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SU(2) monopoles in self-dual axially symmetric, Abelian backgrounds

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Harrison-Neugebauer-type transformations are applied starting from self-dual, axially symmetric, Abelian backgrounds embedded in SU(2). This generalizes a previous construction of multicharged monopoles starting similarly from a Higgs vacuum. For this purpose the necessary solutions for the Harrison pseudopotential are obtained using the spherical R gauge. Some special features for a constant background field are studied in detail. The relevance of our results to various recent speculations concerning vacuums with nonvanishing $F_{\mu\nu}$ and to studies of gauge fields in the presence of sources is pointed out. A generalization to de Sitter space is briefly sketched. It is noted that this latter solution can also be interpreted as instantons in a class of time-dependent backgrounds.

I. INTRODUCTION

It was shown in Ref. 1 how to construct sequences of instantons, having multicharged monopoles as static limits, by applying Harrison-Neugebauer-type transformations. There the seed solution at the zeroth level from which the iteration starts[1] was a Higgs vacuum with $F^{(0)}_{\mu\nu}=0$. (Some useful results of this formalism are recapitulated in the Appendix.) There exists, however, an exact solution[2,3] which generalizes the well-known, finite-action, one-instanton solution by "inserting" it in a self-dual, Abelian background field with

$$F^{(0)}_{12}=F^{(0)}_{03}=B\frac{\sigma_3}{2} \quad (\epsilon_{0123}=1)\ , \tag{1.1}$$

where B is a constant and all other components of $F^{(0)}_{\mu\nu}$ vanish. For $B=0$, one gets back the standard one-instanton solution. For $B\neq 0$, the total action is evidently divergent, but the *difference*

$$\int d^4x \,\mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}-F^{(0)}_{\mu\nu}F^{(0)}_{\mu\nu}\right]=0\ . \tag{1.2}$$

Solutions of this class have been called "chromons." The search for such solutions is related to recent speculations[4] by many authors concerning a "true" QCD vacuum with nonzero $F^{(0)}_{\mu\nu}$. Self-dual constant fields of the type (1.1) are stable with respect to small classical fluctuations.[3] But suitably varying $F^{(0)}_{\mu\nu}$'s might also turn out to be of interest.[5] Such models have their attractive features as well as evident problems. We will not discuss specific models here, but will adopt the point of view that it is worthwhile to construct interesting types of *exact* solutions with a fairly wide class of backgrounds. This would enable us to better understand the content of subclasses of particular potential interest and the effect of perturbations when certain crucial parameters are varied.

In Secs. II and III [considering, as in Ref. 1, only the SU(2) gauge group] we show that it is possible to insert monopoles in the same sense as instantons in the chromon solutions. Moreover this is achieved for self-dual, Abelian $F^{(0)}_{\mu\nu}$ which is far more general. We give a general treatment for self-dual, axially symmetric, Abelian backgrounds. The particular case (1.1) is, however, paid special attention. We provide the necessary technique for generalizing multicharged monopoles in this fashion, though the explicit treatment is limited to charges 1 and 2. The terms "background" and "monopole," which we use throughout for simplicity, are to be interpreted as follows in the context of our formalism. *The background* ($A^{(0)}_{\mu}$) *is not treated as external; we solve for the closed system.* Such backgrounds can alter the basic topological property of monopoles—a monopole can be "unwound" in the same sense as the instanton in chromon solutions, due to the result (1.2). (See the relevant remarks in Sec. V.) *Throughout we will use the term monopole in the sense that when* $F^{(0)}_{\mu\nu}$ *is reduced to zero* [*such as by setting* $B=0$ *in* (1.1)] *one recovers a finite-energy solution with nonzero topological charge.* As in Ref. 1 we consider static, self-dual, Euclidean SU(2) gauge fields. The component A_t of the gauge potentials can be formally replaced in the solutions by a Higgs scalar Φ for Lorentz signature. This well-known equivalence in flat space, for vanishing Higgs potential, will be implicit in Secs. II and III.

Our backgrounds will be sufficiently general to include superpositions of multipole potentials, with arbitrary coefficients, in the expression for $A_t^{(0)}$ (or the Higgs scalar $\Phi^{(0)}$ in the alternative formulation). Hence one can also compare our results to those for gauge fields in the presence of sources, at least to those for a broad class of axially symmetric Abelian sources expansible in multipoles.[6] We show that by adding components $\vec{A}^{(0)}$ which lead to self-duality of the Abelian background, such a "source" need no longer be treated as external. One can solve, formally, for the closed system. The consequences of the singularities introduced through $A_\mu^{(0)}$ must however be examined for each case.

In Sec. IV we briefly indicate a generalization of our formalism to de Sitter space. It is then pointed out, in Sec. V, that the results of Sec. IV can also be interpreted as a finite number of instantons in a class of t-dependent background.

II. HARRISON-NEUGEBAUER TRANSFORMATIONS STARTING FROM AN AXIALLY SYMMETRIC ABELIAN BACKGROUND

We consider a flat Euclidean space and use spherical coordinates. The gauge group is SU(2). The general form of our ansatz for static, axial symmetry is [with $f(r,\theta),\Psi(r,\theta)$ independent of φ and t and $f_r=\partial_r f(r,\theta)$ etc.]

$$\begin{aligned} A_t&=\frac{f_r}{f}\frac{\sigma_3}{2}+\frac{\Psi_r}{f}\frac{\sigma_1}{2}\,,\\ A_r&=\frac{\Psi_r}{f}\frac{\sigma_2}{2}\,,\\ A_\theta&=\frac{\Psi_\theta}{f}\frac{\sigma_2}{2}\,,\\ (\sin\theta)^{-1}A_\varphi&=\frac{f_\theta}{f}\frac{\sigma_3}{2}+\frac{\Psi_\theta}{f}\frac{\sigma_1}{2}\,. \end{aligned} \tag{2.1}$$

This is the "spherical" R gauge of Ref. 1, where its useful and attractive features were discussed in detail. The self-duality constraints are (with $\epsilon_{tr\theta\varphi}=1$)

$$\begin{aligned} &f\tilde{\Delta}f-(\vec{\nabla}f)^2+(\vec{\nabla}\Psi)^2=0\,,\\ &f\tilde{\Delta}\Psi-2(\vec{\nabla}\Psi\cdot\vec{\nabla}f)=0\,, \end{aligned} \tag{2.2}$$

where (since $\partial_\varphi\approx 0$ on the space of our solutions)

$$\vec{\nabla}=\left[\partial_r,\frac{1}{r}\partial_\theta\right] \tag{2.3}$$

and

$$\tilde{\Delta}=\Delta-\frac{2}{r}\partial_r=\partial_r{}^2+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\,. \tag{2.4}$$

As the starting point we take the seed solution

$$f^{(0)}=e^{\Omega(r,\theta)},\quad \Psi^{(0)}=0 \tag{2.5}$$

with

$$\tilde{\Delta}\Omega=0\,. \tag{2.6}$$

Thus

$$\begin{aligned} &A_t^{(0)}=\Omega_r\frac{\sigma_3}{2},\quad A_r^{(0)}=0,\quad A_\theta^{(0)}=0\,,\\ &(\sin\theta)^{-1}A_\varphi^{(0)}=\Omega_\theta\frac{\sigma_3}{2} \end{aligned} \tag{2.7}$$

with

$$\Omega=\sum_{l=0}^{\infty}(a_l r^{l+1}+b_l r^{-l})P_l(\cos\theta)\,, \tag{2.8}$$

where a_l, b_l are arbitrary constants. [One can also, of course, include terms involving $Q_l(\cos\theta)$. But (2.8) is sufficient for the features we aim to discuss.] For $\Omega=b_0$,

$$A_\mu^{(0)}=0\,. \tag{2.9}$$

For

$$\Omega=a_0r+b_0\,, \tag{2.10}$$

we have the "Higgs vacuum"

$$A_t^{(0)}=a_0,\quad A_r^{(0)}=A_\theta^{(0)}=A_\varphi^{(0)}=0;\quad F_{\mu\nu}^{(0)}=0\,. \tag{2.11}$$

This was the starting point in Ref. 1.

For

$$\Omega=a_1r^2\cos\theta+a_0r+b_0\,, \tag{2.12}$$

we have

$$\begin{aligned} &F_{tr}^{(0)}=-2a_1\cos\theta=\frac{1}{r^2\sin\theta}F_{\theta\varphi}^{(0)}\,,\\ &F_{t\theta}^{(0)}=2a_1r\sin\theta=\frac{1}{\sin\theta}F_{\varphi r}^{(0)}\,,\\ &(\sin\theta)^{-1}F_{t\varphi}^{(0)}=0=F_{r\theta}^{(0)}\,. \end{aligned} \tag{2.13}$$

In terms of the Cartesian components (x_0,x_1,x_2,x_3) with $\epsilon_{0123}=1$ the only nonzero components are

$$F_{03}^{(0)}=-2a_1=F_{12}^{(0)}\,. \tag{2.14}$$

Setting $a_1=-B/2$ we get the constant Abelian field of Ref. 3. (Our static $A_\mu^{(0)}$ is related to the gauge potential of Refs. 2 and 3 through an evident gauge transformation followed by a transformation to spherical coordinates.)

For nonzero a_l's with $l>2$ we get space-dependent $F_{\mu\nu}$'s asymptotically more and more singular. Corresponding to the b_l's $(l=1,2,\ldots)$ one has multipole potentials for Ω_r, i.e., for $A_t^{(0)}$ (or for the Higgs scalar in the alternative formulation). This is accompanied by spherical harmonic components in $\vec{A}^{(0)}$ which assure self-duality of $F_{\mu\nu}^{(0)}$.

We now start applying Harrison-Neugebauer transformations on the seed solutions $A_\mu^{(0)}$. The relevant formalism is summarized in the Appendix. From (A2) and (2.5), one has

$$\begin{aligned} M_1^{(0)}&=M_2^{(0)}=\tfrac{1}{2}\partial_+\Omega\ ,\\ N_1^{(0)}&=N_2^{(0)}=\tfrac{1}{2}\partial_-\Omega\ . \end{aligned} \tag{2.15}$$

Hence from (A4)

$$\partial_\pm q=\tfrac{1}{2}e^{\pm i(\omega-\theta)}\partial_\pm\Omega\ . \tag{2.16}$$

Setting

$$q=\tanh Q/2 \tag{2.17}$$

gives

$$\partial_\pm Q=e^{\pm i(\omega-\theta)}\partial_\pm\Omega\ . \tag{2.18}$$

For $\omega=\theta$, i.e., for $c=0$ in (A5),

$$Q=\Omega-K$$

or

$$q=\tanh\tfrac{1}{2}(\Omega-K)\ , \tag{2.19}$$

K being an integration constant. Now from (A6)–(A10), with $K=0$,

$$\begin{aligned} M_1^{(1)}&=BM_1^{(0)}\\ &=\tfrac{1}{2}\left[\coth\frac{\Omega}{2}\right]\partial_+\Omega+\tfrac{1}{2}\partial_+\left[\ln\left[\frac{\sin\theta}{r}\right]\right] \end{aligned}$$

or

$$M_1^{(1)}=\tfrac{1}{2}\partial_+\left[\ln\left[\sinh^2\frac{\Omega}{2}\frac{\sin\theta}{r}\right]\right]\ . \tag{2.20}$$

Similarly,

$$M_2^{(1)}=\tfrac{1}{2}\partial_+\left[\ln\left[\cosh^2\frac{\Omega}{2}\frac{\sin\theta}{r}\right]\right]\ , \tag{2.21}$$

$$N_1^{(1)}=\tfrac{1}{2}\partial_-\left[\ln\left[\sinh^2\frac{\Omega}{2}\frac{\sin\theta}{r}\right]\right]\ , \tag{2.22}$$

and

$$N_2^{(1)}=\tfrac{1}{2}\partial_-\left[\ln\left[\cosh^2\frac{\Omega}{2}\frac{\sin\theta}{r}\right]\right]\ . \tag{2.23}$$

It follows that

$$f^{(1)}=\frac{\sinh\Omega}{r}\sin\theta\ , \tag{2.24}$$

$$\Psi_r^{(1)}=\frac{\sin\theta}{r^2}\Omega_\theta\ , \tag{2.25}$$

$$\Psi_\theta^{(1)}=-\sin\theta\,\Omega_r\ . \tag{2.26}$$

The integrability condition for Ψ is just the self-duality constraint

$$\tilde{\Delta}\Omega=0\ . \tag{2.27}$$

To construct $A_\mu^{(1)}$, $f^{(1)}$, $\Psi_r^{(1)}$, and $\Psi_\theta^{(1)}$ are sufficient. But one can easily integrate for $\Psi^{(1)}$. For

$$\Omega=\sum_{l=0}^{\infty}(a_l r^{l+1}+b_l r^{-l})P_l(\cos\theta)\ ,$$

we have

$$\begin{aligned} \Psi^{(1)}&=\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\left[\sin\theta\frac{d}{d\theta}P_l(\cos\theta)\right]\\ &\quad\times\left[\frac{d}{dr}(a_l r^{l+1}+b_l r^{-l})\right]\\ &=\sum_{l=0}^{\infty}\left[a_l r^l-\frac{l b_l}{(l+1)}r^{-(l+1)}\right]\\ &\quad\times(\cos\theta P_l-P_{l-1})\ . \end{aligned} \tag{2.28}$$

(For $l=0$ only the second form in well defined.) Setting $\Omega=r$, we get the Prasad-Sommerfield monopole in our gauge [Eq. (2.28) of Ref. 1],

$$\begin{aligned} f^{(1)}&=\frac{\sinh r}{r}\sin\theta\ ,\\ \Psi^{(1)}&=\cos\theta\ . \end{aligned} \tag{2.29}$$

For

$$\Omega=r-\frac{B}{2}r^2\cos\theta$$

we get a monopole imbedded in a constant Abelian background [see (2.14)] with

$$\begin{aligned} f^{(1)}&=r^{-1}\sinh\left[r-\frac{B}{2}r^2\cos\theta\right]\sin\theta\ ,\\ \psi^{(1)}&=\cos\theta+\frac{B}{2}r^2\sin^2\theta\ . \end{aligned} \tag{2.30}$$

We now go back to (2.18) and consider $c\neq0$.

Then

$$Q_r = \Omega_r \cos(\omega-\theta) - r^{-1}\Omega_\theta \sin(\omega-\theta) \,,$$
$$r^{-1}Q_\theta = r^{-1}\Omega_\theta \cos(\omega-\theta) + \Omega_r \sin(\omega-\theta) \,, \tag{2.31}$$

where

$$\cos(\omega-\theta) = R^{-1}(r - c\cos\theta) \,,$$
$$\sin(\omega-\theta) = R^{-1} c \sin\theta \,, \tag{2.32}$$

and

$$R = (r^2 + c^2 - 2cr\cos\theta)^{1/2} \,. \tag{2.33}$$

We have obtained the following solutions. They can be easily verified, using the well-known properties of Legendre functions. For

$$\Omega = \Omega_1^{(l)} \equiv r^{l+1} P_l(\cos\theta) \,,$$

we have

$$Q = Q_1^{(l)} = Rc^l \left[\sum_{n=0}^{l} \left[\frac{r}{c} \right]^n P_n(\cos\theta) \right] - K \,, \tag{2.34}$$

where K is a constant. The series involves the first $(l+1)$ terms of the development of R^{-1} for $r < c$. For

$$\Omega = \Omega_2^{(l)} \equiv r^{-l} P_l(\cos\theta) \,,$$

we have

$$Q = Q_2^{(l)} = -\frac{R}{rc^l} \left[\sum_{n=0}^{l-1} \left[\frac{c}{r} \right]^n P_n(\cos\theta) \right] + c^{-l} - K \,. \tag{2.35}$$

The term c^{-l} has been separated from K to ensure the limit (2.19) as $c \to 0$ with K independent of c. For this we note that in (2.35), for $c < r$,

$$Q_2^{(l)} = -\frac{R}{c^l} \left[R^{-1} - r^{-1} \sum_{n=l}^{\infty} \left[\frac{c}{r} \right]^n P_n(\cos\theta) \right] + c^{-l} - K \,. \tag{2.36}$$

For (2.28) or

$$\Omega = \sum_{l=0}^{\infty} (a_l \Omega_1^{(l)} + b_l \Omega_2^{(l)}) \,, \tag{2.37}$$

we have

$$Q = \sum_{l=0}^{\infty} (a_l Q_1^{(l)} + b_l Q_2^{(l)}) \,. \tag{2.38}$$

Now that we have obtained Q for arbitrary c by varying the parameters c and K and using the composition theorem (A11)–(A14) we can iterate an arbitrary number of H (or B) transformations. The specific features will depend on the chosen Ω (i.e., on the a's and the b's). The simplest nontrivial case, when the iteration starts from (2.10), was studied in Ref. 1. In the next section we discuss certain aspects of another interesting case, that of the "constant background field" [from (2.12) to (2.14)].

Here we mention briefly that a complex gauge transformation by [compare Eqs. (2.35)–(2.37) of Ref. 1]

$$U = e^{i\theta\sigma_2/2} \,, \tag{2.39}$$

where

$$\cos\theta = -\coth\Omega \,,$$
$$\sin\theta = (i \sinh\Omega)^{-1} \,, \tag{2.40}$$

leads from (2.5) to

$$f' = (\sinh\Omega)^{-1}, \quad \Psi' = i\coth\Omega \,. \tag{2.41}$$

Using this complex gauge, (2.24)–(2.26) are immediately obtained by using simply the I transformation (A9). As in Ref. 1 we could have used a complex Ehler's transformation to replace (2.39).

III. STUDY OF THE CASE WITH A CONSTANT BACKGROUND FIELD

For this case[2,3] we take

$$\Omega = r - \frac{B}{2} r^2 \cos\theta \,. \tag{3.1}$$

(We fix the scale by normalizing the coefficient of the first term to 1.) Such an Ω has no singularity in the finite region. Let us assume that we can choose the parameters in the H (or B) transformation such that no singularities arise in the gauge-invariant quantities due to them. (We will come back to this point again.) Then the energy in a sphere of radius r' about the origin can be shown to be given by

$$E' = \pi \int_0^\pi \sin\theta \, d\theta \left[r^2 \partial_r \left[\frac{f_r{}^2 + \Psi_r{}^2}{f^2} \right] \right]_{r=r'} \,. \tag{3.2}$$

Making $r' \to \infty$, we have the total energy which, of course, diverges due to nonzero B in (3.1). For the background only one has

$$(f^{(0)})^{-2}\left[\left[f_r^{(0)}\right]^2+\left[\Psi_r^{(0)}\right]^2\right]=\Omega_r{}^2=(1-Br\cos\theta)^2 . \tag{3.3}$$

The corresponding value of E' is

$$E'_{(0)}=\tfrac{4}{3}\pi B^2 r'^3 . \tag{3.4}$$

We want to study the limit

$$\lim_{r'\to\infty}(E'-E'_{(0)}) \tag{3.5}$$

after successive "monopole" insertions. For $B=0$ ($E'_{(0)}=0$) we get the pure monopole case. Otherwise we study the variation with respect to the background level.

For a single B transformation (when one has the PS monopole for $B=0$) one obtains from (2.24) and (2.25)

$$D\equiv\frac{f_r{}^2+\Psi_r{}^2}{f^2}=\left[\Omega_r\coth\Omega-\frac{1}{r}\right]^2+\left[\frac{1}{\sinh\Omega}\frac{1}{r}\Omega_\theta\right]^2 . \tag{3.6}$$

This is regular as $r\to 0$ and for all finite values of r. [As $r\to 0$ one has the behavior of the pure monopole case as is evident from (3.1).] As $r\to\infty$ we have to distinguish two cases for (3.1). For $B=0$,

$$\coth\Omega\to 1 \quad\text{as } r\to\infty . \tag{3.7}$$

For $B\neq 0$, $\lim_{r\to\infty}\coth\Omega$ is no longer independent of θ. *Taking $B>0$ for definiteness* (which does not effect the conclusion to be drawn)

$$\lim_{r\to\infty}\coth\left[r-\frac{Br^2}{2}\cos\theta\right]\to -1 \quad\text{for } \theta<\frac{\pi}{2}$$
$$\to +1 \quad\text{for}+\theta\geq\frac{\pi}{2} . \tag{3.8}$$

(Near $\theta=\pi/2$ or $\cos\theta=\epsilon\to 0$, we take $\frac{1}{2}Br>|\epsilon|^{-1}$.) Hence as $r\to\infty$, with $D_{(0)}=r^2B^2\cos^2\theta$,

$$(D-D_{(0)})\to -2Br\cos\theta+1+\frac{1}{r^2}\mp 2\left[B\cos\theta-\frac{1}{r}\right]+O(e^{-r}) \tag{3.9}$$

or

$$r^2\partial_r(D-D_{(0)})\to -2Br^2\cos\theta\mp 2+\cdots \tag{3.10}$$

accordingly as $\theta<\pi/2$ or $\theta\geq\pi/2$, respectively. For $B=0$, $r^2\partial_r D\to 2$ for all values of θ. The θ integration in (3.2) or (3.5) for any finite value r' of r cancels the first term in (3.10). Thus we see that due to the change of sign in (3.10),

$$\lim_{r'\to\infty}(E'-E'_{(0)})=0 . \tag{3.11}$$

This is to be compared with the result that for $B=0$, $E'_{(0)}=0$, $E=4\pi$, and hence finally one has a magnetic charge 1. The result (3.11) seems to be a general feature for successive iterations. But first one has to verify that no additional singularities appear in the *finite region* for a particular choice of parameters. As a relatively simple nontrivial case we discuss the case of two transformations when for $B=0$ one has a monopole of charge 2.[1] From (A2) and (A16), for this case

$$D=\frac{f_r{}^2+\Psi_r{}^2}{f^2}=4(M_1+N_2)(M_2+N_1)$$
$$=4\left|\frac{Q_1}{Q_2}M_1^{(0)}+\frac{Q_2}{Q_1}N_2^{(0)}-i\frac{(p_1{}^2-p_2{}^2)}{4r\sin\theta}\left[\frac{e^{i\theta}}{Q_2}+\frac{e^{-i\theta}}{p_1p_2Q_1}\right]\right|^2 , \tag{3.12}$$

where Q_1,Q_2 are given by (A15) with

$$p_j=R_j{}^{-1}[(r-c_j\cos\theta)+ic_j\sin\theta] ,$$

$$q_j=\tanh\tfrac{1}{2}\left[R_j-\frac{B}{2}R_j(c_j+r\cos\theta)-K_j\right] , \quad (j=1,2)$$

and

$$R_j=(r^2+c_j{}^2-2c_jr\cos\theta)^{1/2} .$$

$M_1^{(0)}$ and $N_2^{(0)}$ are given by (2.15) and (3.1). For $B=0$ ($\Omega=r$) the two crucial domains where D has to be studied carefully are $\theta\to 0$ and $\theta\to\pi/2$. It is found[1,7,8] that for

$$\begin{aligned} &c_1=-c_2=i\pi/2 ,\\ &K_1=0,\ K_2=i\pi , \end{aligned} \tag{3.13}$$

one has a nonsingular, finite-energy monopole of charge 2 in an explicitly real gauge. For the limit $\theta\to 0$ one has to develop in powers of θ up to θ^2 and the regularity constraint for $\theta=\pi/2$ fixes the value of c_1 ($=-c_2$) as $i\pi/2$. Let us now note the effects of nonzero B ($\Omega=r-\frac{1}{2}Br^2\cos\theta$). *We will maintain the parameters c_j and K_j of (3.13).* One obtains, after careful computations,

$$\lim_{\theta\to 0} D=\left\{(1-Br)\tanh\left[r-\frac{B}{2}\left[r^2+\frac{\pi^2}{4}\right]\right]-\frac{2r}{r^2+\pi^2/4}\right\}^2 \quad (r=z \text{ for} \theta=0) . \tag{3.14}$$

Thus again there is no singularity and as $B\to 0$ one gets back smoothly the expression for a monopole of charge 2. For $\theta=\pi/2$ $[r=(x^2+y^2)^{1/2}]$, defining $\delta\equiv(\pi^2/4-r^2)^{1/2}$ (since we study first the domain $r\le\pi/2$) one has

$$D=\left[\frac{2(\pi^2/4)\cos\delta(\delta\cos\delta-\sin\delta)}{\delta[\delta^2-(\pi^2/4)\sin^2\delta-r^2\sinh^2(\frac{1}{4}B\pi\delta)]}-1\right]^2+\left[\frac{\frac{1}{2}\pi\cos\delta[\delta Br^2\cosh(\frac{1}{4}B\pi\delta)-\pi\sinh(\frac{1}{4}B\pi\delta)]}{\delta[\delta^2-(\pi^2/4)\sin^2\delta-r^2\sinh^2(\frac{1}{4}B\pi\delta)]}\right]^2 . \tag{3.15}$$

For $B=0$, one gets back the corresponding expression for charge-2 monopole.[1,8] There the denominator $[\delta^2-(\pi^2/4)\sin^2\delta]<0$ for $0<r<\pi/2$ and $\approx-\epsilon^2$ as $r\to\epsilon$ or $\delta\to\epsilon$ and $\epsilon\to 0$. These zeros are compensated for by those of $\cos\delta(\delta\cos\delta-\sin\delta)$ in the numerator, giving finite values. Here we note that the presence of B does not modify this aspect of the situation. In the denominator the additional term $\{-r^2\sinh^2[(B\pi/4)\delta]\}$ is always negative and $\approx-\epsilon^2$ at the two limits. The numerator of the second squared term in D also plays the same role as $\cos\delta(\delta\cos\delta-\sin\delta)$ in the first term. Thus B does not introduce any singularity for $\theta=\pi/2$ as is easy to verify again for $r>\pi/2$. Apart from these two values of θ, $\cos\theta$ and $\sin\theta$ being both nonzero, it is not difficult to see that the denominators in (3.12) do not vanish. Due to the simplicity of $F^{(0)}_{\mu\nu}$ it is not unexpected that no essential singularity appears (for finite r). But we have verified the less evident fact that the particular choice of parameters (3.13) need not be modified, the regularity constraints having no B dependence at all in the finite region.

There is however a crucial change in the asymptotic behavior. One has, as $r\to\infty$,

$$D\to\left[(1-Br\cos\theta)\pm\frac{2}{r}\right]^2 \quad \text{as } \theta\lessgtr\frac{\pi}{2} , \tag{3.16}$$

respectively. Hence (retaining only the crucial terms),

$$r^2\partial_r D\to 2Br^3\cos^2\theta\mp 4+\cdots \tag{3.17}$$

since an eventual θ integration eliminates the term proportional to $\cos\theta$. Hence

$$r^2\partial_r(D-D_{(0)})\to\mp 4+\cdots \quad \text{for } \theta\lessgtr\frac{\pi}{2} . \tag{3.18}$$

Hence again

$$\lim_{r'\to\infty}(E'-E'_{(0)})=0 . \tag{3.19}$$

This is to be compared with the result that for $B=0$ $(D_{(0)}=0=E'_{(0)})$

$$r^2\partial_r D\to 4 \quad \text{for } 0\le\theta\le\pi , \tag{3.20}$$

leading to a magnetic charge 2.

The situation should also be compared with that for the simplest Ω involving a negative power of r, namely,

$$\Omega=r-\frac{\lambda}{r}\cos\theta . \tag{3.21}$$

Here the asymptotic behavior is that due to $\Omega=r$. But as $r\to 0$, not only D but also $(D-D_{(0)})$ remains singular. This again is a general feature for higher negative powers of r in Ω. This aspect will, however, not be pursued further in this paper.

IV. GENERALIZATION TO de SITTER SPACE

The technique for such a generalization was given in Ref. 1 and is summarized in the Appendix, (A17)–(A27). The scaling limit of Ref. 1, namely, $\rho=r/\alpha,\tau=t/\alpha$ (hence $A_\tau=\alpha A_t, A_\rho=\alpha A_r$) with $\alpha\to\infty$ gives back the static solutions of Secs. II and III. The t dependence obtained through (A18) is discussed in the following section. The ansatz now is [with the domains (A20) for τ and ρ]

$$\begin{aligned} A_\tau&=\frac{f_\rho}{f}\frac{\sigma_3}{2}+\frac{\Psi_\rho}{f}\frac{\sigma_1}{2} ,\\ A_\rho&=\frac{\Psi_\rho}{f}\frac{\sigma_2}{2} ,\\ A_\theta&=\frac{\Psi_\theta}{f}\frac{\sigma_2}{2} ,\\ (\sin\theta)^{-1}A_\varphi&=\frac{f_\theta}{f}\frac{\sigma_3}{2}+\frac{\Psi_\theta}{f}\frac{\sigma_1}{2} . \end{aligned} \tag{4.1}$$

The self-duality constraints are

$$f\tilde{\Delta}f-(\vec{\nabla}f)^2+(\vec{\nabla}\Psi)^2=0\ ,\qquad f\tilde{\Delta}\psi-2(\vec{\nabla}\Psi\cdot\vec{\nabla}f)=0\ , \tag{4.2}$$

where now

$$\vec{\nabla}=\left[\partial_\rho,\frac{1}{\sinh\rho}\partial_\theta\right] \tag{4.3}$$

and

$$\tilde{\Delta}=\partial_\rho{}^2+\frac{1}{\sinh^2\rho}\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\ . \tag{4.4}$$

The seed solution is

$$f^{(0)}=e^{\Omega(\rho,\theta)},\quad \Psi^{(0)}=0 \tag{4.5}$$

with

$$\tilde{\Delta}\Omega=0\ . \tag{4.6}$$

Instead of (2.8) one has now

$$\Omega=\sum_{l=0}^{\infty}[a_lQ_l(\coth\rho)+b_lP_l(\coth\rho)]P_l(\cos\theta)\equiv\sum_{l=0}^{\infty}\Omega^{(l)},\ \text{say}\ . \tag{4.7}$$

The Legendre functions of the argument $\coth\rho$ give

$$P_0(\coth\rho)=1,\quad Q_0(\coth\rho)=\rho\ , \tag{4.8}$$

$$P_1(\coth\rho)=\coth\rho\ ,\qquad Q_1(\coth\rho)=(\rho\coth\rho-1)\ , \tag{4.9}$$

and so on. Note that now $P_l(\coth\rho)$ is singular as $\rho\to 0$ and not $Q_l(\coth\rho)$. One has as before

$$M_1^{(0)}=M_2^{(0)}=\tfrac{1}{2}\partial_+\Omega\ ,\qquad N_1^{(0)}=N_2^{(0)}=\tfrac{1}{2}\partial_-\Omega\ , \tag{4.10}$$

where

$$\partial_\pm=\tfrac{1}{2}\left[\partial_\rho\pm\frac{i}{\sinh\rho}\partial_\theta\right]\ .$$

Harrison's pseudopotential satisfies

$$\partial_\pm q=e^{i\gamma}(1-q^2)\partial_\pm\Omega\ , \tag{4.11}$$

where γ is given by (A24) and (A25). Defining

$$q=\tanh Q/2\ , \tag{4.12}$$

one obtains

$$Q_\rho=\Omega_\rho\cos\gamma-(\sinh\rho)^{-1}\Omega_\theta\sin\gamma\ , \tag{4.13}$$

$$(\sinh\rho)^{-1}Q_\theta=(\sinh\rho)^{-1}\Omega_\theta\cos\gamma+\Omega_\rho\sin\gamma\ . \tag{4.14}$$

The integrability condition is just (4.6). For $c=0$ ($\gamma=0$) once again

$$q=\tanh\tfrac{1}{2}(\Omega-K)\ . \tag{4.15}$$

Corresponding to (2.24)–(2.26), one has

$$f^{(1)}=\frac{\sinh\Omega}{\sinh\rho}\sin\theta\ , \tag{4.16}$$

$$\Psi_\rho^{(1)}=\frac{\sin\theta}{\sinh\rho}\Omega_\theta\ , \tag{4.17}$$

$$\Psi_\theta^{(1)}=-\sin\theta\Omega_\rho\ . \tag{4.18}$$

Hence

$$\Psi^{(1)}=\frac{1}{l(l+1)}\left[\sin\theta\frac{d}{d\theta}P_l(\cos\vartheta)\right]\times\left[a_l\frac{d}{d\rho}Q_l(\coth\rho)+b_l\frac{d}{d\rho}P_l(\coth\rho)\right] \tag{4.19}$$

for $\Omega=\Omega^{(l)}$. For $\Omega=\sum_{l=0}^{\infty}\Omega^{(l)}$, one has only to take the corresponding superposition for $\Psi^{(1)}$. For $\gamma\neq 0$, the situation is less simple than that for (2.34)–(2.38). Let us consider successively some of the simplest cases, namely,

$$\Omega=\alpha\rho\ , \tag{4.20}$$

$$\Omega=\alpha^{-1}\coth\rho\cos\theta\ , \tag{4.21}$$

$$\Omega=3\alpha^2(\rho\coth\rho-1)\cos\theta\ , \tag{4.22}$$

and

$$\Omega=(6\alpha^2)^{-1}(3\coth^2\rho-1)(3\cos^2\theta-1)\ . \tag{4.23}$$

The coefficients involving α are so chosen that in the limit $\rho=r/\alpha$ with $\alpha\to\infty$ one recovers from (4.20), (4.21), (4.22), and (4.23), respectively,

$$\Omega=r,r^{-1}\cos\theta,\ r^2\cos\theta\ \text{and}\ r^{-2}\tfrac{1}{2}(3\cos^2\vartheta-1)\ . \tag{4.24}$$

Thus, for example, (4.22) is the generalization of the constant background-field case of Sec. II. The "Higgs vacuum" case (4.20) is the one treated in detail in Ref. 1. For (4.21), (4.22), and (4.23) one obtains from (4.13) and (4.14), respectively,

$$Q=\alpha^{-1}\left[-\frac{\sinh\eta}{\sinh c\,\sinh\rho}+\frac{1}{c}\right]-K\ , \qquad (4.25)$$

$$Q=(3\alpha^2)\left[\eta\coth c-\frac{\rho\sinh\eta}{\sinh c\,\sinh\rho}\right]-K\ , \quad (4.26)$$

and

$$Q=-\alpha^{-2}\left[\left[\frac{\sinh\eta}{\sinh c\,\sinh\rho}\right]\times(\coth\rho\cos\theta+\coth c)+\frac{1}{c^2}\right]-K\ . \qquad (4.27)$$

In the limit mentioned above these give $Q_2^{(1)}$, $Q_1^{(1)}$, and $Q_2^{(2)}$ [(2.35) and (2.34)], respectively. Since we have the solutions for arbitrary parameters one can iterate H or B transformations as indicated in the Appendix. We will not attempt here to give a solution Q for the general form (4.7) of Ω.

V. REMARKS

In the preceding section we constructed static solutions. But it is known[9] that, for example, the static Prasad-Sommerfield (PS) monopole can be gauge transformed into a periodic form where it appears as a superposition of an infinite number of instantons. In our conventions, the periodic form is, using 't Hooft's η's,

$$\begin{aligned}A_{\mu a}&=\eta^{(+)}_{\mu\nu a}\partial_\nu(\ln\Lambda)\ ,\\ \Lambda&=\frac{\sinh r}{2r(\cosh r-\cos t)}\\ &=\sum_{m=-\infty}^{\infty}\frac{1}{[(t-2m\pi)^2+r^2]}\ .\end{aligned} \qquad (5.1)$$

From (2.29) one can go over to the regular PS gauge [see Eqs. (2.28)–(2.34) of Ref. 1] by transforming with

$$U_1=e^{-i\varphi\sigma_3/2}e^{-i\theta\sigma_2/2}e^{-i\pi\sigma_2/2}\ . \qquad (5.2)$$

Then another transformation by[9]

$$U_2=e^{-i\Psi\hat{r}\cdot\vec{\sigma}/2}\ ,$$

where

$$\tan\Psi=\frac{\sin t\,\sinh r}{\cosh r\,\cos t-1} \qquad (5.3)$$

gives (5.1).

Hence (2.30) can also be interpreted as the insertion of an infinite, periodic sequence of instantons in the background (2.14). [U_2U_1 will now give a more complicated but still, evidently, a periodic form, which moreover will reduce to (5.1) for $B=0$. A suitable B-dependent generalization of U_1 and U_2 can possibly give a more elegant periodic form, but U_2U_1 is sufficient for the present argument.]

Concerning periodic chromons it was remarked in Ref. 3 that the result (1.2) need not hold. [See the remarks following Eq. (10.8) of Ref. 3.] We see that for the particular type we consider, (3.11) and hence (1.2), continues to hold. (The term that leads to a finite index for $B=0$ is indeed there but it is accompanied by a change of sign at $\theta=\pi/2$ which annihilates the contribution.)

It is noted in Ref. 1 that multiply charged monopoles should correspond to periodic sequences of instantons of higher Atiyah-Ward classes. It would presumably be quite difficult to find the gauges which make this explicit. But implicitly at least, one can consider that through successive Harrison-Neugebauer transformations one inserts here periodic instanton solutions in the backgrounds (2.8).

In Refs. 2 and 3 one had an exact solution for one instanton in a constant, self-dual, Abelian background. Here we have obtained monopoles or periodic instanton sequences in a more general background. But what about a similar insertion of a *finite number* of instantons—namely, solutions that reduce to an arbitrary but a finite index instanton configuration when the background is switched off? In Ref. 1 it was explained how the formalism used in Sec. IV gives (for $\Omega=\alpha\rho$) a finite number of instantons for integral values of α. Here again, transforming back to (t,r) from (τ,ρ) through (1.18) one can similarly interpret our solutions of Sec. IV. One obtains time-dependent (Euclidean time t dependent) instantons in axially symmetric, self-dual, Abelian backgrounds. But in doing so, the background itself becomes t dependent in a complicated fashion. Now one has to substitute in (4.7),

$$\coth\rho=\frac{1+t^2+r^2}{2r}\ . \qquad (5.4)$$

Thus, in general $F^{(0)}_{\mu\nu}$ becomes t dependent through (4.1) and (4.5). (For $\Omega=\alpha\rho$, $F^{(0)}_{\mu\nu}=0$ and hence this aspect did not arise in Ref. 1.) Such backgrounds do not seem to have any evident interpretation. For this reason we have given only a brief treatment of these solutions. This, however, serves to select out backgrounds which can be rendered "τ static" and to which this technique can hence be applied. Secondly if, say, some type of de Sitter bag model is envisaged, the solutions of Sec. IV might have direct significance as static solutions in the internal region. For finite number of instantons in "t static" backgrounds a different technique needs to be found. This will be attempted elsewhere.

The solutions obtained in this paper suggest further study in several directions—fluctuations about such solutions, spinors coupled to such A_μ's, and so on. Solutions not related to instantons or monopoles can also be envisaged. One evident possible generalization would be the removal of the axial-symmetry constraint. But this would again require a different technique.

APPENDIX

We recapitulate here the essential features of Harrison-Neugebauer-type transformations adapted to the spherical R gauge. For details Ref. 1 and the sources quoted therein should be consulted. In particular, in the papers of Forgacs, Horvath, and Palla[7,8] these transformations are used to construct monopoles in a gauge different from ours. (All these papers treat the case without background field, namely, $F^{(0)}_{\mu\nu}=0$.)

We start with the ansatz (2.1) and the self-duality constraints (2.2)–(2.4). We define

$$\partial_\pm=\tfrac{1}{2}\left(\partial_r\pm\frac{i}{r}\partial_\theta\right), \tag{A1}$$

$$\begin{aligned} M_1&=\frac{1}{2f}\partial_+(f+i\Psi), \quad M_2=\frac{1}{2f}\partial_+(f-i\Psi)\\ N_1&=\frac{1}{2f}\partial_-(f-i\Psi), \quad N_2=\frac{1}{2f}\partial_-(f+i\Psi)\end{aligned} \tag{A2}$$

The equations (2.2) are equivalent to

$$\begin{aligned}\partial_-M_1&=-M_1(N_1-N_2)-\frac{ie^{i\theta}}{4r\sin\theta}(N_2-M_1)\,,\\ \partial_-M_2&=-M_2(N_2-N_1)-\frac{ie^{i\theta}}{4r\sin\theta}(N_1-M_2)\,,\\ \partial_+N_1&=-N_1(M_1-M_2)+\frac{ie^{-i\theta}}{4r\sin\theta}(M_2-N_1)\,,\\ \partial_+N_2&=-N_2(M_2-M_1)+\frac{ie^{-i\theta}}{4r\sin\theta}(M_1-N_2)\,.\end{aligned} \tag{A3}$$

Harrison's pseudopotential q satisfies

$$\begin{aligned}\partial_+q&=(M_2-M_1)q+e^{i(\omega-\theta)}(M_2-M_1q^2)\,,\\ \partial_-q&=(N_1-N_2)q+e^{-i(\omega-\theta)}(N_1-N_2q^2)\,,\end{aligned} \tag{A4}$$

where ω is given (for a given parameter c) by

$$R\cos\omega=r\cos\theta-c\,,$$

$$R\sin\omega=r\sin\theta\,,$$

with

$$R=(r^2+c^2-2cr\cos\theta)^{1/2}\,. \tag{A5}$$

This leads to a family of solutions $q(c,K)$ where K is an integration constant. We will not exclude *complex* values of c and K.

Harrison's transformation H adapted to our case is now defined as follows. Let

$$\begin{aligned}\zeta_\pm&=\pm\frac{ie^{\pm i\theta}}{4r\sin\theta}\,,\\ p&=e^{i(\omega-\theta)}\end{aligned} \tag{A6}$$

and

$$\bar{q}=-\frac{p+q}{1+pq}\,. \tag{A7}$$

Then

$$\begin{aligned}HM_1&=\frac{q}{\bar q}M_1+\left[1+\frac{p}{\bar q}\right]\zeta_+\,,\\ HM_2&=\frac{\bar q}{q}M_2+(1+p\bar q)\zeta_+\,,\\ HN_1&=\frac{1}{q\bar q}N_1+\left[1+\frac{1}{p\bar q}\right]\zeta_-\,,\\ HN_2&=q\bar q N_2+\left[1+\frac{\bar q}{p}\right]\zeta_-\,.\end{aligned} \tag{A8}$$

If the set $M_1,\ldots,N_2$ satisfies (A3) and q (A4), then the set $HM_1,\ldots,HN_2$ also satisfies (A3).

The Neugebauer-Kramer mapping I is, for our case,

$$\begin{aligned}If&=\frac{\sin\theta}{rf}\,,\\ I(\Psi_r/f)&=\frac{i\Psi_\theta}{f}\,,\\ I\left[\frac{1}{r}\Psi_\theta/f\right]&=-\frac{\Psi_r}{if}\,.\end{aligned} \tag{A9}$$

If and $I\Psi$ provide new solutions of (2.2). The transformation

$$B = IH \tag{A10}$$

can now be defined. This has been used in (2.20)–(2.26). In applying successive H transformations, the following composition theorem is of crucial importance. Suppose one can solve (A4) for a particular set M_j, N_j $(j=1,2)$. Then for two sets of parameters (c_1, K_1), (c_2, K_2) we write

$$p_j = p(c_j), \quad q_j = q(c_j, K_j) , \tag{A11}$$

with the corresponding $\bar{q}_j$. Then we define

$$q' = \frac{(\bar{q}_1 p_2 - \bar{q}_2 p_1)}{q_1(\bar{q}_1 p_1 - \bar{q}_2 p_2)} ,$$

when (A12)

$$\bar{q}' = \frac{(q_1 p_2 - q_2 p_1)}{\bar{q}_1(q_1 p_1 - q_2 p_2)} .$$

The theorem states that the effect of two *successive* transformations

$$(M_j, N_j) \xrightarrow{H} (M_j', N_j') \xrightarrow{H'} (M_j'', N_j'') \tag{A13}$$

corresponding to the parameters (c_1, K_1) and (c_2, K_2), respectively, is given directly by [see (A8)]

$$H'HM_1 = \frac{q'}{\bar{q}'} M_1 + \left[1 + \frac{p_2}{\bar{q}'} \right] \zeta_+ \tag{A14}$$

and so on. [One does not have to integrate (A4) with M_j', N_j', p_2 all over again in applying H'.] This opens the way for composing successively any number of transformations algebraically after integrating (A4) *once* with a suitable seed solution $M_j^{(0)}, N_j^{(0)}$ but with arbitrary c.

Explicitly two successive H transformations give the following result. Let

$$Q_1 = (q_1 p_1 - q_2 p_2), \quad Q_2 = (q_1 p_2 - q_2 p_1)$$

when

$$\bar{Q}_1 = (\bar{q}_1 p_1 - \bar{q}_2 p_2), \quad \bar{Q}_2 = (\bar{q}_1 p_2 - \bar{q}_2 p_1) . \tag{A15}$$

One obtains finally

$$M_1 = \frac{\bar{Q}_2}{\bar{Q}_1} \left[\frac{Q_1}{Q_2} M_1^{(0)} + \frac{(p_1{}^2 - p_2{}^2)}{Q_2} \zeta_+ \right] ,$$

$$M_2 = \frac{\bar{Q}_1}{\bar{Q}_2} \left[\frac{Q_2}{Q_1} M_2^{(0)} - \frac{q_1 q_2 (p_1{}^2 - p_2{}^2)}{Q_1} \zeta_+ \right] , \tag{A16}$$

$$N_1 = \frac{\bar{Q}_1}{\bar{Q}_2} \left[\frac{Q_1}{Q_2} N_1^{(0)} + \frac{q_1 q_2 (p_1{}^2 - p_2{}^2)}{p_1 p_2 Q_2} \zeta_- \right] ,$$

$$N_2 = \frac{\bar{Q}_2}{\bar{Q}_1} \left[\frac{Q_2}{Q_1} N_2^{(0)} - \frac{(p_1{}^2 - p_2{}^2)}{p_1 p_2 Q_1} \zeta_- \right] .$$

Products of B $(=IH)$ transformations can, as compared to those of H, be reduced essentially to a redefinition of the parameters involved. Composing n transformations one ends up with a remarkable structure in terms of determinants.[7,8] We will not treat here explicitly the higher-order cases.

So far we have been considering the flat Euclidean space and the usual spherical coordinates with

$$ds^2 = dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{A17}$$

and

$$-\infty < t < \infty, \quad 0 \le r < \infty ,$$
$$0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi .$$

The transformation

$$(t + ir) = \tan \left[\frac{\tau + i\rho}{2} \right] \tag{A18}$$

gives

$$ds^2 = (\cosh\rho + \cos\tau)^{-2} [d\tau^2 + d\rho^2 + \sinh^2\rho \times (d\theta^2 + \sin^2\theta \, d\varphi^2)] , \tag{A19}$$

where now

$$-\pi \le \tau \le \pi, \quad 0 \le \rho < \infty . \tag{A20}$$

For

$$ds^2 = (\cosh\rho)^{-2} [d\tau^2 + d\rho^2 + \sinh^2\rho \times (d\theta^2 + \sin^2\theta \, d\varphi^2)] \tag{A21}$$

we have a de Sitter space with nonzero constant curvature, here normalized by choice of scale. In formally constructing gauge field solutions, as in Sec. IV, an overall conformal factor can be ignored. But the factor $\sinh^2\rho$ before the angular terms now plays a crucial role. Harrison-type transformations were treated in detail for such a line element in Ref. 1. Here we note briefly the following modifications as compared to (A1)–(A16). Now in (A1) and (A2) [after starting with (4.1)–(4.4)]

$$\partial_\pm = \tfrac{1}{2} \left[\partial_\rho \pm \frac{i}{\sinh\rho} \partial_\theta \right] \tag{A22}$$

and in (A3) the factors

$$\zeta_\pm = \pm i \frac{e^{\pm i\theta}}{4r \sin\theta}$$

are now replaced, respectively, by

$$\zeta_{\pm}=-\tfrac{1}{4}\left[\frac{\cosh\rho\mp i\cot\theta}{\sinh\rho}\right]. \tag{A23}$$

(These correspond to the notations $X_{\pm}$ of Ref. 1.) The factor $e^{i(\omega-\theta)}$ in (A4) should now be $e^{i\gamma}$ and $R(c)$ in (A5) be η, where

$$\cos\gamma=(\sinh\eta)^{-1}(\cosh c\,\sinh\rho-\sinh c\,\cosh\rho\cos\theta) \tag{A24}$$

and

$$\cosh\eta=(\cosh c\,\cosh\rho-\sinh c\,\sinh\rho\cos\theta)\,. \tag{A25}$$

Now, instead of (A7)

$$p=e^{i\gamma},\;\; \bar{q}=-\frac{p+q}{1+pq}\,. \tag{A26}$$

The H transformations retain the same form using (A23) and (A26). The mapping I is now

$$\begin{aligned} If&=\frac{\sin\theta}{f\sinh\rho}\,,\\ I(\Psi_{\rho}/f)&=\frac{i\Psi_{\theta}}{f\sinh\rho}\,,\\ I\left[\frac{\Psi_{\theta}}{f\sinh\rho}\right]&=-i\frac{\Psi_{\rho}}{f}\,. \end{aligned} \tag{A27}$$

The composition theorem remains formally valid as before with the implicit changes in the definitions of p_j and so on. The composed two-step transformation is thus again given by (A15) and (A16).

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