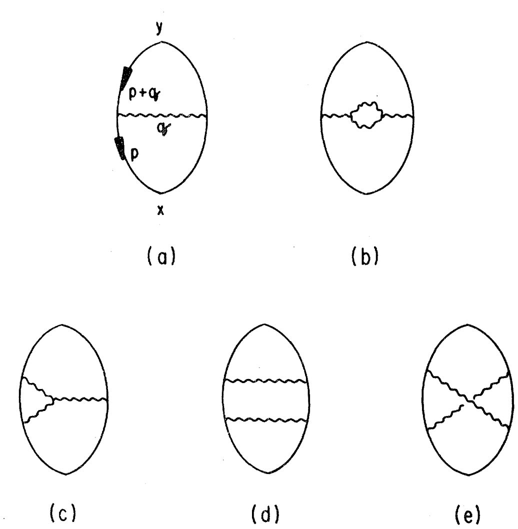

FIG. 2. Relevant graphs through $O(g^4)$ for constructing the gauge-invariant propagator.

$$\Delta_{\mu\nu}=\Delta^{(0)}_{\mu\nu}-\tfrac{1}{2}g^2C_A\left[\frac{P_{\mu\nu}}{q^2}\left[\tfrac{22}{3}I+8I'\right]+\frac{n_\mu n_\nu}{(n\cdot q)^2}(8I+8I')\right], \tag{2.18}$$

where

$$I=\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2(q-k)^2}, \tag{2.19}$$

$$I'=\frac{i}{(2\pi)^4}(n\cdot q)\int\frac{d^4k}{k^2(q-k)^2(n\cdot k)} \tag{2.20}$$

are logarithmically divergent integrals. According to (2.16), I' is independent of n_μ, but when it is dimensionally regularized it has a peculiar divergence in a Feynman-parameter integral (third paper of Ref. 9). This divergence is an artifact of the light-cone gauge and will cancel out in physical quantities.

One conclusion is apparent from (2.18)–(2.20): there is no choice of wave-function renormalization constant Z_3 which, divided into (2.18), makes it finite to $O(g^2)$. This is simply because divergences appear multiplying two independent kinematical tensors, only one of which appears in the free propagator. *The conventional gluon propagator is not multiplicatively renormalizable in the light-cone gauge.*

Now consider graphs 2(c)–2(e). These have uniquely definable parts which are tied to the Φ lines at only two points, and have no Φ propagators in them; we call these single-gluon parts, or SGP's, since they are kinematically equivalent to a propagator part. SGP's arise when longitudinal parts of a propagator numerator $P_{\mu\nu}$ or a vertex are applied to a Φ-Φ-gluon vertex, where they generate an elementary Ward identity. For example, the vertex of Fig. 1(a) is $(2p+q)_\mu$, and we note

$$q\cdot(2p+q)=D^{-1}(p+q)-D^{-1}(p), \tag{2.21}$$

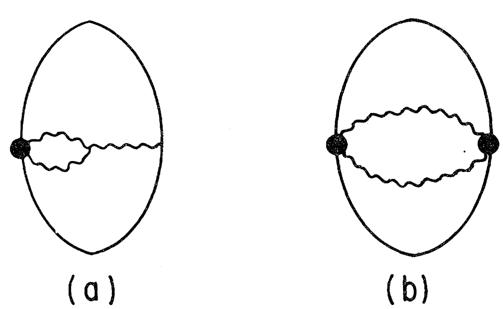

FIG. 3. Structure of the SGP's arising from Fig. 2.

where $D(p)$ is a Φ-propagator. Figure 3 shows the structure of SGP's resulting from Figs. 2(c)–2(e), with Figs. 2(d) and 2(e) both leading to the structure in Fig. 3(b). [In QED, the SGP of 2(d) cancels exactly with that of 2(e).] The SGP's calculated from Fig. 3 are

$$\text{Fig. 3(a):}\quad \tfrac{1}{2}g^2C_A\left[\frac{8P_{\mu\nu}I'}{q^2}+\frac{4n_\mu n_\nu}{(n\cdot q)^2}(2I+I')\right], \tag{2.22}$$

$$\text{Fig. 3(b):}\quad \tfrac{1}{2}g^2C_A\left[\frac{4n_\mu n_\nu I'}{(n\cdot q)^2}\right]. \tag{2.23}$$

When the SGP's are added to the conventional propagator, the result is the modified propagator which is the object of our attention, and which we call $\hat{\Delta}_{\mu\nu}$; its renormalized form is

$$\hat{\Delta}_{\mu\nu}=\frac{P_{\mu\nu}}{q^2}[1-bg^2\ln(-q^2/\mu^2)+\cdots], \tag{2.24}$$

where μ is the renormalization point. $\hat{\Delta}_{\mu\nu}$ can be renormalized multiplicatively because only one kinematic tensor, namely $P_{\mu\nu}$, appears in it. This feature persists to all orders of perturbation theory, as we show below.

B. Two- and higher-loop graphs

There are some useful things which are easily shown to be true to all orders, but as is not uncommon in graphical studies some aspects have been thoroughly investigated only through two loops and these extrapolation to all orders is speculative.

First we show to all orders that $\hat{\Delta}_{\mu\nu}$, defined as the sum of all propagator parts and SGP's, has an inverse which can be written

$$\hat{\Delta}_{\mu\nu}{}^{-1}=\Delta^{(0)-1}_{\mu\nu}-(q^2g_{\mu\nu}-q_\mu q_\nu)\hat{\Pi}(q^2). \tag{2.25}$$

The significance of this is that in general other kinematic forms of the self-energy are allowed, as in fact happened with the one-loop calculation of $\Delta_{\mu\nu}$; these other forms interfere with renormalizability and gauge invariance.

Clearly, $\hat{\Delta}_{\mu\nu}$ is symmetric in μ and ν, and a bit of thought shows that $n^\mu\hat{\Delta}_{\mu\nu}=0$. This is because forming an SGP does not affect the fact that the index μ or ν is part of a projection operator like

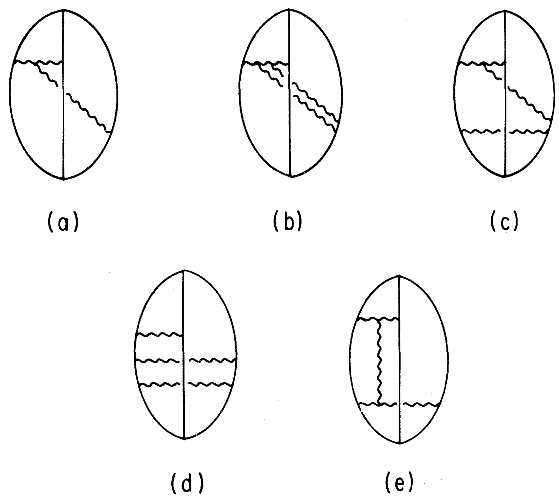

FIG. 4. Some of the graphs relevant to the new vertex $\hat{\Gamma}_{\alpha\beta\gamma}$.

$P_{\mu\alpha}$ associated with a free propagator. It then follows that $\hat{\Delta}_{\mu\nu}$ has the general form

$$\hat{\Delta}_{\mu\nu}=AP_{\mu\nu}+B\frac{n_\mu n_\nu}{(n\cdot q)^2}-\eta\frac{q_\mu q_\nu}{(n\cdot q)^2}\ , \qquad (2.26)$$

where the term in η comes from the free propagator. Simple algebra shows that

$$\hat{\Delta}_{\mu\nu}{}^{-1}= -A^{-1}\left[g_{\mu\nu}-\frac{g_\mu g_\nu}{q^2}\right] -\frac{B}{Aq^2(B+Aq^2)}\left[q_\mu-\frac{n_\mu q^2}{n\cdot q}\right]\left[q_\nu-\frac{n_\nu q^2}{n\cdot q}\right] -\frac{n_\mu n_\nu}{\eta} \qquad (2.27)$$

from which (2.25) follows if $B=0$. But B must vanish by gauge invariance, since the n-dependent terms in the gauge-invariant Green's function (2.17) which would arise from a term $\sim n_\mu n_\nu$ cannot be cancelled by other terms in the Green's function which are not SGP's. In contrast, the n dependence coming from $P_{\mu\nu}$ is associated with a longitudinal gradient which mixes, through Ward identities, SGP's, and other non-SGP graphs to cancel the n dependence.

The proper self-energy given in (2.25) is not only conserved, but completely independent of n_μ—that is, it is gauge invariant, just as in QED. The only remaining gauge dependence of $\hat{\Delta}_{\mu\nu}$ is in the free propagator, which must be gauge dependent in order to define the inverse propagator. We would arrive at the same proper self-energy $\hat{\Pi}_{\mu\nu}$ in any gauge.

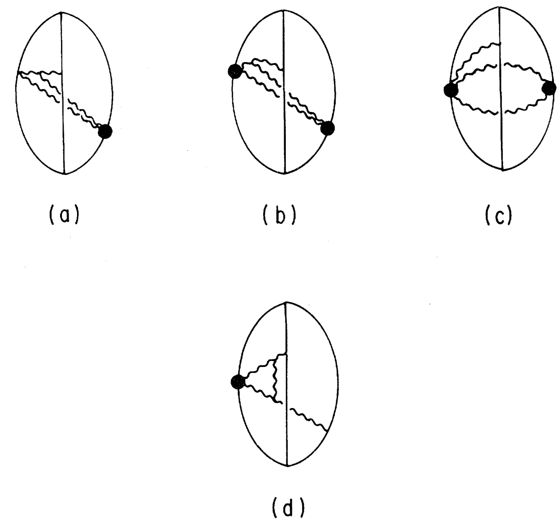

FIG. 5. Contributions of the graphs of Fig. 4 to the new (improper) vertex.

To understand the structure of $\hat{\Pi}_{\mu\nu}$ at the two-loop level and beyond we need to know something about a new vertex $\hat{\Gamma}_{\mu\nu\alpha}$ (and a corresponding seagull vertex which we will not discuss explicitly) and the Ward identity it satisfies. This vertex, like $\hat{\Delta}_{\mu\nu}$, receives contributions from pinching out Φ propagators, but this time we look at gauge-invariant processes with three lines, such as shown in Fig. 4. [For group-theoretical reasons, the lowest-order graph of Fig. 4(a) vanishes when the three lines are identical, even when these represent quarks,[24] so one has to think in terms of three different Φ fields or, say $\bar{q}q\Phi$ fields.] By pinching out lines, one comes to graphs like those of Fig. 5, which represent contributions to the improper vertex (schematically, $\hat{\Delta}\hat{\Delta}\hat{\Delta}\hat{\Gamma}$). There are many Feynman graphs which contribute to a single graph for $\hat{\Gamma}$, just as Figs. 2(d) and 2(e) give rise to Fig. 3(b).

It is straightforward to show that Γ obeys the Ward identity like (2.14) to all orders:

$$k_1^\alpha\hat{\Gamma}_{\alpha\beta\gamma}=\hat{\Delta}_{\beta\gamma}(k_2)^{-1}-\hat{\Delta}_{\beta\gamma}(k_3)^{-1}\ . \qquad (2.28)$$

(Incidentally, this shows the equality of wave-function and vertex renormalization constants.) Actually it is easiest to derive the Ward identity for the improper vertex graphs of Fig. 5. The canonical light-cone gauge Ward identities follow (third paper of Ref. 9) from applying $-\eta^{-1}(n\cdot k_1)n_\alpha$ to the improper vertex, with results easily seen to be equivalent to applying $k_{1\alpha}$ to the proper vertex [see (2.11)]. If $\eta^{-1}(n\cdot k)n_\alpha$ is applied to any of the heavily dotted vertices in Fig. 5 the

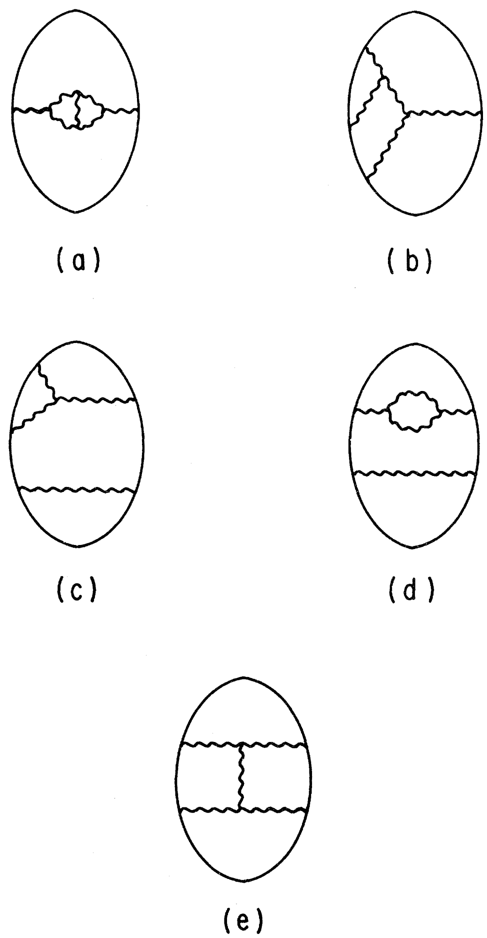

FIG. 6. Some relevant graphs at $O(g^6)$ for the Green's function (2.17).

result is zero. To see what happens when this is used on a normal line, it is simplest to generate the Ward identity *before* pinching out lines; that is, one uses the elementary Ward identity (2.14) (for $m=0$) in Fig. 4, and then pinches out lines (not using, of course, the propagator which had $\eta^{-1}(n\cdot k)n_\alpha$ used on it). Comparing the results with SGP's (e.g. Fig. 3) establishes the Ward identity. A similar Ward identity is derived for the new seagull vertex.

We are now in a position to discuss *systematically* the Schwinger-Dyson equation for $\hat{\Delta}$. In fact, this will not be done in detail because we lack a systematic discussion of transverse corrections to the vertex $\hat{\Gamma}$ (see the next section). This lack is not one of principle, but results only from computational difficulties, and a systematic treatment of transverse corrections could also be given.

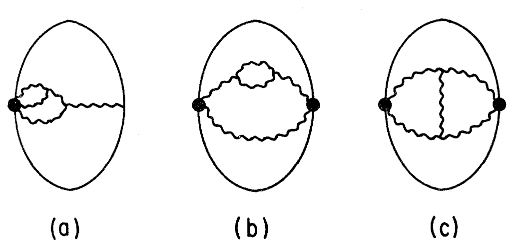

FIG. 7. SGP's coming from Fig. 6.

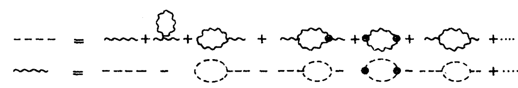

FIG. 8. Graphical representation of the gauge-invariant propagator $\hat{\Delta}$ (dashed line) in terms of the free propagator (wiggly line), and the inverse relationship.

The first step is to look at two- and higher-loop SGP's contributing to $\hat{\Delta}$. A few two-loop Feynman graphs are shown in Fig. 6, and their SGP's in Fig. 7. Note that (crossed) ladder graphs are missing; it is not hard to show that these have no SGP's if they have three or more rungs. In Fig. 7 we recognize the appearance of one-loop contributions to $\hat{\Delta}$ as internal lines, as well as one-loop vertex parts which appear in $\hat{\Gamma}$. However, the SGP of Fig. 3(b) never appears as an internal part of a two-loop SGP. This is as it should be, although it is not obvious.

In Fig. 8 we introduce a dotted line to represent the propagator $\hat{\Delta}$, which is expressed as a sum of Feynman graphs as shown by the top line. This is solved for the free propagator $\hat{\Delta}^{(0)}$ as on the bottom line of Fig. 8, and a dynamical equation derived by substituting this solution in the top equation of Fig. 8 everywhere except for the first term on the right. Similarly, one solves for the bare vertex $\Gamma^{(0)}$ in terms of $\hat{\Gamma}$ (and also for the seagull vertex). The result is an equation for $\hat{\Pi}_{\mu\nu}$ [see (2.25)] expressed in terms of $\hat{\Delta}$, $\hat{\Gamma}$, ... This is not necessarily as simple as the usual Schwinger-Dyson equation for Δ, the conventional propagator. We

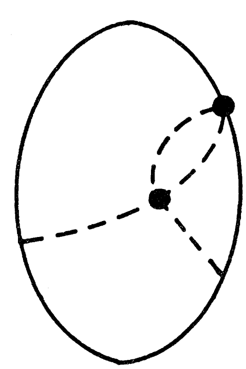

FIG. 9. A graph, written in terms of $\hat{\Delta}$, which might naively contribute to an SGP.

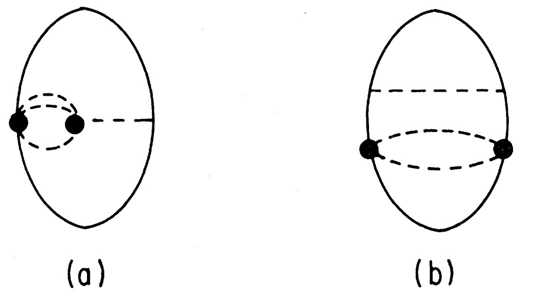

FIG. 10. (a) The nonexistent SGP which would naively stem from Fig. 9. (b) A graph with no SGP.

have looked at the two-loop equation generated by this procedure, and find many cancellations, which stem in part from the appearance of one-loop SGP's and terms in $\hat{\Gamma}$ as internal parts of the two-loop graphs. Other simplifications arise because the part of $\hat{\Delta}$ shown in Fig. 3(b) cannot generate SGP's, consistent—as noted above—with the fact that Fig. 3(b) does not appear as part of a two-loop SGP. In detail, note that Fig. 9 arises in the course of rewriting the Feynman graphs in terms of $\hat{\Delta}$; then one might expect to find the SGP of Fig. 10(a). But Fig. 10(a) does not exist, because Fig. 3(b) has the kinematic structure $n_\alpha n_\beta$ while the other propagators have $P_{\mu\nu}$ as numerators. Straightforward calculation shows that there can be no SGP in such a case. Similarly, Fig. 3(b) does not appear as an internal part in Fig. 7 because Fig. 10(b), which also arises in the course of rewriting the Feynman graphs in terms of $\hat{\Delta}$, has no SGP (as noted above, ladders with more than two rungs have no SGP's). Of course, these remarks about the contribution of specific graphs are only true in the light-cone gauge, and we have not discussed the details of the modified seagull vertex.

The upshot of our investigation at the two-loop level is that it is possible (but not yet proven) for

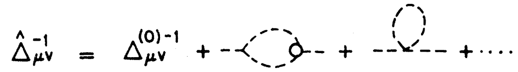

FIG. 11. The first few terms in the Dyson equation for $\hat{\Delta}$.

the Schwinger Dyson equation for $\hat{\Delta}$ to be as simple, in a sense, as for the conventional propagator. That is, the equation starts off as shown in Fig. 11, where we have not written two-loop graphs which have explicit modified seagull vertices (this truncation is gauge invariant; such graphs are higher order in g^2, in perturbation theory). The omitted terms, analogous to the new terms in Fig. 8, are such as to render $\Pi_{\mu\nu}$ gauge invariant, and we speculate that their effect can be found by omitting from the equation of Fig. 11 all terms which refer to n_μ [including the peculiar divergences exemplified in (2.20)]. This speculation is consistent with our approximate solution of the Ward identity $\hat{\Gamma}$, but it need not be used in the context of this approximation, since we will be able to show *explicitly* the cancellation of n_μ-dependent terms.

The reader who has bothered to read this section in any detail will be struck by the awkwardness and opacity of our graphical analysis. There surely must be a deeper principle underlying an algorithm which enforces such elementary requirements as multiplicative renormalizability and gauge invariance. In this vein, we offer another speculation. In ghost-free gauges, the field strengths uniquely determine the potential (at least in perturbation theory):

$$A_\mu = \frac{1}{n\cdot\partial} n^\nu G_{\mu\nu} \;. \tag{2.29}$$

A gauge-invariant propagator can be written by inserting a string operator between two field strengths, and using (2.29), so we speculate that

$$i\Delta_{\mu\nu}(x,x') = \frac{1}{n\cdot\partial n\cdot\partial'} \frac{n^\alpha n^\beta}{C_A} \left\langle 0 \left| T\left\{ G^a_{\mu\alpha}(x) P\left[\exp\left[ig\int_x^{x'} dz\cdot A\right]\right]_{ab} G^a_{\nu\beta}(x')\right\}\right| 0\right\rangle \,, \tag{2.30}$$

where the string factor is in the adjoint representation and the path is a straight line from x to x'. It is very simple to verify (2.30) to zeroth order in g^2, but not so simple at higher orders because of the string factors and associated divergences; however, calculations are in progress which show suggestive correspondences with the extra graphical pieces in Fig. 3.

III. "SOLVING" THE WARD IDENTITY

Ward identities can be solved either by directly working with differences of propagators, or by using spectral techniques based on the Lehmann representation.[21] In certain elementary cases, they give the same result, and in any event they differ only by transverse parts which are, for a theory with a mass gap, of higher order in the gauge-particle momentum when this momentum is small. However, in a theory with no mass gap (e.g., perturbative QCD) transverse vertex parts are found to be less singular only by a single logarithm than the longitudinal parts.[20] It has been pointed out[20] that the eikonal approximation to gauge theories (which is justified only if the charged particles are massive) is exactly equivalent to the propagator-difference solution to Ward identities with neglect of transverse parts. For example, in scalar QED the improper vertex in the eikonal limit is, to all orders,

$$D(p)D(p+q)\Gamma_\mu(p,p+q)=\frac{ip_\mu}{p\cdot q}[D(p)-D(p+q)]\ , \quad (3.1)$$

where D is the propagator of the massive charged scalar, and q is much less than p, which is nearly on shell. Clearly (3.1) obeys the Ward identity. Similar differences of propagators were used by Baker *et al.*,[22] but their theory has no mass gap and the neglect of transverse parts in the infrared might be hard to justify.

The Lehmann representation for D,

$$D(p^2)=\int d\lambda^2\frac{\rho(\lambda^2)}{p^2-\lambda^2+i\epsilon} \quad (3.2)$$

gives an essentially equivalent solution to (3.1):

$$D(p)D(p+q)\Gamma_\mu(p,p+q)=i\int d\lambda^2\frac{\rho(\lambda^2)(2p+q)_\mu}{[p^2-\lambda^2+i\epsilon][(p+q)^2-\lambda^2+i\epsilon]}=\frac{i(2p+q)_\mu}{2p\cdot q+q^2}[D(p)-D(p+q)]\ . \quad (3.3)$$

We will generalize (3.3) to the QCD case. Figure 11 shows that for $\hat{\Pi}_{\mu\nu}$ we need the improper vertex with one leg truncated, that is, something corresponding to $\hat{\Delta}\hat{\Delta}\hat{\Gamma}$.

There are actually two classes of solutions to the Ward identity (2.28): those with longitudinally coupled massless scalar fields, as in the massive gauge-theory vertex (2.13), and those without such fields. The second class has been used by Baker *et al.*,[22] while we will examine the first class. The distinction between them is all important, since adding the massless scalar fields allows for the kinematical description of gluon mass and hence short-range correlations of fluctuating color fields.

The Lehmann representation for $\hat{\Delta}_{\mu\nu}$,

$$\hat{\Delta}_{\mu\nu}(q)=P_{\mu\nu}\int d\lambda^2\frac{\rho(\lambda^2)}{q^2-\lambda^2+i\epsilon}\ , \quad (3.4)$$

shows that $\hat{\Delta}$ can be thought of as a linear superposition of free, massive gauge propagators [see (2.11)]. Since the massive vertex (2.13) obeys a Ward identity (2.14) of exactly the type we need, and since this Ward identity when multiplied by two $\hat{\Delta}$'s becomes linear in $\hat{\Delta}$, the required generalization of (3.3) is almost obvious. We define the longitudinal part of $\hat{\Gamma}$ by $\hat{\Gamma}^L$:

$$\hat{\Delta}^{\beta\beta'}(k_2)\hat{\Gamma}^L_{\alpha\beta'\gamma'}\hat{\Delta}^{\gamma\gamma'}(k_3)=\int d\lambda^2\rho(\lambda^2)\left[\frac{P^{\beta\beta'}(k_2)}{(k_2{}^2-\lambda^2+i\epsilon)}\Gamma^{(\lambda)}_{\alpha\beta'\gamma'}(k_1,k_2,k_3)\frac{P^{\gamma\gamma'}(k_3)}{(k_3{}^2-\lambda^2+i\epsilon)}\right]\ , \quad (3.5)$$

where $\Gamma^{(\lambda)}$ is the massive vertex (2.13) for mass $m=\lambda$. It is very simple to verify the correct Ward identity (2.28) by multiplying by $k_{1\alpha}$. Although $\hat{\Gamma}^L$ has massless poles, these couple as pure gauge fields and never appear in physical quantities; they do not affect our general conclusion that if there is a mass gap, transverse corrections to $\hat{\Gamma}^L$ are small by at lease one power of k_1 when k_1 is small.

Let us discuss the relationship of $\hat{\Gamma}^L$ to perturbation theory. In zeroth order, $\rho(\lambda^2)=\delta(\lambda^2)$, and then $\hat{\Gamma}^L$ is precisely the free massless vertex (the massless scalar fields decouple). Thus transverse corrections are of $O(g^2)$. These appear as $O(g^4)$ corrections in $\hat{\Delta}$, and their main effect is in the ultraviolet regime, where they are important for dealing with overlapping divergences. Another concern might be the applicability of the Lehman representation in a theory with only massless particles, such as perturbative QCD. However, one can find singular spectral functions which yield the perturbative propagator and give zero for seagull graphs. For example, the one-loop result (2.24) for $\hat{\Delta}$ is consistent with the spectral function

$$\rho=\delta(\lambda^2)(1-bg^2\ln\epsilon\mu^{-2})-\frac{bg^2}{\lambda^2}\theta(\lambda^2-\epsilon) \quad (3.6)$$

in which the limit $\epsilon\to 0$ is taken after integrating over λ. Of course, this ρ used in (3.5) does not generate the correct $O(g^2)$ corrections to $\hat{\Gamma}$; transverse parts must be added separately.

We are concerned not only with the Ward identity (2.28) for Γ, but also with enforcing the gauge-invariant structure (2.25) for $\widehat{\Pi}_{\mu\nu}$, in which one of the two possible kinematic structures [see (2.27)] is missing. Obviously these two issues are closely related, and any approximation made in $\widehat{\Gamma}$ will be reflected in an analogous approximation for $\widehat{\Pi}_{\mu\nu}$, which receives not only the usual contributions listed in Fig. 11, but also from (dressed) SGP graphs. Now the graphs of Fig. 11 are precisely linear superpositions of one-loop graphs for the massive gauge theory, each mass λ being weighted by $\rho(\lambda^2)$. If this prescription is used for the other one-loop SGP's (see Figs. 3 and 8), it will not be a surprise that $\widehat{\Pi}_{\mu\nu}$ is, in fact, of the n-independent and conserved form shown in (2.25). Explicit calculation shows this to be true, and in the next section we give the resulting one-dressed-loop value for $\widehat{\Pi}_{\mu\nu}$ and discuss its renormalization.

IV. THE DYNAMICAL EQUATION AND ITS RENORMALIZATION

Introduce the notation

$$\widehat{\Delta}_{\mu\nu}(q^2)=P_{\mu\nu}\widehat{\Delta}(q^2) \ . \tag{4.1}$$

The dynamical equation (2.25) (see Fig. 8) for $\widehat{\Delta}$,

$$\widehat{\Delta}^{-1}(q^2)=q^2-\widehat{\Pi}(q^2) \ , \tag{4.2}$$

plus the solution of the Ward identity in Sec. III requires us to repeat the one-loop calculation of Sec. II using massive propagators and one massive vertex, then to sum over masses λ^2 with weight $\rho(\lambda^2)$. A rather lengthy calculation gives the following result for the *renormalized* propagator in terms of the renormalized coupling constant g^2:

$$\widehat{\Delta}^{-1}(q^2)=Z_3\left\{q^2\left[1+\frac{ibg^2}{\pi^2}\int d^4k\int d\lambda^2\frac{\rho(\lambda^2)}{(k^2-\lambda^2)[(k+q)^2-\lambda^2]}\right]\right.$$
$$\left.+\frac{ibg^2}{11\pi^2}\int d^4k\int d\lambda^2\frac{\lambda^2\rho(\lambda^2)}{(k^2-\lambda^2)[(k+q)^2-\lambda^2]}-\frac{4ibg^2}{11\pi^2}\int d^4k\widehat{\Delta}(k^2)\right\} , \tag{4.3}$$

where b is given in (1.2). In writing (4.3) we have assumed, as directed by the Ward identity (2.28), that $Z_1=Z_3$, where Z_1 is the vertex renormalization constant. But because we have dropped all transverse vertex parts we have mishandled overlapping divergences, so that the actual relation between Z_1 and Z_3 is ambiguous and depends on the way we choose to resolve this mishandling.

It is convenient to isolate all the divergences in the k integrations at $q=0$. In so doing, we will encounter the integral

$$\int d^4k\int d\lambda^2\frac{\lambda^2\rho(\lambda^2)}{(k^2-\lambda^2)^2}=-\int d^4k\left[1+k^2\frac{\partial}{\partial k^2}\right]\widehat{\Delta}(k^2) \ . \tag{4.4}$$

If this integral converged, it would be elementary to show that

$$-\int d^4k\left[1+k^2\frac{\partial}{\partial k^2}\right]\widehat{\Delta}(k^2)=\int d^4k\widehat{\Delta}(k^2) \ . \tag{4.5}$$

Even though the integrals are divergent, (4.5) is true in dimensional regularization (at $d=4$) and we will use it freely below. Then (4.3) can be rewritten as [using $\int d\lambda^2\rho(\lambda^2)=0$, which follows from perturbation theory]

$$\widehat{\Delta}^{-1}(q^2)=Z_3\left\{q^2\left[1-\frac{ibg^2}{\pi^2}\int d^4k\frac{\partial}{\partial k^2}\widehat{\Delta}(k^2)\right]+bg^2\int_0^1 d\beta\int d\lambda^2\rho(\lambda^2)\ln\left[1-\frac{\beta(1-\beta)q^2}{\lambda^2}\right]\right.$$
$$\left.+\frac{bg^2}{11}\int_0^1 d\beta\int d\lambda^2\lambda^2\rho(\lambda^2)\ln\left[1-\frac{\beta(1-\beta)q^2}{\lambda^2}\right]-\frac{3ibg^2}{11\pi^2}\int d^4k\widehat{\Delta}(k^2)\right\} . \tag{4.6}$$

Note that (4.3) or (4.6) makes sense in Euclidean space, becuase all dependence on the lightlike vector n_μ is gone. Also note in (4.3) that the massless scalar fields which appeared in the vertex do not appear in the integrands; their only role is to allow for the possibility that $\widehat{\Delta}^{-1}(q^2=0)\neq 0$.

Next we go to Euclidean space, defining the Euclidean propagator d by

$$d(q^2)=-\hat{\Delta}(-q^2)\ . \tag{4.7}$$

In what follows, q^2 is the positive square of a Euclidean four-vector. It is possible to do the β integrations in (4.6) explicitly, and the result is

$$d^{-1}(q^2)=Z_3\left\{q^2\left[1+bg^2\int_0^{q^2/4}dz\left[1-\frac{4z}{q^2}\right]^{1/2}d(z)+\frac{bg^2}{\pi^2}\int d^4k\frac{\partial}{\partial k^2}d(k^2)\right]\right.$$
$$\left.+\frac{bg^2}{11}\int_0^{q^2/4}dzz\left[1-\frac{4z}{q^2}\right]^{1/2}d(z)+\frac{3bg^2}{11\pi^2}\int d^4kd(k^2)\right\}\ . \tag{4.8}$$

There are now two potential infinities inside the curly brackets. The last term, which gives a mass, is actually finite after a physically appropriate regularization; we will discuss it in the next section. The other infinite is multiplied by q^2, and is the first term of an infinite series which is to be removed by the Z_3 factor. Since Z_3 is divergent, it is clearly impossible that (4.8) is finite to all orders in g^2. The exact version of (4.8) would have an intricate structure of overlapping divergences allowing it to be finite order by order. The best we can do at present is to interpret the renormalization algorithm *subtractively* rather than *multiplicatively*, that is, to ignore infinities of $O(g^4)$ and higher. We thus interpret Z_3 as a factor which renders the product

$$Z_3\left[1+\frac{bg^2}{\pi^2}\int d^4k\frac{\partial}{\partial k^2}d(k^2)\right]\equiv K \tag{4.9}$$

finite, and set $Z_3=1$ elsewhere. The final renormalized equation is therefore

$$d^{-1}(q^2)=q^2\left[K+bg^2\int_0^{q^2/4}dz\left[1-\frac{4z}{q^2}\right]^{1/2}d(z)\right]+\frac{bg^2}{11}\int_0^{q^2/4}dzz\left[1-\frac{4z}{q^2}\right]^{1/2}d(z)+d^{-1}(0)\ , \tag{4.10}$$

$$d^{-1}(0)=\frac{3bg^2}{11\pi^2}\int d^4kd(k^2)\ . \tag{4.11}$$

K is fixed by imposing the renormalization condition

$$d^{-1}(\mu^2)=\mu^2 \tag{4.12}$$

and we will always take $\mu^2 \gg \Lambda^2$, where Λ is the renormalization-group mass of a few hundred MeV.

Now (4.10) differs from (4.8) in a profound way; (4.10) comes with a built-in renormalization group whose parameters must be deduced from that equation, and not simply taken from perturbation theory. One way of making this difference clear is to note that $d(q^2)$, although gauge invariant, is not physical because it depends on the renormalization point μ. But just as in QED (where $Z_1=Z_2$) the fact that $Z_1=Z_3$ allows us to write a propagator which is both gauge invariant and renormalization-group invariant; it is

$$D(q^2)=g^2d(q^2)\ . \tag{4.13}$$

If one substitutes this in (4.8) and uses

$$Z_3=g^2g_0^{-2}\ , \tag{4.14}$$

where g_0 is the bare coupling constant, all reference to g^2 and hence to μ is removed, as expected. But the price paid for this is the appearance of the divergent quantity g_0^{-2} in the equation. Of course this divergence is canceled by other terms in (4.8), which leads us back to the ambiguities of renormalizing it.

The substitution (4.13) in the subtractively renormalized equation (4.10) does not remove the dependence on g^2. Instead, it is clear that (with a suitable choice of K) (4.10) has no explicit g dependence when written in terms of $g\,d(q^2)$; that is, this quantity is μ independent on the basis of (4.10). This is precisely what happens if instead of $Z_1=Z_3$ these constants are related by $Z_1=Z_3{}^{1/2}$. A formal way of understanding this peculiar result is to write the renormalization-group equations

$$\left[\mu\frac{\partial}{\partial_\mu}+\beta\frac{\partial}{\partial g}-2\gamma\right]d^{-1}=0\ , \tag{4.15}$$

$$\left[\mu\frac{\partial}{\partial_\mu}+\beta\frac{\partial}{\partial g}-2\gamma\right]Z_3=0 \tag{4.16}$$

and check for consistency with (4.8) or (4.10). Here γ is the anomalous dimension associated with Z_3. Calculation shows that (4.15) and (4.16) are consistent with (4.8) if $\beta=\gamma g$, which is the same as

$Z_1=Z_3$, while for (4.10) the necessary relation is $\beta=2\gamma g$, or $Z_1=Z_3^{1/2}$. In more detail, we find

$$\beta=-2bg^3\ ,\quad \gamma=-bg^2\ . \tag{4.17}$$

The renormalization-group coefficients are falsely represented because we dropped transverse vertex parts. The same thing happens in Baker *et al.*,[22] where 2 is replaced by $\frac{16}{11}$.

An easy way to see what is happening is to write a simplified version of (4.10), which is exact for large q:

$$d^{-1}(q^2)=q^2\left[1+bg^2\int_{\mu^2}^{q^2}dz d(z)\right] \tag{4.18}$$

whose solution is

$$d^{-1}(q^2)=q^2[1+2bg^2\ln(q^2/\mu^2)]^{1/2}\ . \tag{4.19}$$

This agrees, as it must, with the exact result to $O(g^2)$ but disagrees at higher order.

The upshot is that dropping transverse vertex parts forces us to a subtractively renormalized equation (4.10) which is free of infinities, but which has a false renormalization group. We could, of course, use (4.8) which respects the true renormalization group but which is not easily made finite. One might do this by setting $Z_3=g^2g_0^{-2}$, and guessing that correctly incorporating transverse vertex parts will convert the infinity in g_0^{-2} of the form (in lowest order)

$$g_0^{-2}=b\ln(\Lambda_{\rm uv}^2/\Lambda^2) \tag{4.20}$$

($\Lambda_{\rm uv}$ is the ultraviolet cutoff) into a finite factor

$$g_0^{-2}\to b\xi\ , \tag{4.21}$$

where $\xi>1$ is both finite and μ independent; we expect ξ to be logarithmically sensitive to our mishandling of the true ultraviolet behavior. Instead, we will use (4.10). It turns out that this amounts, mathematically, precisely to using (4.8) and (4.21), except that ξ will be construed to depend logarithmically on μ and the physical quantity $D=g^2d$ will have a μ dependence roughly like $(\ln\mu)^{1/2}$. This appearance of μ dependence is mathematical and formal, not physical, since we can *systematically* improve the deficiencies of (4.8) or (4.10) by adding transverse vertex parts of $O(g^2)$ and higher.

V. REGULARIZATION OF THE SEAGULL GRAPH AND OF $\langle TrG_{\mu\nu}^2\rangle$

Regularizing the seagull graph—that is, giving a finite value to (4.11)—and regularizing $\langle {\rm Tr}G_{\mu\nu}^2\rangle$ are very closely related concepts. In physical terms, a gluon "mass" may lead to a vortex condensate with $\langle {\rm Tr}G_{\mu\nu}^2\rangle\neq0$ (second paper of Ref. 6), and conversely $\langle {\rm Tr}G_{\mu\nu}^2\rangle\neq0$ generates a gluon mass, so knowledge of a regularized value for one of these quantities may furnish a regularization for the other.[25]

We will discuss two kinds of regularization schemes: those stemming from first principles, and which are necessary for a *systematic* treatment of the mass gap; and more phenomenologically oriented schemes. The first-principle schemes depend in an essential way on the vanishing of the gluon mass at large momentum, while one phenomenological model we use is severely regulated that even a constant mass gives a finite value to (4.11). Other semiphenomenological schemes for estimating $\langle {\rm Tr}G_{\mu\nu}^2\rangle$ (or equivalently the vacuum self-energy) will be elaborated. One of these is based on the semiclassical vortex condensate of the massive theory and requires a number which is very difficult to calculate: the areal density of vortices. A self-consistency argument for this density leads us back to the glueball regularization scheme.

We begin with a discussion of the vanishing of the dynamical mass at large momenta.

A. Behavior of dynamically generated mass at short distance

It is the essence of a dynamically generated mass that is is not constant but vanishes at large momentum.[11] This vanishing—expressed as the positivity of the anomalous dimension of the mass operator—reflects in a quantitative way the attractiveness of forces in the channel carrying the mass quantum numbers. This soft behavior at short distances, when coupled with dimensional regularization or an equivalent gauge-invariant prescription which denies the perturbative generation of mass, ensures from first principles the finiteness of the mass (seagull) integral given in (4.11). To speak heuristically, one might anticipate that a mass term in $d(k^2)$ appears in some such way as

$$d(k^2)\sim\frac{1}{k^2+m^2(k^2)}\ . \tag{5.1}$$

The dimensional regularization rules (1.6) allow (4.11) to be written as

$$\int d^4k\left[d(k^2)-\frac{1}{k^2}\right]=-\int d^4k\frac{m^2(k^2)}{k^2[k^2+m^2(k^2)]}\ . \tag{5.2}$$

If $m^2(k^2)$ vanishes sufficiently rapidly, this integral is finite. Unfortunately, this approach is too simple minded, because it gives the wrong sign for the mass term. Something more elaborate than (5.1) happens for small k^2, but the basic idea which leads to ultraviolet finiteness is expressed in (5.2).

An asymptotically free gauge theory is somewhat more complicated, with its powers of logarithms. We can find the anomalous mass-operator dimension in lowest-order perturbation theory by adding a bare mass $m_0{}^2$ to the fundamental equation (4.6) (which is in Minkowski space; we use its Euclidean analog), and then choosing $\rho(\lambda^2)=\delta(\lambda^2-m_0{}^2)$. This works because (4.6) becomes precisely the one-loop result for the massive gauge-invariant Lagrangian of Sec. II. The dimensionally regularized result for the renormalized propagator is, for $q^2, \mu^2 \gg m^2$,

$$d^{-1}\left[1+\frac{2}{\epsilon}bg^2\right]\left[q^2\left[1-\frac{2}{\epsilon}bg^2(q^2/\mu^2)^{-\epsilon/2}\right]+m_0{}^2\left[1-\frac{8}{11\epsilon}bg^2(m_0{}^2/\mu^2)^{-\epsilon/2}\right]+m_0{}^2\frac{2}{11\epsilon}bg^2(q^2/\mu^2)^{-\epsilon/2}\right], \tag{5.3}$$

where $\epsilon=d-4$ and the factor in front is Z_3 to $O(g^2)$. To this order, the physical, finite mass m^2 is determined by

$$m_0{}^2=m^2\left[1-\frac{16}{11\epsilon}bg^2(m_0{}^2/\mu^2)^{-\epsilon/2}\right] \tag{5.4}$$

which gives a pole in (5.3) at $-q^2=m^2+$ finite corrections). (These corrections vanish when the correct low-energy version of (5.3) is used.) The renormalized propagator is, with (5.4).

$$d^{-1}=[1+bg^2\ln(q^2/\mu^2)]\left[q^2+m^2\left[1-\tfrac{12}{11}bg^2\ln(q^2/m^2)\right]\right] \tag{5.5}$$

which will be recognized as the lowest-order term in a renormalization-group- improved expansion:

$$d^{-1}=[1+bg^2\ln(q^2/\mu^2)]\{q^2+m^2[1+bg^2\ln(q^2/m^2)]^{-12/11}\}\ . \tag{5.6}$$

As we mentioned in the Introduction, the power $\frac{12}{11}$ is π^{-1} times the Coulombic strength of attraction in the 0^+ channel with quantum numbers of the mass operator. The mass term in (5.6) is the solution of a standard linearized mass equation,[26] which also has another solution[17] behaving like $q^{-2}(\ln q^2)^{+12/11}$. This latter behavior is likely to be correct, but either solution works in the qualitative discussion which follows.

Now we argue that there is a sense in which the seagull integral vanishes in perturbation theory:

$$\int d^4k d_{\text{pert}}(k^2)=0\ , \tag{5.7}$$

where

$$d_{\text{pert}}{}^{-1}(k^2)=k^2[1+bg^2\ln(k^2/\mu^2)] \tag{5.8}$$

is the renormalization-group-improved propagator. Even though this quantity has a Landau ghost making the interpretation of (5.7) problematical, (5.7) is necessary to ensure the nongeneration of mass (through a seagull term) in perturbation theory. It is then trivial that the regulated seagull

$$\int d^4k[d(k^2)-d_{\text{pert}}(k^2)] \tag{5.9}$$

is finite, when d is taken from (5.6).

Let us comment briefly on the three-dimensional case. Asymptotic freedom is replaced by superrenormalizability, so the powers of logarithims are gone, and the dynamical mass need have no special behavior at large k. One might then use (5.1) and (5.2) with $d=3$ to conclude finiteness of the seagull. It is not quite this easy, because the actual behavior of the gauge-invariant propagator at large k is

$$d^{-1}(k^2)=k^2-\pi bg^2k+O(\ln k^2)\ , \tag{5.10}$$

where $b=15C_A(32\pi)^{-1}$. If now we add to (5.10) a constant mass term, and form (5.2) in three dimensions, a logarithmic divergence appears. But this is a purely *perturbative* divergence, which is necessarily canceled in nonseagull two-loop graphs. So finiteness of a dynamical mass in $d=3$ QCD involves more than just the seagull and more than just the one-dressed-loop approximation we use in this paper.

Next we discuss some semiphenomenological regularization ideas. To begin with, we derive an exact relation between the vacuum self-energy and $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$.

B. Regularizing $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$ with a finite vacuum energy

Let us temporarily rescale the fields:

$$A_\mu^a = g^{-1} \hat{A}_\mu^a \ , \quad G_{\mu\nu}^a = g^{-1} \hat{G}_{\mu\nu}^a \ . \tag{5.11}$$

The basis of the regularization will be to set the energy of the perturbation vacuum to zero, so we define the vacuum partition function Z and vacuum self-energy Ω by

$$Z = Z_p{}^{-1} \int (d\hat{A}_\mu) \exp\left[-g^{-2} \int d^4x \tfrac{1}{4} \sum_a (\hat{G}_{\mu\nu}^a)^2\right]$$

$$= e^{-V\Omega} \ , \tag{5.12}$$

where V is the volume of Euclidean space-time and Z_p is the functional integral written as a formal power series in g. By differentiation with respect to g it follows that

$$\frac{\partial \ln Z}{\partial g} = \frac{1}{2g} \int d^4x \left\langle \sum_a (G_{\mu\nu}^a)^2{}_{\mathrm{reg}} \right\rangle = -\frac{V \partial \Omega}{\partial g} \ , \tag{5.13}$$

where the subscript reg indicates that the regularization is by subtraction of the perturbative expectation value. The integral over x in (5.13) gives a factor of V which can be canceled out.

We will show explicitly in the next section that when a dynamical mass m is generated, $\Omega = \Omega(\mu, g^2, m)$ is a finite negative function of its arguments—finite because the perturbative contribution has been subtracted out. Of course, Ω is renormalization-group invariant, so

$$\left[\mu \frac{\partial}{\partial_\mu} + \beta \frac{\partial}{\partial g}\right] \Omega = 0 \ . \tag{5.14}$$

By simple dimensional analysis,

$$\left[\mu \frac{\partial}{\partial_\mu} + m \frac{\partial}{\partial m} - 4\right] \Omega = 0 \ . \tag{5.15}$$

Combining (5.13)–(5.15) yields (with $\beta \simeq -bg^3$):

$$\left\langle g^2 \sum_a (G_{\mu\nu}^a)^2 \right\rangle = -2b^{-1}(4 - m\partial/\partial m)\Omega \tag{5.16}$$

(the subscript reg is understood). Finally, observe that Ω can be expressed as a functional of the propagator $d(k^2)$, and that the only dependence of Ω on m is through d. Since the functional derivatives of Ω with respect to d vanishes when d is the solution to the Schwinger-Dyson equation,[27] it follows that $m\,\partial\Omega/\partial m = 0$ and finally

$$\left\langle g^2 \sum_a (G_{\mu\nu}^a)^2 \right\rangle = -8b^{-1}\Omega \tag{5.17}$$

which is positive since $\Omega < 0$. (Incidentally, these equations when combined with the Euclidean trace anomaly[28] yields the result $\langle \theta_{\mu\mu} \rangle = -4\Omega$, with $\langle \theta_{\mu\mu} \rangle$ being interpreted as the positive pressure of the gluon condensate.) The bag constant B is equal to $-\Omega$, and (5.17) yields $B^{1/4} = 250$ MeV for the phenomenological value of $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$ in Ref. 12.

We remind the reader that relations like (5.17), although derived for bare quantities, survive renormalization intact; and that Ω is gauge invariant.

C. Glueball regularization.

As discussed in the Introduction, one way of looking at the physics behind mass generation is that the (composite) field ϕ which creates O^+ glueballs has a finite vacuum expectation value (VEV). To describe this, we use a phenomenological effective Lagrangian with terms describing a VEV for ϕ, a coupling between ϕ and the gauge fields, and a Hartree-type term coming from the four-field coupling in the gauge Lagrangian.

Our ultimate goal is in describing the VEV $\langle \mathrm{Tr} G_{\mu\nu}{}^2 \rangle$; since $G_{\mu\nu}$ is described from a gauge potential, it will be necessary to speak of a VEV bilinear in the gauge potentials. This VEV must, of course, be gauge invariant. We have at our disposal a completely gauge-invariant form of the potential, using the auxiliary fields U introduced in the massive Lagrangian (2.2). It is

$$\tilde{A}_\mu \equiv U^{-1} A_\mu U - g^{-1}(\partial_\mu U^{-1}) U \tag{5.18}$$

which can be shown, using the equations for U, to be conserved. In lowest order of g,

$$\tilde{A}_\mu = A_\mu - \partial_\mu \Box^{-1} \partial \cdot A \ . \tag{5.19}$$

More generally, it must be the case that propagators of $\tilde{A}_\mu$ involve the gauge-invariant propagator $\hat{\Delta}_{\mu\nu}$ [see (2.24)], except that the projection operator $P_{\mu\nu}$ of (2.12) is replaced with the conserved projection based on (5.19). Thus we make the (Euclidean-space) identification

$$\langle \tilde{A}^a_\mu(x)\tilde{A}^b_\nu(0)\rangle =\delta_{ab}\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]dk^2) , \tag{5.20}$$

where $d(k^2)$ is the gauge-invariant propagator of Sec. IV, and $\tilde{A}^a_\mu$ the component form of the matrix $\tilde{A}_\mu$. The VEV of interest we write as, for gauge group SU(N),

$$Q\equiv\left\langle\sum_a[\tilde{A}^a_\mu(x)]^2\right\rangle =3(N^2-1)\int \frac{d^4k}{(2\pi)^4}d(k^2) . \tag{5.21}$$

The effective Lagrangian involving ϕ and Q is

$$\mathscr{L}_{\text{eff}}=\frac{\lambda^2}{4}\phi^4-\frac{\mu^2}{2}\phi^2+\frac{1}{2}R\phi Q+\frac{1}{12}\frac{g^2NQ^2}{(N^2-1)} . \tag{5.22}$$

The first two terms summarize processes which give a VEV to ϕ, the third term is a *renormalizable* coupling of ϕ to two gauge fields, and the last term is essentially the Hartree approximation to the four-gluon term in the gauge Lagrangian. However, the coefficient of Q^2 is not what one would expect from purely classical considerations (which would be larger by a factor of $\frac{9}{4}$; instead it has been chosen so that the mass generated for small fluctuations around Q is given by (4.11) [with $3\rightarrow N$ for SU(N)] and (5.21). This renormalization is necessary because it is not only seagull terms which contribute to the mass; also contributing are the massless poles in the vertex (2.13).

Straightforward minimization of (5.22) with respect to ϕ and Q gives a minimum at $\phi=\langle\phi\rangle$, where

$$\langle\phi\rangle=\pm\frac{1}{\lambda}\left[\mu^2+\frac{3(N^2-1)R^2}{2g^2N}\right]^{1/2} , \tag{5.23}$$

$$\frac{g^2NQ}{3(N^2-1)}=-R\langle\phi\rangle=|R\langle\phi\rangle| . \tag{5.24}$$

Positivity of Q requires $R\langle\phi\rangle<0$, of course. As we have already said, Q is related to $d^{-1}(0)$ by (4.11) and (5.21), which yields an important relation:

$$d^{-1}(0)=|R\langle\phi\rangle| . \tag{5.25}$$

Another useful relation follows from an explicit description of ϕ as a composite field:

$$\phi(x)=\sum_a[\tilde{A}^a_\mu(x)]^2\left[e^{-ip\cdot x}\left\langle 0\left|\sum(\tilde{A}_\mu)^2\right|p\right\rangle\right]^{-1} . \tag{5.26}$$

The normalization is such that $\langle 0|\phi(x)|p\rangle=^{-ip\cdot x}$, where $|p\rangle$ is a glueball state momentum p. It is more convenient to normalize at $p=0$, simply by continuing the term in large parentheses in (5.26) to zero momentum. This is calculated from the vertex implicit in $\mathscr{L}_{\text{eff}}$ (assuming R = constant in momentum space, which a renormalization group analysis shows to be very nearly true). We may now calculate $\langle\phi\rangle$ directly from (5.26), with the result

$$R\langle\phi\rangle=-\int d^4kd(k^2)\left[\int d^4kd^2(k^2)\right]^{-1} . \tag{5.27}$$

This furnishes a finite, positive regulator for Q, because the second factor in (5.27) is convergent (even for constant mass). The reason for this convergence is that $d\sim(k^2\ln k^2)^{-1}$ at large k. Combining (5.25) and (5.27) yields the explicit regularization

$$\int d^4kd(k^2)\Big|_{\text{reg}}=d^{-1}(0)\int d^4kd^2(k^2) . \tag{5.28}$$

This has the virtues of being finite, positive, consistent with the renormalization group, and zero in perturbation theory [where $d^{-1}(0)=0$].

D. Vortex-condensate regularization.

The massive Lagrangian (2.2) has vortex solutions (second paper of Ref. 6). The gauge potentials are regular at the center of the vortex, but the invariant potential $\tilde{A}_\mu$ has a singularity. For a stationary vortex running along the z axis we have [in SU (3)]

$$A_j^{(v)}=\pm i\frac{\hat{Q}}{g}\hat{\phi}_j\left[\frac{1}{\rho}-mK_1(m\rho)\right] , \tag{5.29}$$

where $j=(x,y)$, $\hat{\phi}_j$ is the unit vector in the ϕ direction (angle around the z axis), $\hat{Q}$ is the quark charge matrix, and K_1 is a modified Hankel function. The invariant potential $\tilde{A}_j^{(V)}$ is gotten from (5.29) by dropping the (pure-gauge) ρ^{-1} term. The resulting singularity ($mK_1\sim\rho^{-1}$) leads to a short-distance logarithmic singularity in the vortex action, coming from the mass term in (2.2). We will discuss in detail elsewhere how the fact that a dynamically generated mass (in effect, $m \to 0$ as $\rho \to 0$) removes this singularity, giving the vortex a finite action proportional to the area of its world sheet [in the present case, the (zt) plane].

In two dimensions, where vortices are point particles, we can calculate the classical gauge-invariant propagator corresponding to (5.20), using standard collective-coordinate calculations:

$$\left\langle \sum_a \widetilde{A}_i^a(x)\widetilde{A}_j^a(y) \right\rangle_\phi = \sum \frac{e^{-s}}{U} \int d^2a\,(-2\,\mathrm{Tr})\widetilde{A}_i^{(v)}(x-a)\widetilde{A}_j^{(v)}(y-a)$$

$$=(4\mathrm{Tr}\hat{Q}^2)\frac{e^{-s}}{Ug^2}\int d^2k e^{-ik\cdot(x-y)}\frac{k^2}{(k^2+m^2)^2}\left[\delta_{ij}-\frac{k_ik_j}{k^2}\right]. \quad (5.30)$$

The translation collective coordinate a has been explicitly indicated, while the summation sign refers to all other coordinates [including group degrees of freedom and summing over $\pm$ in (5.29)]. In (5.30), S is the vortex action and U an area calculated from quantum fluctuations. It will be very important to calculate $U^{-1}e^{-S}$ someday, but it is beyond our powers now. An estimate is not hard to come by, since this number is simply the areal density of vortices. A vortex has area $\sim \pi m^{-2}$, so we expect $U^{-1}e^{-S}$ to be m^2/π times a number ranging from, say, 0.1 to 0.5. Below we give a self-consistency argument which yields values in this range.

The four-dimensional calculation corresponding to (5.30) is much harder, but we can see the general features it must have. Of course, the projection tensor $\delta_{ij}-k_ik_jk^{-2}$ is immediately generalized to four dimensions. The quantity $U^{-1}e^{-S}$ retains its interpretation as an areal density of vortices. The most important charge is that we can incorporate in S the entropy arising from summing over different sizes and shapes of vortices (this entropy must be at least as great as the vortex action if there is to be condensation). Note that, because of the k^2 in the numerator (5.30), we can readily interpret the integral as four dimensional. Finally, as will be discussed elsewhere, the effect of radiative corrections on this classical formula is essentially to replace $(k^2+m^2)^{-1}$ by the full propagator, appropriately normalized at small k:

$$(k^2+m^2)^{-1} \to [m^2 d(0)]^{-1} d(k^2)\,. \quad (5.31)$$

The upshot of all these considerations is an approximation to the propagator which is *regulated* at small distances, regularized because no contributions solely stemming from perturbation theory have been kept. We find $[4\,\mathrm{Tr}\hat{Q}^2=\frac{8}{3}$ for SU(3)]

$$\left\langle \sum_a \widetilde{A}_\mu^a(x)\widetilde{A}_\nu^a(0) \right\rangle_{\mathrm{reg}} = \frac{8e^{-S}}{3\pi g^2 U}[m^2 d(0)]^{-2}\int d^4k e^{-ikx} d^2(k)\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right] \quad (5.32)$$

which is finite at $x=0$. This regularization is in the same spirit as (5.28), differing only by a constant factor associated with the unknown e^{-S}/U. We can recover (5.28) precisely by making the not unreasonable demand that the regulated propagator should agree with the true propagator at large distances (small k). The Fourier transform of (5.32) at $k=0$ yields upon comparison with (5.20)

$$\frac{e^{-s}}{U}=\frac{3\pi m^4}{(2\pi)^4}g^2 d(0) \quad (5.33)$$

which correctly shows the left-hand side to be renormalization-group independent. We will see in the next section that (5.33) gives $e^{-s}/U = (m^2/\pi)\times(0.1-0.3)$, in agreement with expectations.

Similarly, the vortex condensate can be used to estimate $\langle \mathrm{Tr}G_{\mu\nu}{}^2 \rangle$. We record the contribution to $G_{\mu\nu}$ from $\partial_\mu A_\nu - \partial_\nu A_\mu$; the part coming from the non-Abelian term $gA_\mu \times A_\nu$ can easily be estimated and turns out to be numerically rather small. Note that it is important to use A_μ and not $\widetilde{A}_\mu$; these differ by a singular gauge transformation which affects $G_{\mu\nu}$. The result is

$$g^2\left\langle \sum_a (G_{\mu\nu}^a)^2 \right\rangle_{\mathrm{reg}} \simeq 16\frac{e^{-S}}{U}\int d^2k \frac{m^4}{(k^2+m^2)^2}$$

$$=16\pi m^2 \frac{e^{-S}}{U}\,. \quad (5.34)$$

The non-Abelian terms add about $0.5m^4$ to this. Estimating $e^{-S}/U \sim m^2/4\pi$ and using the phenomenological value[12] of 0.47 GeV^4 for the left-hand side of (5.34) gives $m \simeq 600$ MeV.

VI. SOLUTION OF THE SCHWINGER-DYSON EQUATION AND ESTIMATES OF THE GLUON MASS

The equation to be solved is (4.10), which we repeat for convenience:

$$d^{-1}(g^2)=g^2\left[K+bg^2\int_0^{q^2/4}dz\left[1-\frac{4z}{q^2}\right]^{1/2}d(z)\right]+\frac{bg^2}{11}\int_0^{q^2/4}dz\,z\left[1-\frac{4z}{q^2}\right]^{1/2}d(z)+d^{-1}(0)\,. \tag{6.1}$$

K is determined by the requirement $d^{-1}(q^2=\mu^2)=\mu^2$, where $\mu^2 \gg d^{-1}(0)$, and $d^{-1}(0)$, of course undetermined by (6.1), will later be found from either (5.17) or (5.28). As we know from Sec. IV, (6.1) has the unpleasant feature that g^2d, which should be independent of g (or μ), in fact depends roughly on g because of the renormalization procedure we were forced to adopt. Now g is not very sensitive to μ, changing by about 30% as μ/Λ ranges from 20 to 200. But this sort of ambiguity which is intrinsic to (6.1) means that an exact numerical solution is of less importance than an approximate solution which respects the true renormalization group. That is, we are better off using the first significant term of the expansion of the exact large-q solution (4.19) than we are using (4.19) itself; the error committed will be small as long as $bg^2\ln(q^2/\mu^2) \ll 1$. This one-term expansion can be renormalized according to the true renormalization group. For very small values of q (6.1) is exact, aside from renormalization-group ambiguities (choice of K), so we want to find numerical solutions to (6.1) and then fit them to approximate trial functions which are chosen to be renormalized according to the true renormalization group.

[Although we will not study it in detail here, we give the corresponding dynamical equation for the three-dimensional case:

$$d^{-1}(q^2)=q^2\left[1-\frac{bg^2}{\pi}\int_0^\infty dzK(z,q^2)d(z)\right]+\frac{2bg^2}{15\pi^2}\int d^3kd(k^2)\,, \tag{6.2}$$

$$K(z,q^2)=q^{-1}\ln\left|\frac{2z^{1/2}+q}{2z^{1/2}-q}\right|\,, \tag{6.3}$$

and $b=15C_A/32\pi$. This equation—or more precisely the two-loop version of it—has no ultraviolet divergences (see the discussion of the seagull in Sec. VI), but it appears to compensate for such good large-momentum behavior by a more virulent infrared behavior than is found in the four-dimensional case. This equation has only massive solutions, and the minimum mass for which there is a solution is $O(bg^2)$.[24] For SU(2), $b\sim 0.3$, and $m=0.3g^2$ is the gluon mass found in lattice calculations.[23] The perturbative solution of (6.2) and (6.3) agrees with the calculations of Ref. 5, when these are made gauge invariant according to the prescription of Sec. VII.]

It turns out that the following trial propagator, which incorporates the correct ultraviolet behavior (including the momentum-dependent mass term) and which is renormalized according to the true renormalization group, is an excellent fit to numerical solutions of (6.1):

$$d_T{}^{-1}(q^2)=[q^2+m^2(q^2)]bg^2\ln\left[\frac{q^2+4m^2(q^2)}{\Lambda^2}\right]\,, \tag{6.4}$$

$$m^2(q^2)=m^2\left[\ln\left[\frac{q^2+4m^2}{\Lambda^2}\right]\Big/\ln\frac{4m^2}{\Lambda^2}\right]^{-12/11}\,. \tag{6.5}$$

It is manifest that g^2d_T is independent of g. The relationship between g, μ, Λ, and m is *defined* by $d_T{}^{-1}(q^2=\mu^2)=\mu^2$; aside from mass terms, this relationship is the usual one-loop result. The appearance of $4m^2$ in (6.4) and (6.5) is suggested by simple theoretical arguments. Another possible trial form would follow from the first iteration of (6.1) using a free massive propagator as input; this is closely related to but not identical to (6.2). Note that d_T becomes singular for $m<\Lambda/2$, which is related to the fact that (6.1) becomes a singular equation for $d^{-1}(0)=0$. We repeat that Λ can only be defined precisely when at least the two-loop correction is taken into account, but this ambiguity in Λ is exactly of the same character as the dependence of g^2d on μ, as discussed in Sec. IV, and need not concern us until the two-loop corrections to the basic equation (6.1) are incorporated.

In Sec. IV we saw that the dependence of (6.1) on g can be formally scaled out. Introduce the new quantities

$$K=g\bar{K},\quad y=q^2/m^2\,, \tag{6.6}$$

and define F by

$$d^{-1}=gm^2F(y,\bar{K}) . \tag{6.7}$$

Then (6.1) becomes

$$F(y,\bar{K})=\bar{K}(y+1) +yb\int_0^{y/4}d\rho\left[1-\frac{4\rho}{y}\right]^{1/2}F^{-1}(\rho,\bar{K}) +\frac{b}{11}\int_0^{y/4}d\rho\left[1-\frac{4\rho}{y}\right]\rho F^{-1}(\rho,\bar{K}) . \tag{6.8}$$

The lack of dependence on g is only formal, since the normalization of F (choice of $\bar{K}$) will depend on g:

$$F\left[\frac{\mu^2}{m^2},\bar{K}\right]=\frac{\mu^2}{gm^2} . \tag{6.9}$$

Equation (6.8) is easily solved on a computer, and we will discuss such a solution later. But it is instructive to develop a power-series solution for small y:

$$F(y,\bar{K})=\bar{K}\left[1+\sum_1 \alpha_N y^N\right] . \tag{6.10}$$

The coefficients are found to be

$$\begin{aligned} \alpha_1&=1 ,\\ \alpha_2&=\frac{b}{6\bar{K}^2}(1+\tfrac{1}{110}) ,\\ \alpha_3&=-\frac{b}{60\bar{K}^2}(1+\tfrac{1}{77}) ,\\ \alpha_4&=\frac{b}{420\bar{K}^2}(1+\tfrac{1}{66})\left[1-\frac{b}{6\bar{K}^2}(1+\tfrac{1}{110})\right] ,\\ \alpha_N&=(-)^N\frac{b}{\bar{K}^2}4^{1-N}\frac{\Gamma(N-1)\Gamma(3/2)}{\Gamma(N+1/2)}\\ &\quad\times\left[1+\frac{N}{22(2N+3)}\right]\left[1+O\left[\frac{b}{\bar{K}^2}\right]\right] . \end{aligned} \tag{6.11}$$

The radius of convergence of the power series is $y=4$, as long as $b/6\bar{K}^2$ is not large compared to 1. But when this quantity exceeds a value of order unity, (6.8) appears to develop singular, unphysical behavior. When $\bar{K}^2 \gg b$, the equation is easily solved, but the mass m will be too large to achieve self-consistency through the regulated value of $d^{-1}(0)$ given in (5.28). In practice, $0.2<\bar{K}<0.4$ is the interesting range for SU(3) ($b\simeq 0.07$), which will correspond to mass values $m/\Lambda\simeq 1.5-2$.

One may solve (6.8) with the help of the power series for any desired value of y as follows. Use the power series for $y<y_0<4$, then substitute in the right-hand side of (6.8) to generate the solution up to $y=4y_0$, etc. This helps us to understand why the trial form (6.4) agrees well with the exact solution, at least for $0.2<\bar{K}<0.4$: the first iterate of (6.8) based on saving only the first two terms in the power series is reasonably accurate at least to $y\lesssim 16$, and has the same asymptotic behavior as (6.4) for larger y.

In Fig. 12 we show computer-generated solutions of (6.8) for $\bar{K}=0.3$, 0.4, as well as values of $m^2g^2d_T$ from (6.4) for $m/\Lambda=1.5$, 2. From (6.7), $m^2g^2d=gF^{-1}$, so g is the ratio between the dotted and solid curves; Fig. 12 yields g in the range 1.5–2, corresponding to μ/Λ in the range from 6 to 25. The precise relationship between $\bar{K}$ and g follows from equating the first two terms of the trial form (6.4) to the first two terms in the power series for *small y*:

$$\bar{K}=bg\left[\ln\frac{4m^2}{\Lambda^2}+\frac{1}{4}\right] . \tag{6.12}$$

(The $\frac{1}{4}$ is numerically unimportant.) In principle, this should be the same condition as (6.9), which relates $\bar{K}$ and g at high energies ($q=\mu$), but in practice there is a difference unless μ is chosen to be rather small. For large μ/m and $q^2\simeq\mu^2$ it is straightforward to show from (6.1) that $K=1-bg^2\ln(\mu^2/4m^2)+O(bg^2)$, which yields (6.12) up to terms of $O(bg^2)$. Part of the numerical discrepancy between (6.9) and (6.12) arises, of course, becase d_T is not exactly a solution to (6.1). It is better to use (6.12) than (6.9), because the pro-

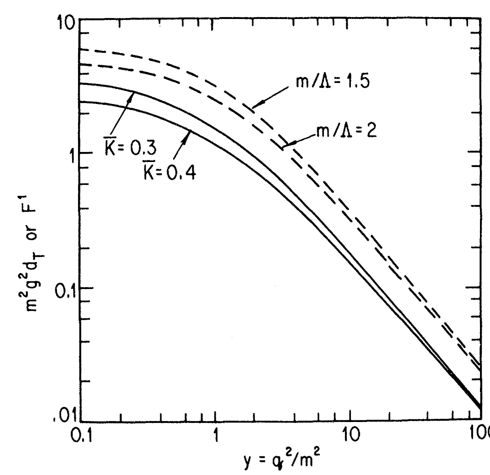

FIG. 12. Numerical solutions of Eq. (6.1) (solid lines) for $\bar{K}=0.3$ 0.4, and the trial propagator d_T of Eqs. (6.4) and (6.5).

pagator is largest for small y; an error in the ultraviolet is much less important than are in the infrared.

We are now in a position to estimate the gluon mass, in two different ways: a self-consistent determination, based on (5.28), or an estimate in terms of a phenomenological value of $\langle \mathrm{Tr} G_{\mu\nu}^2 \rangle$ using (5.17). In both cases we will use the trial propagator (6.4). For the self-consistent determination we do not even need to use the momentum-dependent mass of (6.5) for convergence, but this form is essential for (5.17).

Consider first the self-consistent case. The mass equation, coming from using (5.28) in the right-hand side of (4.11), is

$$1=\frac{3bg^2}{11\pi^2}\int d^4k d^2(k)\ . \tag{6.13}$$

Here d means d_T of (6.4) with $m^2(q)\equiv m^2$. Just like (6.1), this condition is g independent only for the *false* renormalization group, where gd is independent of g. It would be nice to use the exact solution to (6.1) in (6.13) because of the g independence of the results; unfortunately, (6.13) then diverges because the solution to (6.1) behaves like $k^{-2}(\ln k^2)^{-1/2}$ for large k. The divergence is very tame ($\sim \ln\ln\Lambda_{\mu\nu}$). We have calculated masses using (6.13) and the exact solution to (6.1) for a range of cutoff values, and get results ($m/\Lambda\simeq 1.5-2$) consistent with the use of (6.4) in (6.13).

To calculate the self-consistent mass, pick a value for bg^2, find a value for m/Λ, then use $d_T^{-1}(q^2=\mu^2)=\mu^2$ to eliminate g^2 in favor of μ/Λ. We find in this way that the renormalization-point range $5<\mu/\Lambda<100$ corresponds to $m/\Lambda=1.3\pm 0.5$. That is, a range of a factor of 20 in μ/Λ corresponds to $\pm 40\%$ spread in m/Λ, illustrating the rather weak dependence of m on μ which inevitably follows from the mishandling of overlapping divergences inherent in the one-loop equation (6.1). This dependence can be fit as logarithmic in μ, or as a small (~ 0.2) fractional power dependence, over the above-mentioned range of μ/Λ.

For very large values of μ/Λ, it is straightforward to show that (6.13) leads to

$$m=\frac{\Lambda}{2}e^{3/22bg^2}\ . \tag{6.14}$$

If μ is large enough $\Lambda\simeq\mu e^{-1/2bg^2}$, so (6.14) can be written

$$m\simeq\mu e^{-1/2b'g^2} \tag{6.15}$$

with $b'=(\frac{11}{8})b$ (compare to the false renormalization-group coefficient $2b$ as discussed in Sec. IV). So the dependence of m on μ can be thought of as a simple consequence of using the wrong value of b, and m is independent of μ for another value of b as in (6.15).

There is another sense in which m can be regarded as independent of μ. Note that the original equation (4.8) from which (6.1) was derived by subtractive renormalization had its mass term multiplied by Z_3, which we set equal to one in the subtractive renormalization procedure. We might instead replace Z_3 by $g^2 g_0^{-2}$, in which case the self-consistency condition (6.13) is independent of g (hence independent of μ) when $g^2 d$ is independent of g, as is the case for d_T from (6.4). We then have to deal with the divergent quantity g_0^{-2}, so μ dependence has been traded for cutoff dependence. Correct treatment of the overlapping divergences in the mass term will then replace g_0^{-2} by a finite but still μ-independent quantity. What we have termed μ dependence of the mass ratio m/Λ is entirely equivalent to our ignorance of what finite but μ-independent value we should use for g_0^{-2}. That ignorance will be much reduced when the two-loop version of (4.8) or (6.1) is studied.

Going to the two-dressed-loop version of (6.1), which is a *systematic* improvement of that equation, will do two things: first, the renormalization mismatch will be smaller by a factor of roughly bg^2, and second, we will be able to say what Λ is with some accuracy. It appears from the other considerations we are about to give that $m=(2\pm 0.5)\Lambda$ with $\Lambda\simeq 300$ MeV, consistent with momentum-space renormalization with four quark flavors.

Now that we have a specific form for the propagator, namely d_T from (6.4), we return to Eq. (5.33) and (5.34) which deal with vortex-condensate regulation. The first of these equations estimates the areal density of vortices $e^{-S/u}$; using (6.4) yields

$$\frac{e^{-s}}{U}=\frac{m^2}{\pi}\left[\frac{3}{11\ln 4m^2/\Lambda^2}\right] \tag{6.16}$$

and the second gives

$$g^2\left\langle\sum_a (G_{\mu\nu}^a)^2\right\rangle=\frac{48m^4}{11\ln(4m^2/\Lambda^2)}\ . \tag{6.17}$$

The phenomenological value for the left-hand side of 0.47 GeV^4 gives rise to $m\simeq 700$ MeV.

Finally, consider a direct estimate of the vacuum energy Ω, based on the Hartree approximation.

The vacuum energy is given by[27]

$$V\Omega=-\tfrac{1}{2}\,\mathrm{Tr}[\ln d d_0{}^{-1}-d d_0{}^{-1}+1] + 2\mathrm{PI\ graphs}\ -(d\to d_p)\ . \qquad (6.18)$$

Here d is the gauge-invariant propagator, d_0 the free-field propagator, and d_p the perturbative counterpart of d. The 2PI (two-particle irreducible) graphs are not standard because of the appearance of d in place of the conventional propagator, but the graphical structure in terms of d can easily be worked out.

The first and most important point to be made is that Ω is finite, given a mass $m(k^2)$ that decreases at large k. The reason was given long ago in a related context[26]: clearly Ω vanishes at $m=0$, and has a formal expansion in powers of m^2. There could be a quadratic divergnece coming from the term $\sim m^2$, but the coefficient must be zero. This is because we can expand d as $d_p+O(m^2)$ at large momentum, and then the potential divergence in Ω is $O(m^2)\times\delta\Omega/\delta d$. This functional derivative vanishes when d is the propagator which solves the Schwinger-Dyson equation. The m^4 terms are convergent because the mass decreases at large k, and higher powers converge even with a constant mass.

To estimate Ω we will save only the Hartree term corresponding to the last term of the effective Lagrangian (5.22):

$$\Omega=-\frac{3(N^2-1)}{2(2\pi)^4}\int d^4k\,[\ln(d d_0{}^{-1})-d d_0{}^{-1}+1] + \tfrac{3}{4}g^2N(N-1)\left[\int\frac{d^4k}{(2\pi)^4}d(k^2)\right]^2 . \qquad (6.19)$$

We have substituted $d_p\to d_0$ and dropped the perturbative contribution. The variational derivative is

$$\delta\Omega/\delta d=0\sim d^{-1}-d_0{}^{-1}-\frac{g^2N}{(2\pi)^4}\int d^4k\,d(k^2) \qquad (6.20)$$

which at zero momentum ($d_0{}^{-1}=0$) gives back the mass equation (4.11). To evaluate Ω we use a simple form for d: $d^{-1}=k^2+m^2$. Using the variational equation (6.20) for d in (6.19) allows the rearrangement[27] of Ω to

$$\Omega=+\frac{3(N^2-1)}{2(2\pi)^4}\int d^4k\ln\left[\frac{k^2+m^2}{k^2}\right] - \frac{3(N^2-1)}{4Ng^2}d^{-2}(0)\ . \qquad (6.21)$$

The divergent integral over $\ln d$ must be regulated, which we do with the help of (4.5), yielding

$$\Omega=-\frac{3(N^2-1)}{2(2\pi)^4}\int d^4k\left[\ln\left[\frac{k^2+m^2}{k^2}\right]-\frac{m^2}{k^2+m^2}\right] - \frac{3(N^2-1)}{4Ng^2}d^{-2}(0)\ . \qquad (6.22)$$

Now the integral is at most logarithmically divergent (the integrated goes like m^4k^{-4} at large k). But we know that it cannot diverge, once the radiative corrections so far omitted cause m to decrease at large k, so we will simply substitute $m^2\to m^2(k^2)$ by hand, leaving detailed justification to a later work. Numerical evaluation of (6.22) for SU(3) gives roughly $-0.3m^4$ [depending, of course, on m/Λ and on g^2, since the last term of (6.22) is not renormalization-group invariant] and thus a rather low mass of $\simeq$350 MeV when compared to phenomenology [$\Omega=-(250\text{ MeV})^4$].

One may also compare directly (6.17) and $-8b^{-1}\Omega$ from (6.22), to find a formula for m in terms of Λ. It gives $m\simeq\Lambda$, a low value judged by other estimates. This is probably because our crude calculations somewhat overestimate $-\Omega m^{-4}$.

So we have several rough estimates for m, ranging from, say, 300 MeV ($m\simeq\Lambda$) to 700 MeV. This spread illustrates the inherent inaccuracy of the present calculations. It is amusing to note that this spread may be greatly reduced by the simple *(ad hoc)* expedient of multiplying the estimate (6.16) of the vortex areal density $e^{-S/U}$ by a factor approximately equal to 2. This same factor will then appear on the right-hand side of the mass equation (6.13), leading to $m\simeq2\Lambda$ or possibly 600 MeV, while the comparison to $\langle \mathrm{Tr}G_{\mu\nu}{}^2\rangle$ in (6.17) lowers m from 700 MeV to around 600 MeV.

VII. PHYSICAL IMPLICATIONS OF THE GLUON MASS

In this summary we discuss a half-dozen or so different ways of relating the gluon mass m to measurable quantities (including those measured on the lattice). In each instance, the theoretical errors in the relationship are of the order of 50%, and in one or two cases the relationship is nothing more than a reasonably sophisticated guess. No single connection between m and another physical parameter is quantitatively persuasive by itself, but taken together they strongly suggest that $m=(500\pm200)$ MeV. We also discuss several other ways in which hadron physics is directly influenced by the gluon mass, and which require further investigation.

The first two ways of determining m have already been discussed: with the phenomenological glueball regularization of Sec. V, $m=(1.59\pm0.9)\Lambda$, where Λ is a renormalization-group scale not determined by one-loop calculations. For consistency with other values of m found later, we should take $\Lambda\simeq300$ MeV, consistent with $\Lambda_{\rm MOM}$ in a four-quark theory.[29]

The second way is through $\langle {\rm Tr}G_{\mu\nu}{}^2\rangle$ which is related to the vacuum self-energy Ω, which we could only crudely estimate. This gave $m\simeq350-700$ MeV. Each of these two ways has an accuracy of only about $\pm50\%$, but this can in principle be greatly reduced by writing down and solving the two-dressed-loop version of (6.1) (with a correspondingly complex evaluation of Ω). Calculations in this direction are now in progress.

The third way is to relate m to the glueball spectrum expected in the continuum theory. It is at least conceivable that the dynamics of massive-gluon bound states may be usefully approximated with a Schrödinger equation, and A. Soni and the author are presently calculating glueball masses using potentials derived from single-gluon Yukawa forces as well as a breakable string force which characterizes adjoint Wilson loops (see below). We find that a gluon mass of $m=500$ MeV yields a 2^{++} glueball of mass ~1600 MeV and a 0^{-+} mass of ~1400 MeV. We have fitted these values because there are experimental candidates for glueballs of these quantum numbers at the appropriate masses.[30] The theoretical errors in this determination—even if one granted no experimental uncertainties—are not negligible because gluon systems are reasonably relativistic and because of mixing with $q\bar{q}$ states.

The fourth way is to compare these glueball calculations with lattice calculations, which have the advantage of no $\bar{q}q$ mixing as well as a clearly identified 0^{++} glueball. Disadvantages are that lattice measurements of glueball masses are themselves subject to $\pm50\%$ error, and that they are done for SU(2) instead of SU(3). The 0^{++} glueball calculated [for SU(3)] by Soni and the author has a mass of $\simeq2.2m$, while lattice calculations[31,32] give a mass $(3\pm1)\sqrt{K_F}$. In the continuum theory, at least, there is no marked difference between glueballs for the color groups SU(2) and SU(3), so these numbers ought to be comparable within errors. For $\sqrt{K_F}=400$ MeV, we find $m\simeq(550\pm200)$ MeV. One group[32] has gone on to calculate the 0^{-+} and 2^{++} masses in SU(2), getting 4.1 $\sqrt{K_F}\simeq1640$ MeV and 4.4 $\sqrt{K_F}\simeq1760$ MeV, respectively, with rather large errors. The continuum theory fits these values with $m\simeq600$ MeV.

A fifth way has already been mentioned. In a lattice calculation, Bernard[8] has measured the energy needed to pop two gluons out of the vacuum (i.e., to break the adjoint string), which should be of order $2m$. He finds $m\gtrsim530$ MeV (a lower bound because of finite-size effects). It is not clear how much the nominal value $2m$ is changed by gluon-binding effects, so we should assign $\pm50\%$ errors to this comparison also.

The last two ways are really no better than educated guesses, but we record them because of their intrinsic interest. In the Introduction we showed how a vacuum field with fluctuating color fields led to confinement. A rough estimate for the string tension for QCD comes from (1.4), with an extra factor of $1/2N$ because ${\rm Tr}\frac{1}{2}\lambda_a\frac{1}{2}\lambda_b=(1/2N)\delta_{ab}{\rm Tr}1$:

$$K_F\simeq\frac{1}{4N}\Delta\sigma g^2\langle G^2\rangle\,, \tag{7.1}$$

where $\langle G^2\rangle$ is related to $\langle {\rm Tr}G_{\mu\nu}^2\rangle$ by

$$\left\langle\sum_a G_{\mu\nu}^a(0)G_{\alpha\beta}^a(0)\right\rangle=(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\nu\alpha}\delta_{\mu\beta})\langle G^2\rangle\,, \tag{7.2}$$

$$\langle G^2\rangle=\frac{1}{12N}\left\langle\sum_a G_{\mu\nu}^a)^2\right\rangle\,. \tag{7.3}$$

This gives

$$K_F\simeq\frac{\Delta\sigma}{48N}g^2\left\langle\sum_a (G_{\mu\nu}^a)^2\right\rangle \quad [\simeq0.003\ ({\rm GeV})^4\Delta\sigma \text{ for SU(3)}]\,. \tag{7.4}$$

Here $\Delta\sigma$ is the area over which fluctuations are correlated. We might guess that $\Delta\sigma$ is the inverse of the areal vortex density e^{-S}/U, in which case (5.34) gives $K_F\simeq(\pi/9)m^2$. For $K_F\sim0.16$ GeV^2, we find $m\sim670$ MeV.

Finally, we have looked at the behavior of adjoint Wilson loops, which show a perimeter law (because an adjoint string always breaks by popping gluon pairs out of the vacuum). The expectation value of the Wilson loop is estimated by filling it with tangled vortices, in just the same way as done earlier for quark Wilson loops (2nd paper of Ref. 6). The vortices have only short-ranged interactions with an adjoint Wilson loop, and for very large loops one can do the continuum analog of Bernard's[8] lattice calculation. In this way we find $m\simeq600$ MeV with errors impossible to assess, but large. Details of this work will be published elsewhere.

These estimates are useful not only for the light they shed on the value of m, but to indicate how one relates a mass which cannot be directly measured by an experimentalist to physical quantities like the string tension. The most fundamental relation, perhaps, is that between K_F and m, and it is one of the most poorly determined, unfortunately. There are other relationships which have hardly been explored at all, which we now list.

Gluon masses should affect the transverse-momentum structure of gluon jets, and (both through phase-space effects and infrared stabilization of the running charge) quarkonium decays. This last point has been made by Parisi and Petronzio,[33] who guessed at a gluon mass of 800 MeV. The most recent study of three-gluon quarkonium decay,[34] however, is based on massless gluons, and points to a rather small value for Λ ($\Lambda_{\overline{\rm MS}} \lesssim 100$ MeV). But including gluon-mass effects will raise Λ, and it is most important to redo this calculation with massive gluons.

Each hadron made of $q\bar{q}$ or qqq will be the lowest member of a series in which the next member is $q\bar{q}g$ or $qqqg$ (g for gluon). The next-lowest members of the series should have opposite parity to the ground state and lie about 600 MeV higher; their angular momentum may or may not change. This sort of series should be distinguishable from Regge trajectories and from radial excitations (for which there is no parity change). If the gluon is allowed to have orbital angular momentum, there will also be states acting very much like radial excitations; indeed, they may not even differ in principle, since a radial string excitation must be coupled strongly to virtual gluons (vacuum polarization). Similar properties of $\bar{q}qg$ and $qqqg$ states are expected in the bag model (J. Kuti, private communication).

Finally, there are some implications of gluon mass of a more theoretical nature. We have already pointed out (second paper of Ref. 6) that the massive theory has screened instantons, whose size cannot exceed m^{-1}; an alternative study of the vacuum using such screened instantons in place of vortices might shed some light on what the QCD vacuum really looks like. As mentioned in the Introduction, massive gluons lead to terms in the operator-product expansion of the sort usually associated with nonperturbative effects[12] (i.e., inverse powers of q^2 rather than powers of $\ln q^2$). A simple example is the electromagnetic vacuum-polarization tensor (or photon self-energy). If this is computed with massless quarks and gluons to $O(g^2)$, only powers of $\ln q^2$ emerge, but with massive gluons there are powers of m^2/q^2. In particular, the m^4/q^4 term may be identified with the $\langle {\rm Tr} G_{\mu\nu}^2 \rangle q^{-4}$ term of Ref. 12, giving yet another determination of the value of this expectation value in terms of m^4. Calculations are in progress on this point.

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