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Nonlocal conservation laws for strings

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A finite number of nonlocal conservation laws are found in the Nambu-Goto string model. An infinite number of conserved currents may be obtained by embedding the string in more than 3 + 1 space-time dimensions. These currents resemble the nonlocal currents found in two-dimensional chiral models.

I. INTRODUCTION

Recently, there has been much interest in two-dimensional field theories which possess an infinite number of nonlocal conservation laws.[1–5] At the quantized level of the theory, these conservation laws are known to give rise to new sets of Ward identities.[2] Further, in the case of the nonlinear σ model, they have been shown to lead to the absence of particle production and the factorization of the multiparticle S matrix.[3]

Several different criteria have been proposed to ensure the existence of nonlocal conservation laws. In Refs. 4 and 5 they correspond to zero-curvature conditions on a set of locally defined currents J_a. The currents J_a take values in a Lie algebra associated with some Lie group G. The nonlocal conserved currents are then constructed via repeated integrals which involve J_a.

In this paper we show that the above zero-curvature conditions are satisfied for the case of the Nambu-Goto string model.[6] Here, the currents J_a are associated with the momentum and angular momentum densities along the string. Unlike all the previous cases, there exists only a *finite number* of linearly independent nonlocal currents which can be constructed from J_a. This occurs because the group G is a *nilpotent* group for the string.

The new conservation laws tend to indicate the existence of certain hidden symmetries for the string model. There appear to be problems, however, associated with implementing these symmetries within the context of a Hamiltonian description. This follows since we find that the charges associated with the new currents are either not constants of the motion or are not expressible in terms of local functions of a time variable. In the case of open strings, the new conserved currents and their associated charges are expressible as local functions of time. However, due to the fact that the end points of the string are not fixed, the associated charges are not constants of the motion. Alternatively, for closed strings, we find that the new conserved currents are not expressible as local functions in time, whereas their associated charges are always constants of the motion.

We also consider the case of strings embedded in more than 3 + 1 space-time dimensions, i.e., Kaluza-Klein strings. Such theories and their supersymmetric extensions have been of considerable interest in the literature.[7,8] For us the additional space dimensions are associated with a compact group manifold. This system is shown to contain an infinite number of nonlocal conserved currents in addition to those found previously for the Nambu-Goto string. Here too there are problems associated with defining charges which are both local function of time and constants of the motion.

In Sec. II we give the criteria for generating the nonlocal conserved currents, as well as review some of their properties. In Sec. II we also examine the conditions necessary for the new charges to be (a) local functions of time and (b) constants of the motion. These considerations are applied to the relativistic string in Sec. III and its higher-dimensional generalizations in Sec. IV. In the Appendix we show in general how the nonlocal currents can be constructed in a manner which is independent of the particular group representation for G.

II. GENERATING NONLOCAL CURRENTS AND CHARGES

A. Conditions for the existence of nonlocal conserved currents

Let $J_a^A(\sigma)$, $A=1,2,\dots,N$ correspond to a set of N conserved currents. Here $\sigma=(\sigma^0,\sigma^1)$ parametrizes space-time and $a=0,1$ is a space-time index. Nonlocal conserved currents can be generated provided that the following condition is satisfied[4,5]:

$$\partial_a J_b^A - \partial_b J_a^A = -2C_{BC}^A J_a^B J_b^C \; . \tag{2.1}$$

Here the C_{BC}^A's are constants (not all zero) which are antisymmetric in the lower indices. We also require that

$$C_{BC}^A C_{AD}^E + C_{CD}^A C_{AB}^E + C_{DB}^A C_{AC}^E = 0 \; . \tag{2.2}$$

It follows that the C_{BC}^A's can be identified as the structure constants associated with some N-dimensional Lie group G. For convenience we define

$$J_a = J_a^A T(A) \; , \tag{2.3}$$

where $T(A)$'s are the group generators. The condition (2.1) can then be written

$$D_a J_b - D_b J_a = 0 \; , \qquad D_a = \partial_a + [J_a, \;] \; . \tag{2.4}$$

Since J_a is conserved, we also have

$$\partial_a(\sqrt{-g}J^a) = 0 \; , \tag{2.5}$$

where we assume in general that space-time is not flat and $g=\det[g_{ab}]$, g_{ab} being the components of a 2×2 metric tensor.

The proof that new conserved currents $J_{(i)}^a$ $(i=1,2,\dots)$ can be generated from currents J^a satisfying (2.4) and (2.5) is given in Refs. 4 and 5 (also see the Appendix). The currents $J_{(i)}^a$ are found to be functions of the variables $\chi_1,\chi_2,\dots,\chi_i$. The variable χ_n in turn is defined in terms of $J_{(n-1)}^a$ according to

$$\epsilon^{ab}\partial_b \chi_n = \sqrt{-g}\, J_{(n-1)}^a \; . \tag{2.6}$$

Here

$$J_{(0)}^a \equiv -2J^a \tag{2.7}$$

and ϵ^{ab} is the usual antisymmetric tensor density, with $\epsilon^{01}=1$. From (2.6), χ_n is a nonlocal function of σ. For instance,

$$\chi_1(\sigma) = 2\int_{\bar\sigma}^{\sigma} d\sigma'^a \epsilon^{ab}[-g(\sigma')]^{1/2} J^b(\sigma') \; , \tag{2.8}$$

where $\bar\sigma$ is some fixed point in space-time. It follows that the currents $J_{(i)}^a$, $i\geq 1$ are nonlocal functions of σ.

In the treatments given in Refs. 4 and 5, a matrix representation is required for the group generators $T(A)$. In the Appendix, we give an algorithm for constructing $J_{(i)}^a$ in a manner which is independent of a particular group representation. The results for the first two sets of nonlocal currents $J_{(1)}^a$ and $J_{(2)}^a$ are

$$J_{(1)}^a = D^a \chi_1 \; , \tag{2.9}$$

$$J_{(2)}^a = D^a \chi_2 - \tfrac{1}{6}[[J^a,\chi_1],\chi_1] \; . \tag{2.10}$$

Our currents $J_{(i)}^a$ take values in the Lie algebra associated with G. Thus we may write $J_{(i)}^a \equiv J_{(i)}^{aA} T(A)$, and the components $J_{(i)}^{aA}$ are determined from equations like (2.9) and (2.10). Although an infinite number of currents $J_{(i)}^{aA}$ may be generated, it is possible that for some G only a finite number of linearly independent $J_{(i)}^{aA}$ exist. This will be demonstrated in Sec. III.

B. Condition for $J_{(i)}^a$ to be local in time

Note that the currents defined in (2.9) and (2.10) are nonlocal in the time parameter σ^0 as well as the space parameter σ^1. In order that $J_{(1)}^a$ be definable on any single time slice, we must impose the additional requirement that there exists a curve C, given by $\sigma^1 = f(\sigma^0)$, such that

$$\sqrt{-g}\, J^1 \,|_C = 0 \; . \tag{2.11}$$

If we define $\bar\sigma$ to lie along the curve C, then Eq. (2.8) can be replaced by

$$\chi_1(\sigma) = -2\int_{f(\sigma^0)}^{\sigma^1} d\sigma'^1 [-g(\sigma^0,\sigma'^1)]^{1/2} \times J^0(\sigma^0,\sigma'^1) \; , \tag{2.12}$$

which is local in the time parameter σ^0.

In order to express all the currents $J_{(i)}^a$ as local functions of time, we must in general replace (2.11) by the stronger condition

$$J^a \,|_C = 0 \; , \quad a=0,1 \; . \tag{2.13}$$

C. Conditions for the existence of conserved charges

The charges associated with the currents $J_{(i)}^a$ are

$$Q_{(i)} = \int_S d\sigma^1 \sqrt{-g}\, J_{(i)}^0 \; , \tag{2.14}$$

where the integration is performed over an entire time slice S. In order that they be constants of the motion, we must require that J^a, $a=0,1$ vanishes at the boundaries ∂S (if there are boundaries). Note that it automatically follows that $J^a_{(i)}$ and $Q_{(i)}$ are expressible as local functions in σ^0 since we can set C in (2.13) equal to one of the boundaries. Thus $\partial_0 Q_{(i)}=0$ implies for us that $Q_{(i)}$ can be locally defined in time, although the converse is not in general true. The latter will be illustrated in Sec. III B.

The above is not valid if S has no boundaries, i.e., all time-slice surfaces are closed. In this case $Q_{(i)}$ is always a constant of the motion. However, if there exists no curve C for which (2.13) holds, then it is in general not possible to define $Q_{(i)}$ on a single time slice. Alternatively, it may be possible to define a related quantity $Q'_{(i)}$ which is a local function of σ^0, but whose time derivative is not zero. Instead, $\partial_0 Q'_{(i)}$ is a function of the fields along some curve C'. This will be illustrated in Sec. III C.

III. CONSERVATION LAWS FOR THE FREE STRING

A. Generating nonlocal conserved currents

We now apply the preceding formalism to a free relativistic string.[6] Again σ^0 and σ^1 $(0\leq\sigma^1\leq 2\pi)$ correspond to time and space parameter, respectively. The dynamical fields in this case are the string coordinates $z^\mu(\sigma)$, out of which the induced metric tensor can be formed,

$$g_{ab}=\partial_a z^\mu \partial_b z_\mu \ . \tag{3.1}$$

The free-string dynamics is determined from the action

$$\mathscr{S}=\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-g} \ . \tag{3.2}$$

Equations of motion resulting from variations in z^μ are given by

$$\partial_a(\sqrt{-g}\,g^{ab}\mathscr{P}^\mu_b)=0 \ , \quad \mathscr{P}^\mu_a=\partial_a z^\mu \ . \tag{3.3}$$

It follows that

$$\begin{aligned}&\partial_a(\sqrt{-g}\,g^{ab}\mathscr{M}^{\mu\nu}_b)=0 \ ,\\ &\mathscr{M}^{\mu\nu}_a\equiv z^\mu\partial_a z^\nu - z^\nu\partial_a z^\mu \ .\end{aligned} \tag{3.4}$$

Computing the curl of $\mathscr{P}^\nu_a$ and $\mathscr{M}^{\mu\nu}_a$, we find

$$\begin{aligned}&\partial_a\mathscr{P}^\mu_b-\partial_b\mathscr{P}^\mu_a=0 \ ,\\ &\partial_a\mathscr{M}^{\mu\nu}_b-\partial_b\mathscr{M}^{\mu\nu}_a=2(\mathscr{P}^\mu_a\mathscr{P}^\nu_b-\mathscr{P}^\nu_a\mathscr{P}^\mu_b)\end{aligned} \tag{3.5}$$

Upon identifying $\{\mathscr{P}^\mu_a,\mathscr{M}^{\mu\nu}_a\}$ with the set of currents $\{J^A_a\}$ of Sec. II, we note that condition (2.1) is satisfied. Here all structure constants C^A_{BC} vanish except

$$C^{(\mu\nu)}_{\rho\ \ \sigma}=-\delta^\mu_\rho\delta^\nu_\sigma+\delta^\mu_\sigma\delta^\nu_\rho \ . \tag{3.6}$$

If we define X_μ and $X_{\mu\nu}(=-X_{\nu\mu})$ to be the associated group generators, then

$$\begin{aligned}&[X_{\mu\nu},X_{\rho\sigma}]=[X_{\mu\nu},X_\rho]=0 \ ,\\ &[X_\rho,X_\sigma]=2X_{\sigma\rho} \ .\end{aligned} \tag{3.7}$$

Commutation relations (3.7) define a nilpotent Lie algebra[9] N and due to this property we can generate only a finite number of linearly independent nonlocal currents $J_{a(i)}=\mathscr{P}^\mu_{a(i)}X_\mu+\mathscr{M}^{\mu\nu}_{a(i)}X_{\mu\nu}$.

Because repeated commutators vanish, i.e.,

$$[[\cdot,\cdot],\cdot]=0 \ , \tag{3.8}$$

the expressions for $J^a_{(i)}$ [cf., e.g., (2.9), (2.10)] may be simplified to

$$J^a_{(i)}=D^a\chi_i, \quad i\geq 1 \ . \tag{3.9}$$

Again χ_i are computed by integrating (2.6). The proof of (3.9) follows by induction and is similar to that of Ref. 4. From Eq. (2.9) it is valid for $i=1$. Assuming the conservation of the ith current and computing the divergence of the $(i+1)$th current, we find

$$\begin{aligned}\partial_a(\sqrt{-g}\,J^a_{(i+1)})&=D_a(\sqrt{-g}\,\partial^a\chi_{i+1})\\ &=\epsilon^{ab}D_aJ_{b(i)}\\ &=\epsilon^{ab}D_aD_b\chi_i \ ,\end{aligned} \tag{3.10}$$

where we have used the identity

$$\partial_a\sqrt{-g}\,D^a=D_a\sqrt{-g}\,\partial^a \ . \tag{3.11}$$

Using (2.4), (3.10) reduces to

$$\partial_a(\sqrt{-g}\,J^a_{(i+1)})=-\epsilon^{ab}[[J_a,J_b],\chi_i] \ . \tag{3.12}$$

The right-hand side of (3.12) vanishes from (3.8), thus completing the proof.

To compute the first set of currents we write $\chi_1=\chi^\mu_1X_\mu+\chi^{\mu\nu}_1X_{\mu\nu}$. Substituting into (3.9) yields

$$\begin{aligned}&\mathscr{P}^\mu_{a(1)}=\partial_a\chi^\mu_1 \ ,\\ &\mathscr{M}^{\mu\nu}_{a(1)}=\partial_a\chi^{\mu\nu}_1-\mathscr{P}^\mu_a\chi^\nu_1+\mathscr{P}^\nu_a\chi^\mu_1 \ .\end{aligned} \tag{3.13}$$

Applying (2.6) and (2.7)

$$\mathscr{P}^{a\mu}_{(1)}(\sigma)=-2\epsilon^{ab}\mathscr{P}^{\mu}_{b}(\sigma)/[-g(\sigma)]^{1/2}\,,$$
$$\mathscr{M}^{a\mu\nu}_{(1)}(\sigma)=-2\epsilon^{ab}\mathscr{M}^{\mu\nu}_{b}(\sigma)/[-g(\sigma)]^{1/2}-2\int_{\bar\sigma}^{\sigma}d\sigma'^{c}\epsilon^{cb}[-g(\sigma')]^{1/2}\times[\mathscr{P}^{a\mu}(\sigma)\mathscr{P}^{b\nu}(\sigma')-(\mu\rightleftarrows\nu)]\,. \tag{3.14}$$

Similarly, the ith currents can be given in terms of the $(i-1)$th currents:

$$\mathscr{P}^{a\mu}_{(i)}(\sigma)=\epsilon^{ab}\mathscr{P}^{\mu}_{b(i-1)}(\sigma)/[-g(\sigma)]^{1/2}\,,$$
$$\mathscr{M}^{a\mu\nu}_{(i)}(\sigma)=\epsilon^{ab}\mathscr{M}^{\mu\nu}_{b}(\sigma)/[-g(\sigma)]^{1/2}+\int_{\bar\sigma}^{\sigma}d\sigma'^{c}\epsilon^{cb}[-g(\sigma')]^{1/2}\times[\mathscr{P}^{a\mu}(\sigma)\mathscr{P}^{b\nu}_{(i-1)}(\sigma')-(\mu\rightleftarrows\nu)]\,. \tag{3.15}$$

We now show that only twenty two of the currents generated from (3.14) and (3.15) are linearly independent. Applying (3.15) twice, we find

$$\begin{aligned}\mathscr{P}^{a\mu}_{(i)}(\sigma)&=-\mathscr{P}^{a\mu}_{(i-2)}(\sigma)\,,\\ \mathscr{M}^{a\mu\nu}_{(i)}(\sigma)&=-\mathscr{M}^{a\mu\nu}_{(i-2)}(\sigma)+2\int_{\bar\sigma}^{\sigma}d\sigma'^{c}[\mathscr{P}^{a\mu}(\sigma)\mathscr{P}^{\nu}_{c(i-2)}(\sigma')-(\mu\rightleftarrows\nu)]\\ &\quad+\int_{\bar\sigma}^{\sigma}d\sigma'^{c}\epsilon^{cb}[-g(\sigma')]^{1/2}[\mathscr{P}^{a\mu}_{(1)}(\sigma)\mathscr{P}^{b\nu}_{(i-2)}(\sigma')-(\mu\rightleftarrows\nu)]\,.\end{aligned} \tag{3.16}$$

Upon setting $i=2n$, where n is an integer, we find the recursion relations

$$\mathscr{P}^{a\mu}_{(2n)}=2(-1)^{n+1}\mathscr{P}^{a\mu}\,,$$
$$\mathscr{M}^{a\mu\nu}_{(2n)}=-\mathscr{M}^{a\mu\nu}_{(2n-2)}+(-1)^{n+1}(-2\mathscr{M}^{a\mu\nu}+\mathscr{M}^{a\mu\nu}_{(2)})\,. \tag{3.17}$$

It follows that all currents with i equal to an even integer are linear combinations of $\mathscr{P}^{a\mu}$, $\mathscr{M}^{a\mu\nu}$, and $\mathscr{M}^{a\mu\nu}_{(2)}$. For $i=2n+1$,

$$\mathscr{P}^{a\mu}_{(2n+1)}=(-1)^{n}\mathscr{P}^{a\nu}_{(1)}\,,$$
$$\mathscr{M}^{a\mu\nu}_{(2n+1)}=-\mathscr{M}^{a\mu\nu}_{(2n-1)}+(-1)^{n+1}\left[\mathscr{M}^{a\mu\nu}_{(1)}+\mathscr{M}^{a\mu\nu}_{(3)}\right]\,. \tag{3.18}$$

Thus all currents with i equal to an odd integer are linear combinations of $\mathscr{P}^{a\mu}_{(1)}$, $\mathscr{M}^{a\mu\nu}_{(1)}$, and $\mathscr{M}^{a\mu\nu}_{(3)}$. We thus conclude that from the initial conserved currents $\mathscr{P}^{a\mu}$ and $\mathscr{M}^{a\mu\nu}$ only a finite number of linearly independent currents can be generated, namely, $\mathscr{P}^{a\mu}_{(1)}$ and $\mathscr{M}^{a\mu\nu}_{(i)}$, $i=1,2,3$.

B. Open strings

Upon specializing the above to open strings we must supplement Eq. (3.3) and (3.4) with the boundary conditions

$$\sqrt{-g}\,\mathscr{P}^{1\mu}=\sqrt{-g}\,\mathscr{M}^{1\mu\nu}=0\quad\text{at }\sigma^{1}=0,2\pi\,. \tag{3.19}$$

In view of Sec. II B, these conditions will permit us to write $\mathscr{M}^{a\mu\nu}_{(1)}$ on a single time slice:

$$\mathscr{M}^{a\mu\nu}_{(1)}(\sigma^{1})=-2\epsilon^{ab}\mathscr{M}^{\mu\nu}_{b}(\sigma^{1})/[-g(\sigma^{1})]^{1/2}+2\int_{0}^{\sigma^{1}}d\sigma'^{1}[-g(\sigma'^{1})]^{1/2}[\mathscr{P}^{a\mu}(\sigma^{1})\mathscr{P}^{0\nu}(\sigma'^{1})-(\mu\rightleftarrows\nu)]\,. \tag{3.20}$$

Using the definition of $\mathscr{P}^{\mu}_{a}$, we can express the remaining currents, as well, as local functions in time:

$$\mathscr{M}^{a\mu\nu}_{(2)}(\sigma^{1})=2\int_{0}^{\sigma_{1}}d\sigma'^{1}[-g(\sigma'^{1})]^{1/2}[\mathscr{P}^{a\mu}_{(1)}(\sigma^{1})\mathscr{P}^{0\nu}(\sigma'^{1})-(\mu\rightleftarrows\nu)]\,, \tag{3.21}$$

$$\mathscr{M}^{a\mu\nu}_{(3)}(\sigma^{1})=-\mathscr{M}^{a\mu\nu}_{(1)}(\sigma^{1})+2\int_{0}^{\sigma^{1}}d\sigma'^{1}[\mathscr{P}^{a\mu}(\sigma^{1})\mathscr{P}^{\nu}_{1(1)}(\sigma'^{1})-(\mu\rightleftarrows\nu)]\,. \tag{3.22}$$

The charges $Q^{\mu\nu}_{(i)}$ and $Q^{\mu}_{(1)}$ associated with the currents $\mathscr{M}^{a\mu\nu}_{(i)}$ and $\mathscr{P}^{a\mu}_{(1)}$ may be written

$$Q^{\mu\nu}_{(1)}=-2\int_{0}^{2\pi}d\sigma^{1}\mathscr{M}^{\mu\nu}_{1}(\sigma^{1})+4\int_{0}^{2\pi}\int_{0}^{2\pi}d\sigma^{1}d\sigma'^{1}\epsilon(\sigma^{1}-\sigma'^{1})[-g(\sigma^{1})]^{1/2}[-g(\sigma'^{1})]^{1/2}\mathscr{P}^{0\mu}(\sigma^{1})\mathscr{P}^{0\nu}(\sigma'^{1})\,, \tag{3.23}$$

$$Q^{\mu\nu}_{(2)}=M^{\mu\nu}+P^{\mu}z^{\nu}\,|_{\sigma^{1}=0}-P^{\nu}z^{\mu}\,|_{\sigma^{1}=0}\,, \tag{3.24}$$

$$Q^{\mu\nu}_{(3)}=z^{\mu}|_{\sigma^1=2\pi}z^{\nu}|_{\sigma^1=0}-z^{\mu}|_{\sigma^1=0}z^{\nu}|_{\sigma^1=2\pi}\,, \tag{3.25}$$

$$Q^{\mu}_{(1)}=z^{\mu}|_{\sigma^1=2\pi}-z^{\mu}|_{\sigma^1=0}\,, \tag{3.26}$$

where P^{μ} and $M^{\mu\nu}$ are the conserved momenta and angular momenta for the string and

$$\epsilon(x)=\begin{cases}1, & x>0,\\ -1, & x<0\end{cases}\,.$$

From the definition of $\mathscr{P}^{\mu}_{a}$ it was possible to reduce all the charges except $Q^{\mu\nu}_{(1)}$ to functions of $M^{\mu\nu}$, P_{μ}, and the coordinates of the boundary. Because all components of the currents $\mathscr{P}^{\mu}_{a}$ and $\mathscr{M}^{\mu\nu}_{a}$ do not vanish at the boundaries, the charges (3.23)–(3.26) are not constants of the motion. Instead, their evolution is determined by the motion of the boundaries. This is obvious for (3.24)–(3.26). For (3.23), we find

$$-\tfrac{1}{2}\partial_0 Q^{\mu\nu}_{(1)}=\mathscr{M}^{\mu\nu}_{0}|_{\sigma^1=2\pi}-\mathscr{M}^{\mu\nu}_{0}|_{\sigma^1=0}\,. \tag{3.27}$$

C. Closed strings

In the case of closed strings the boundary conditions (3.19) are replaced by periodic ones, i.e.,

$$\begin{aligned}&z^{\mu}|_{\sigma^1=0}=z^{\mu}|_{\sigma^1=2\pi}\,,\\ &\mathscr{P}^{\mu}_{a}|_{\sigma^1=0}=\mathscr{P}^{\mu}_{a}|_{\sigma^1=2\pi}\,.\end{aligned} \tag{3.28}$$

It follows that the conserved currents $\mathscr{M}^{a\mu\nu}_{(i)}$ given by (3.14) and (3.15) can no longer be reduced to (3.20)–(3.22); i.e., $\mathscr{M}^{a\mu\nu}_{(i)}$ will always contain terms which are nonlocal in both σ^0 and σ^1. Since there are infinitely many homotopically inequivalent paths connecting $\bar{\sigma}$ and σ, infinitely many conserved currents $\mathscr{M}^{a\mu\nu}_{(i)}$ can be constructed for a single value of i. [Cf. Eqs. (3.14) and (3.15).] However, it can be shown that all such currents are equivalent up to integral multiples of trivially conserved quantities.

Because the closed string has no boundaries the charges associated with the conserved currents $\mathscr{M}^{\mu\nu}_{a(i)}$ are constants of the motion. On the other hand, they cannot be expressed as local functions of time. Conversely, it may be possible to define new quantities associated with $\mathscr{M}^{\mu\nu}_{a(i)}$ which are local functions of σ^0 but are not constants of the motion. An example of such a quantity would be $Q^{\mu\nu}_{(1)}$, given in (3.23). From (3.27) it appears that $Q^{\mu\nu}_{(1)}$ may be conserved for closed strings. However, the open-string boundary conditions (3.19) were assumed in the derivation of (3.27). A straightforward calculation shows that for closed strings

$$\partial_0 Q^{\mu\nu}_{(1)}=-4[\sqrt{-g}\,\mathscr{P}^{1\mu}|_{\sigma^1=0}P^{\nu}-(\mu\rightleftarrows\nu)]\,. \tag{3.29}$$

Consistent with the discussion in Sec. II C, the evolution of $Q^{\mu\nu}_{(1)}$ is determined by the motion of one point $z^{\mu}(\sigma^0,0)$ on the string (and P^{μ}).

IV. KALUZA-KLEIN STRINGS

We now consider relativistic strings in the context of a Kaluza-Klein theory,[7,8] i.e., we extend Minkowski space-time M^4 to $M^4\times H$, where we take H to be a compact group manifold. We shall also assume that H is a semisimple group. Let $\Gamma=\{u\}$ be a faithful unitary representation of H. The associated Lie algebra $\underline{\Gamma}$ has a basis $L(\alpha)$, $\alpha=1,2,\dots,k$ with $L(\alpha)^{\dagger}=L(\alpha)$ and

$$[L(\alpha),L(\beta)]=iC^{\gamma}_{\alpha\beta}L(\gamma)\,, \tag{4.1}$$

$$\mathrm{Tr}L(\alpha)L(\beta)=\delta_{\alpha\beta}\,. \tag{4.2}$$

We now replace the induced metric tensor (3.1) by

$$g_{ab}=\partial_a z^{\mu}\partial_b z_{\mu}+\lambda\,\mathrm{Tr}\partial_a u^{\dagger}\partial_b u\;, \tag{4.3}$$

where $u=u(\sigma)\in\Gamma$ and λ is a constant. For the action we once again take (3.2). By varying z_{μ} in (4.3) we find as usual that $\mathscr{P}^{a\mu}$ and $\mathscr{M}^{a\mu\nu}$ are conserved currents. Variations of u are of the form

$$\delta u=i\rho u,\quad \rho=\rho_{\alpha}L(\alpha)\,. \tag{4.4}$$

Consequently,

$$\delta g_{ab}=i\lambda\,\mathrm{Tr}(\partial_a u^{\dagger}\partial_b\rho\, u-u^{\dagger}\partial_a\rho\partial_b u) \tag{4.5}$$

and

$$\delta\mathscr{S}=-i\lambda\int\sqrt{-g}\,g^{ab}\mathrm{Tr}(\partial_b u u^{\dagger}\partial_a\rho)d^2\sigma\,. \tag{4.6}$$

Upon minimizing the action we thus find a new set of conserved currents I^a:

$$I_a=\tfrac{1}{2}\partial_a u u^{\dagger}\,. \tag{4.7}$$

Taking the curl of I_a, we find

$$\begin{aligned}\partial_a I_b-\partial_b I_a&=\tfrac{1}{2}(\partial_a u\partial_b u^{\dagger}-\partial_b u\partial_a u^{\dagger})\\ &=-2[I_a,I_b]\,,\end{aligned} \tag{4.8}$$

where we have used $u\partial_a u^{\dagger}=-\partial_a u u^{\dagger}$. Equation

(4.8) agrees with condition (2.4). Thus in addition to the nonlocal currents of Sec. III,[10] we can generate a new set of nonlocal currents from I^a. In fact, an infinite number of such currents $I^a_{(i)}$ exists. The first set of generated currents are

$$I^a_{(1)}(\sigma) = -2\epsilon^{ab} I_b(\sigma)/[-g(\sigma)]^{1/2} - 2\int_{\bar{\sigma}}^{\sigma} d\sigma'^c \epsilon^{cb}[-g(\sigma')]^{1/2} \times [I^a(\sigma), I^b(\sigma')] \ . \tag{4.9}$$

Upon setting $g_{ab} = \eta_{ab}$ (η_{ab} being the usual flat-metric tensor) and requiring (2.13), all the nonlocal currents reduce to the nonlocal conserved currents found for two-dimensional chiral models. However, condition (2.13) appears to be too restrictive for our dynamical model. In specializing to open strings, we have the weaker condition

$$\sqrt{-g}\, I^1 \,|_{\sigma^1=0,2\pi} = 0 \ . \tag{4.10}$$

From Sec. II B, (4.10) implies that, in general, only the currents $I^a_{(1)}$ are expressible as local functions in σ^0. Furthermore, none of the charges associated with the nonlocal currents $I^a_{(i)}$ are conserved. Instead, their evolution is determined by the motion of the boundaries of the string within the group manifold H.

For the case of closed strings, we must replace (4.10) by

$$I^a \,|_{\sigma^1=0} = I^a \,|_{\sigma^1=2\pi} \ . \tag{4.11}$$

Now none of the generated currents $I^a_{(i)}$ are expressible as local functions in σ^0. On the other hand, all the charges associated with these currents are constants of the motion. As before it is possible to replace these charges by related quantities which are local functions of σ^0, but are not constants of the motion. Instead, their evolution is determined by the values of I^a along some timelike curve on the string. An example of such a quantity is $Q'_{(1)}$:

$$Q'_{(1)} = \int_0^{2\pi} d\sigma^1 I_1 - \int_0^{2\pi} d\sigma^1 \int_0^{\sigma^1} d\sigma'^1 [-g(\sigma^1)]^{1/2} [-g(\sigma'^1)]^{1/2} [I^0(\sigma^1), I^0(\sigma'^1)] \ , \tag{4.12}$$

which may be thought of as the charge associated with $I^a_{(1)}$ minus terms which are nonlocal in σ^0. Using (4.8), we find

$$\partial_0 Q'_{(1)} = 2[\sqrt{-g}\, I^1 \,|_{\sigma^1=0}, Q] \ , \tag{4.13}$$

where Q is the conserved charge associated with the primary current I^a.

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APPENDIX

Here we show in general how the nonlocal currents $J^a_{(i)}$ may be generated from a set of currents J^a satisfying (2.4) and (2.5). Unlike in previous treatments we require no particular matrix representation for G. This is evident since for us $J^a_{(i)}$ takes values in the Lie algebra associated with G.

From (2.4) we may write[11]

$$J_a = \tfrac{1}{2} \partial_a f f^{-1} \ , \tag{A1}$$

where f is a function which takes values in G. Following Ref. 5 we introduce a one-parameter set of potentials $A_a^{(\lambda)}$,

$$A_a^{(\lambda)} = \frac{1}{1-\lambda^2} \left[f^{-1}\partial_a f + \lambda \frac{\epsilon_{ab}}{\sqrt{-g}} f^{-1} \partial^b f \right] , \tag{A2}$$

which are curvature free, i.e.,

$$\partial_a A_b^{(\lambda)} - \partial_b A_a^{(\lambda)} + [A_a^{(\lambda)}, A_b^{(\lambda)}] = 0 \ . \tag{A3}$$

Here λ is a real parameter. Equation (A3) serves as the integrability condition for the linear equation

$$\partial_a \psi^{(\lambda)} + A_a^{(\lambda)} \psi^{(\lambda)} = 0 \ , \tag{A4}$$

where $\psi^{(\lambda)}$ is a λ-dependent function of space-time taking values in G.

We now show that upon expanding Eq. (A4) in powers of λ, recursion relations can be found which lead to an infinite set of conservation laws. We expand $\psi^{(\lambda)}$ as follows:

$$[\psi^{(\lambda)}(\sigma)]^{-1} = h^{(\lambda)}(\sigma) f(\sigma) \ , \quad h^{(\lambda)}(\sigma) = \exp\left[\sum_{n=1} \lambda^n \chi_n(\sigma) \right] . \tag{A5}$$

Here both $h^{(\lambda)}$ and f take values in the group G. Unlike the analogous expansion in Ref. 5, all the coefficients $\chi_n(\sigma)$ take values in the Lie algebra associated with G.

Using (A5), Eq. (A4) may be written

$$h^{(\lambda)-1}\partial_a h^{(\lambda)} = -\partial_a f f^{-1} + f A_a^{(\lambda)} f^{-1} . \tag{A6}$$

From (A5)

$$h^{(\lambda)} = 1 + \lambda\chi_1 + \frac{\lambda^2}{2}(\chi_1^2 + 2\chi_2) + \frac{\lambda^3}{6}[\chi_1^3 + 3(\chi_1\chi_2 + \chi_2\chi_1) - 6\chi_3] + \cdots . \tag{A7}$$

It follows that the left-hand side of (A6) can be expanded to

$$\lambda\partial_a\chi_1 + \lambda^2(\partial_a\chi_2 + \tfrac{1}{2}[\partial_a\chi_1,\chi_1]) + \lambda^3(\partial_a\chi_3 + \tfrac{1}{2}[\partial_a\chi_1,\chi_2] + \tfrac{1}{2}[\partial_a\chi_2,\chi_1] + \tfrac{1}{6}[[\partial_a\chi_1,\chi_1],\chi_1]) + \cdots . \tag{A8}$$

Using (A1) and (A2), the right-hand side of (A6) can be expanded to

$$-2\left[\lambda\frac{\epsilon_{ab}}{\sqrt{-g}}J^b + \lambda^2 J_a + \lambda^3\frac{\epsilon_{ab}}{\sqrt{-g}}J^b \cdots \right] . \tag{A9}$$

Upon equating coefficients in (A8) and (A9), we find the following recursion relations for χ_n:

$$\partial_a\chi_1 = -2\frac{\epsilon_{ab}}{\sqrt{-g}}J^b , \tag{A10}$$

$$\partial_a\chi_2 = -2J_a - \tfrac{1}{2}[\partial_a\chi_1,\chi_1] , \tag{A11}$$

$$\partial_a\chi_3 = -2\frac{\epsilon_{ab}}{\sqrt{-g}}J^b - \tfrac{1}{2}[\partial_a\chi_1,\chi_2] - \tfrac{1}{2}[\partial_a\chi_2,\chi_1] - \tfrac{1}{6}[[\partial_a\chi_1,\chi_1],\chi_1] . \tag{A12}$$

We can now define a new set of conserved currents $J^a_{(n-1)}$ according to Eq. (2.6). The explicit form for the first three currents $J^a_{(0)}$, $J^a_{(1)}$, and $J^a_{(2)}$ can be determined after repeated substitutions of (A10)–(A12). For them we find Eqs. (2.7), (2.9), and (2.10).

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[8]For couplings of Kaluza-Klein strings to gauge fields, see N. K. Nielsen, Nucl. Phys. B167, 249 (1980).

[9]See, for example, N. Jacobson, *Lie Algebras* (Interscience, New York, 1962).

[10]The nonlocal currents of Sec. III are conserved for Kaluza-Klein strings after replacing the metric (3.1) by (4.3).

[11]This statement is true in general only locally. For global considerations, see A. P. Balachandran and G. Marmo, Lett. Nuovo Cimento 27, 13 (1980).