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Consistency check for asymptotic convergence of classical Yang-Mills fields in the Lorentz gauge

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We derive the class of restricted local gauge transformations of Yang-Mills fields that leave the four-divergence of these fields invariant. They form a symmetry of the Yang-Mills equations combined with the Lorentz condition and lead, in Euclidean space-time, to the Gribov ambiguity. We make use of this symmetry in order to check possible inconsistencies arising from the supposition of asymptotic convergence of classical Yang-Mills fields, obeying the Lorentz condition, to free fields. Inconsistencies would support the common conjecture of color confinement. We show that residual gauge transformations of the interpolating fields induce Abelian gauge transformations and global SU_n rotations on the hypothetical asymptotic fields thereby revealing no inconsistency of the hypothetical asymptotic convergence.

I. INTRODUCTION

The set of solutions of a classical Abelian gauge theory obeying the Lorentz condition decomposes into gauge equivalence classes induced by the invariance transformations $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ with Λ obeying the equation $\Box\Lambda = 0$ in order to preserve the Lorentz condition. An analogous feature, however much more complicated, has been pointed out and discussed by Gribov[1] for the case of Yang-Mills fields obeying the Lorentz condition. In this work, by showing that the field equations and the Lorentz condition exhibit some residual local gauge invariance, he argued[1] that the Lorentz condition does not single out a unique element in each gauge class of solutions of the Yang-Mills equations. In Ref. 1 the non-Abelian analog to $\Box\Lambda = 0$ was studied in Euclidean space-time in order to find its consequences for the path-integral formulation of the quantized Yang-Mills field. In this way, the implication of the residual local gauge invariance for the classical field theory defined by the Yang-Mills equations remained unexplored. It is the aim of our work to study aspects of this residual gauge invariance as a feature of the classical Yang-Mills theory.

Since the Yang-Mills equations supplemented by the Lorentz condition are known to pose, at least locally, a well-defined initial-value problem,[2] the possibility for asymptotic convergence of the Yang-Mills field to a free, and therefore Abelian, vector field for $|t| \rightarrow \infty$ exists. No rigorous results, however, concerning the asymptotic convergence in the sense of classical nonlinear scattering theory[3] seem to exist for the case of the Yang-Mills field. If asymptotic convergence actually occurs, one can ask for the transformations induced on the symptotic fields by the residual local non-Abelian gauge transformations of the interpolating field. They can only be global SU_n rotations and/or Abelian local gauge transformations since these are the maximal internal symmetries of the linear (Abelian) gauge fields with values in the SU_n Lie algebra. It is, however, not obvious that the asymptotic fields do indeed transform in this way. Since lack of asymptotic convergence of the color charge carrying a Yang-Mills field is crucial, at least on the quantum level, for the widely suspected occurrence of color confinement in non-Abelian gauge theories, it seems interesting to see whether the assumption of asymptotic convergence leads to any inconsistencies, thereby disproving asymptotic convergence and supporting color confinement. It is a check of this kind, at the classical level, that we intend to present with this work. We examine the consistency of asymptotic convergence with the existence of the residual local gauge transformations by investigating the transformations that these induce on the tentatively defined asymptotic fields. By proving a lemma concerning decay properties of smooth wave-packet solutions of the free massless wave equation we can derive the transformation law of the asymptotic fields. It turns out to be an Abelian gauge transformation, so that no inconsistency of the assumed asymptotic convergence can be seen. Clearly this lack of inconsistency does not prove asymptotic convergence.

The paper is organized as follows. In Sec. II the residual non-Abelian gauge invariance is derived in a version slightly different from the one given by Gribov.[1] Section III contains the discussion of the transformation behavior of the asymptotic fields. In Sec. IV a summary of our findings and the conclusions we draw from them are given.

II. THE RESIDUAL GAUGE INVARIANCE

First, we state which class of models we consider. The fundamental dynamical variables of the SU_n Yang-Mills theory are n^2-1 real vector fields A_μ^a (with $a=1,\ldots,n^2-1$) and a fundamental SU_n multiplet of matter fields $\vec{\phi}$ coupled minimally to A_μ^a and, for our purposes, demanding no further specification. The generators of the fundamental representation of SU_n are called T^a. In terms of the Hermitian matrix field $A_\mu \equiv A_\mu^a T^a$ the local SU_n-covariant derivatives and field strengths are defined by

$$D_\mu(A) \equiv \partial_\mu - ig[A_\mu,\cdot\,] ,$$
$$F_{\mu\nu} \equiv A_{\nu|\mu} - A_{\mu|\nu} - ig[A_\mu,A_\nu], \quad A_{\mu|\nu} \equiv \partial_\nu A_\mu .$$

The Yang-Mills field equations then read

$$D_\nu(A)F^{\nu\mu} = S^\mu(A,\vec{\phi}) . \tag{2.1}$$

The source S^μ which is due to the presence of matter fields has, as a consequence of minimal coupling, vanishing gauge-covariant divergence,

$$D_\mu(A)S^\mu(A,\vec{\phi}) = 0 . \tag{2.2}$$

Local non-Abelian gauge transformations read as given in Eq. (2.3):

$$A_\mu \to \hat{A}_\mu \equiv U\left[A_\mu + \frac{i}{g}U_{|\mu}{}^{-1}U\right]U^{-1}, \quad \vec{\phi} \to \hat{\vec{\phi}} \equiv U\vec{\phi} . \tag{2.3}$$

The SU_n element

$$U(x) \equiv \exp[-ig\epsilon(x)], \quad \epsilon(x) \equiv \epsilon^a(x)T^a$$

has arbitrary space-time-dependent group parameters. It is known that each of the gauge equivalence classes, generated by the transformation (2.3), contains at least one element (modulo global SU_n rotations) obeying the Lorentz condition.[4,2] The question to be considered now is whether there exists more than one such element in each gauge class. (The trivial case of global SU_n rotations is excluded from now on.)

To investigate this question we consider the change of ∂A under (2.3). Calling

$$l_\mu(U) \equiv -\frac{i}{g}U_{|\mu}{}^{-1}U ,$$

one obtains

$$\begin{aligned}\partial^\mu A_\mu &\to \partial^\mu[U(A_\mu - l_\mu)U^{-1}] \\ &= U[D^\mu(l)(A_\mu - l_\mu)]U^{-1} \\ &= U[\partial A - D^\mu(A)l_\mu(U)]U^{-1} .\end{aligned}$$

This shows that ∂A transforms covariantly—and, as a special case, remains zero—for transformation matrices U obeying the scalar second-order nonlinear partial differential equation for $U(\epsilon)$ given by

$$D^\mu(A)l_\mu(U) = 0 , \tag{2.4}$$

or equivalently

$$\Box U^{-1} + U_{|\mu}{}^{-1}U^{|\mu}U^{-1} - ig[A^\mu, U_{|\mu}{}^{-1}U]U^{-1} = 0 . \tag{2.5}$$

Since $l_\mu(U)$ as well as $D^\mu(A)l_\mu(U)$ are Hermitian for $U^\dagger = U^{-1}$, Eqs. (2.4) and (2.5) are compatible with the subsidiary condition $U^\dagger U = UU^\dagger = \mathbb{1}$. For infinitesimal group parameters ϵ^a one obtains the following linear wave equation with A_μ as an external field:

$$\Box\epsilon - ig[A^\mu, \epsilon_{|\mu}] = 0 . \tag{2.6}$$

Equations (2.4) to (2.6) are different versions of the non-Abelian analog to the Abelian equation $\Box\Lambda = 0$. For given initial data $(U(0,\vec{x}), U_{|0}(0,\vec{x}))$ the solution $U(x;A)$ obviously depends on A_μ in a complicated way. For each given field A_μ the matrices $U(x;A)$, solving Eq. (2.4) and corresponding to variable initial data $(U,U_{|0})|_{x^0=0}$, induce local non-Abelian gauge transformations of A_μ under which ∂A transforms covariantly. In particular, all these transformations preserve the Lorentz condition. We will call these transformations "restricted local gauge transformations."

We conclude this section with a remark concerning a possible implication of the restricted local gauge transformations for the quantized Yang-Mills field. Obviously these transformations constitute an invariance of the Lagrangian

$$\mathscr{L} = -\tfrac{1}{2}\operatorname{Tr}[F_{\mu\nu}F^{\mu\nu} + 2\alpha(\partial A)^2] + \mathscr{L}_{\text{matter}} ,$$

which provides the formal starting point for the covariant canonical quantization of the Yang-Mills fields. It is subject to further investigation whether these transformations imply some sort of Ward-Takahashi identities which might open an alternative to the usual quantization procedure consisting in the introduction of ghost fields in order to "restore" part of the gauge invariance in form of the crucial Becchi-Rouet-Stora (BRS) transformation invariance, which then allows us to derive the Slavnov-Taylor identities leading to a unitary S matrix.

III. THE TRANSFORMATION LAW OF THE ASYMPTOTIC FIELDS

Writing Eq. (2.1) explicitly in terms of A_μ one realizes that the Lorentz condition $\partial A=0$ allows one to express $(A_{\mu|00})|_t$ in terms of the Cauchy data $(A_\mu, A_{\mu|0})|_t$ at any time t. This makes it possible that Eq. (2.1) together with the Lorentz condition poses a well-defined global initial-value problem. For the case of scalar matter fields, in fact, rigorous results in one and two space dimensions[2] support this conjecture. Concerning the classical nonlinear scattering theory of the Yang-Mills field no rigorous results seem to exist as they do, e.g., for the scalar ϕ^4 theory.[5]

In what follows we assume that the solutions of the system

$$D_\nu(A)F^{\nu\mu}=S^\mu(A,\vec{\phi}), \quad \partial A=0 , \tag{3.1}$$

with sufficiently small and smooth initial data exhibit asymptotic convergence to the free fields defined through the Yang-Feldman equation

$$A_\mu^{\text{in(out)}}(x)\equiv A_\mu(x) - \int d^4y\, \Delta^{\text{ret(adv)}}(x-y)\Box_y A_\mu(y) \tag{3.2}$$

with $\Box\Delta^{\text{ret(adv)}}=\delta^4$.

It is easy to check that $A_\mu^{\text{in(out)}}$ solves the linearized version of (3.1), namely,

$$\Box A_\mu^{\text{as}}=0, \quad \partial A^{\text{as}}=0 . \tag{3.3}$$

Furthermore, we assume that the solutions of Eq. (2.4) converge asymptotically:

$$U^{\text{in(out)}}(x)\equiv e^{-ig\epsilon^{\text{in(out)}}(x)} \tag{3.4}$$

with

$$\epsilon^{\text{in(out)}}(x)\equiv\epsilon(x)-\int d^4y\, \Delta^{\text{ret(adv)}}(x-y)\Box_y\epsilon(y) .$$

Starting from these assumptions, which ultimately must be proven or disproven, we investigate the transformation behavior of A_μ^{as} under a restricted local gauge transformation of A_μ. To this end we perform the substitution (2.3) in definition (3.2) and expand U in its power series in ϵ. This leads to the substitution

$$A_\mu^{\text{as}}\to\hat{A}_\mu^{\text{as}}=(1-\Delta^{\text{as}}*\Box)\{A_\mu-\partial_\mu\epsilon+ig[A_\mu,\epsilon]-ig[\epsilon_{|\mu},\epsilon]+o(g)\} . \tag{3.5}$$

The term $o(g)$ is due to products containing three or more factors of ϵ, $\epsilon_{|\mu}$, and A_μ.

Our aim is to show that due to the assumed asymptotic convergence of the fields A_μ and ϵ the quadratic and higher-order terms in Eq. (3.5) do not contribute to $\hat{A}_\mu^{\text{as}}$ so that one obtains the Abelian transformation behavior for A_μ^{as} as given by

$$A_\mu^{\text{as}}\to\hat{A}_\mu^{\text{as}}=(1-\Delta^{\text{as}}*\Box)(A_\mu-\partial_\mu\epsilon)=A_\mu^{\text{as}}-\partial_\mu\epsilon^{\text{as}} . \tag{3.6}$$

To this end we partially integrate the differential operator $\Box$ in Eq. (3.5) to apply it on Δ^{as} and keep the surface terms to obtain

$$\hat{A}_\mu^{\text{in(out)}}(t,\vec{x})=+(-)\lim_{t'\to-(+)\infty}\int d^3y\,[\Delta^{\text{ret(adv)}}(t-t',\vec{x}-\vec{y})\overleftrightarrow{\partial}_{t'}\hat{A}_\mu(t',\vec{y})] \tag{3.7}$$

with

$$f\overleftrightarrow{\partial}_t g\equiv f_{|t}g-fg_{|t} .$$

Using

$$\Delta^{\text{ret(adv)}}(x^0,\vec{x})=-(+)\Delta(x^0,x)\theta(\pm x^0)$$

and

$$\Delta(x^0,\vec{x})=-\frac{1}{4\pi|\vec{x}|}[\delta(|\vec{x}|-x^0)-\delta(|\vec{x}|+x^0)] ,$$

one gets (treating only the case $\hat{A}_\mu^{\text{out}}$)

$$\hat{A}_\mu^{\text{out}}(t,\vec{x})=\lim_{t'\to+\infty}\int d^3y\left[\left[\frac{\partial}{\partial t'}\delta(|\vec{x}-\vec{y}|+t-t')\right]\frac{\hat{A}_\mu(t',\vec{y})}{4\pi|\vec{x}-\vec{y}|}-\frac{\delta(|\vec{x}-\vec{y}|+t-t')}{4\pi(t'-t)}\frac{\partial}{\partial t'}\hat{A}_\mu(t',\vec{y})\right]$$

$$=\lim_{t'\to\infty}\left[\frac{\partial}{\partial t}\int d^3y\,\delta(|\vec{x}-\vec{y}|+t-t')\frac{\hat{A}_\mu(t',\vec{y})}{4\pi(t'-t)}-\int d^3y\,\delta(|\vec{x}-\vec{y}|+t-t')\frac{\partial_{t'}\hat{A}_\mu(t',\vec{y})}{4\pi(t'-t)}\right]. \quad (3.8)$$

Since the integral in (3.8) is taken at $t'\to+\infty$ and due to the assumed asymptotic convergence we may replace A_μ and ϵ in the expansion

$$\hat{A}_\mu=A_\mu-\partial_\mu\epsilon+ig[A_\mu,\epsilon]-ig[\epsilon_{|\mu},\epsilon]+o(g)\ , \quad (3.9)$$

which enters Eq. (3.8) by A_μ^{out} and ϵ^{out}. In this way Eq. (3.8) contains only free fields, and the integrals can be estimated with the help of an asymptotic estimate on the decay of A_μ^{out} and ϵ^{out} which we prove in the last part of this section.

This estimate for sufficiently smooth solutions of the free wave equation reads as follows:

$$\sup_{\vec{x}}|\epsilon^{\text{out}}(t,\vec{x})|\le c|t|^{-1}\ \text{for}\ |t|\to\infty\ ,$$
$$\sup_{\vec{x}}|A_\mu^{\text{out}}(t,\vec{x})|\le c'|t|^{-1}\ \text{for}\ |t|\to\infty\ . \quad (3.10)$$

The same hold for $\partial_t\epsilon^{\text{out}}$ and $\partial_t A_\mu^{\text{out}}$ since these functions also obey the free wave equation.

The integrals entering Eq. (3.8) can then be estimated as follows:

$$\left|\int d^3y\,\delta(|\vec{x}-\vec{y}|+t-t')\frac{\phi(t',\vec{y})}{(t'-t)}\right|\le\frac{\sup_{\vec{y}}|\phi(t',\vec{y})|}{(t'-t)}\int d^3y\,\delta(|\vec{x}-\vec{y}|+t-t')$$
$$\underset{t'\to\infty}{\longrightarrow}\sup_{\vec{y}}|\phi(t',\vec{y})|\,t'\ .$$

This estimate shows that any field $\phi(t',\vec{y})$ dropping off faster than $1/t'$ for $t'\to\infty$ cannot contribute to the integrals. And this is indeed the case for all the quadratic and higher-order terms of the expansion (3.9) inserted into Eq. (3.8) as a consequence of the estimate (3.10).

To finish our proof of the Abelian transformation law (3.6) we finally have to derive the estimate (3.10). We do this following closely the ideas used by Ruelle[5] in proving the asymptotic decay properties of smooth solutions of the Klein-Gordon equation for $m\neq0$. As we will see, the decay of a solution for $m=0$ is slower than in the case $m\neq0$. We first state the estimate in a more precise form for the general case of $r\ge2$ space dimensions and then prove it.

Lemma: Let $f:M^{1+r}\to C$ be a positive-frequency solution of $\Box f=0$ with initial data $f(0,\vec{x})\in\mathscr{S}(\mathsf{R}^r)$ (C^∞ function of fast decrease) such that the Fourier transform of $f(0,\vec{x})$,

$$\int d^rx\frac{e^{-i\vec{p}\cdot\vec{x}}}{(2\pi)^{r/2}}f(0,\vec{x})\equiv\frac{\tilde{f}(\vec{p})}{\sqrt{2\pi}2|\vec{p}|}\ ,$$

is an element of $\mathscr{D}(\mathsf{R}^r)$, i.e., C^∞ and compact support. Let $u^\mu\in M^{1+r}$ with $u^\mu\neq0$ and $t\in\mathsf{R}$, then for $r\ge2$ the following asymptotic estimate holds:

$$|f(t\cdot u)|\le c|t|^{-(r-1)/2}$$
$$\text{for}\ |t|\to\infty\ \ \text{for}\ u_\mu u^\mu=0\ ,$$

and $f(t\cdot u)\in\mathscr{S}(\mathsf{R})$ for $u_\mu u^\mu\neq0$.

Proof: The positive-frequency solutions are given by

$$f(x)=(2\pi)^{-n/2}\int_{\mathsf{R}^n}d^np\,e^{-ipx}\delta(p^2)\theta(p^0)\tilde{f}(\vec{p})$$
$$=(2\pi)^{-n/2}\int_{\mathsf{R}^r}d^rp\,e^{-i\check{p}x}\frac{\tilde{f}(\vec{p})}{2|\vec{p}|}$$

with $n\equiv1+r$ and $\check{p}^\mu\equiv(|\vec{p}|,\vec{p})$.

The Fourier transform of $f(t\cdot u)\equiv f_u(t)$ reads

$$F_u(s)\equiv(2\pi)^{-1/2}\int_{\mathsf{R}}dt\,e^{ist}f_u(t)$$
$$=(2\pi)^{-r/2}\int_{\mathsf{R}^n}d^np\,\delta(s-pu)\delta(p^2)$$
$$\times\theta(p^0)\tilde{f}(\vec{p})\ .$$

The behavior of $f_u(t)$ for $|t|\to\infty$ is determined by the smoothness properties of $F_u(s)$. To discuss these we have to distinguish the cases (a) $u_\mu u^\mu>0$, (b) $u_\mu u^\mu=0$, (c) $u_\mu u^\mu<0$.

(a) By virtue of the Lorentz invariance we may choose $u^\mu=(1,\vec{0})$ so that $F_u(s)$ takes the form

$$F_u(s)=(2\pi)^{-r/2}\int_{\mathbb{R}^r} d^r p\, \delta(s^2-p^2)\theta(s)\tilde{f}(\vec{p})$$
$$=\frac{(2\pi)^{-r/2}}{2}\theta(s)\int_0^\infty d|\vec{p}|\,|\vec{p}|^{r-1}\frac{\delta(s-|\vec{p}|)}{|\vec{p}|}\int_{\Omega^{r-1}} d\Omega^{r-1}\tilde{f}(\vec{p})$$
$$=\frac{(2\pi)^{-r/2}}{2}\theta(s)s^{r-2}g(s)\,.$$

With

$$g(|\vec{p}|)\equiv\int_{\Omega^{r-1}} d\Omega^{r-1}\tilde{f}(\vec{p})\,,$$

Ω^n is an n-dimensional unit sphere. Now $f(\vec{p})/|\vec{p}|\in\mathscr{D}(\mathbb{R}^r)$ implies that f and all its derivatives of arbitrary order vanish at $\vec{p}\to 0$, a property which carries over to the function $g(|\vec{p}|)$. This again implies

$$\theta(s)s^{r-2}g(s)\in\mathscr{D}(\mathbb{R}) \text{ and } f_u(t)\in\mathscr{S}(\mathbb{R})\,.$$

(b) Let $u^\mu=(1,\vec{u})$, $\vec{u}^2=1$, $\vec{p}\cdot\vec{u}\equiv|\vec{p}|\cos\theta$, so that $F_u(s)$ takes the form

$$F_u(s)=\frac{(2\pi)^{-r/2}}{2}\int_0^\infty d|\vec{p}|\,|\vec{p}|^{r-3}\int_0^\pi d\theta(\sin\theta)^{r-2}\delta\left[\cos\theta-\left[1-\frac{s}{|\vec{p}|}\right]\right]g(|\vec{p}|,\cos\theta)$$

with

$$g(|\vec{p}|,\cos\theta)\equiv\int_{\Omega^{r-2}} d\Omega^{r-2}\tilde{f}(\vec{p})\,.$$

Again

$$\frac{g(|\vec{p}|,\cos\theta)}{|\vec{p}|}\in\mathscr{D}(\mathbb{R}^+\times[-1,1])\,.$$

Now $\tilde{f}(\vec{p})/|\vec{p}|\in\mathscr{D}(\mathbb{R}^r)$ implies that there exists a K such that $\tilde{f}(\vec{p})=0$ for all $\vec{p}$ with $|\vec{p}|\geq K$. This implies $F_u(s)=0$ for all $s\notin[0,2K]$. Since $F_u(s)$ is of compact support we need only investigate its smoothness properties. By changing the integration variable to $x\equiv 2|\vec{p}|-s$ and performing the $d\theta$ integration $F_u(s)$ becomes

$$F_u(s)=\frac{(2\pi)^{-r/2}}{2}\theta(s)s^{(r-3)/2}\int_0^\infty dx\, x^{(r-3)/2}g\left[\frac{x+s}{2},\frac{x-s}{x+s}\right]\,.$$

For $s\neq 0$ the function $F_u(s)$ and all of his higher derivatives exist. Only the point $s=0$ needs a more careful analysis.

(i) $r=2$: The integrand of the dx integration has an integrable singularity at $x=0$ so that $F_u(s)$ behaves as

$$F_u(s)=O(1/\sqrt{s})\ \text{ for } s\to 0+$$

due to the factor $s^{(r-3)/2}$ in front. The long-range behavior of the Fourier transform can be estimated by "extracting" the singularity from $F_u(s)$ and $dF_u(s)/ds$ at $s=0$:

$$F_u(s)=\frac{(2\pi)^{-1}}{2}\theta(s)\left[\frac{1}{\sqrt{2}}\{I(s)-I(0)e^{-s}-[I(0)+I'(0)]se^{-s}\}+\frac{1}{\sqrt{2}}\{I(0)-[I(0)+I'(0)]\}e^{-s}\right],$$

with

$$I(s)\equiv\int_0^\infty dx\frac{1}{\sqrt{x}}g\left[\frac{x+s}{2},\frac{x-s}{x+s}\right]\,.$$

The first term is of the order $s^{3/2}$ for $s\to 0+$ and therefore of type $C^1(\mathbb{R}^+)$ which implies by applying partial integration that its Fourier transform $f_u(t)$ decreases faster than $1/|t|$ for $|t|\to\infty$.[6] The Fourier transform of the second term which carries the singularity at $s\to 0+$ can be calculated explicitly,[7] thereby showing that

$$f_u(t)=O\left[\frac{1}{\sqrt{|t|}}\right] \quad \text{for } |t|\to\infty \ .$$

(ii) A similar analysis for $r\geq 3$ shows for $s\to 0+$

$$\frac{d^n}{ds^n}F_u(s)=\begin{cases} O(1) \ \text{for odd } r \text{ and } n>\left[\frac{r-3}{2}\right], \\ O(s^{(r-3)/2-n}) \ \text{for even } r \text{ and } n>\left[\frac{r-3}{2}\right]. \end{cases}$$

$$\frac{d^n}{ds^n}F_u(s)=O(s^{(r-3)/2-n}) \quad \text{for } n\leq\left[\frac{r-3}{2}\right],$$

where $[x]\equiv$integer part of x, and

The effect of this behavior at $s\to 0+$ on the Fourier transform $f_u(t)$ is the following: For odd r one applies the method of partial integration[8] immediately to show

$$f_u(t)=O\left[\frac{1}{|t|^{(r-1)/2}}\right] \quad \text{for } |t|\to\infty \ .$$

For even r one extracts, after a $[(r-3)/2]$-fold partial integration, the same singularity structure as in the case $r=2$ and obtains

$$f_u(t)=O\left[\frac{1}{|t|^{(r-1)/2}}\right] \quad \text{for } |t|\to\infty \ .$$

(c) We skip the case of spacelike u^μ which is similar to the case treated in (a).

A final comment should be made concerning a comparison with the analogous lemma of the massive Klein-Gordon equation. The decay is given in that case by[5]

$$f_u(t)=O\left[\frac{1}{|t|^{r/2}}\right] \quad \text{for } |t|\to\infty \ .$$

IV. CONCLUSION

We have examined the consequences of assuming that classical Yang-Mills fields obeying the Lorentz condition converge asymptotically to free fields, in the hope that an inconsistency would be found. Such an inconsistency would have been a strong indication of color confinement at the classical level. Unfortunately, no inconsistency was found. This does not prove that classical Yang-Mills fields do not confine and it certainly says even less about the confinement of quantized Yang-Mills fields.

Should such asymptotic convergence, as we have assumed, indeed occur, then certain conclusions follow. Namely, if the Lorentz condition were to single out a unique representative A_μ^L in every gauge class of Yang-Mills fields, the object A_μ^L would provide us with an unambiguous and unique characterization of each gauge class. A_μ^L would play the role of a fundamental observable and any other observable could be expressed as a function of A_μ^L. As we have argued in Sec. II, this is not the case. This leaves open the question as to which other gauge-invariant object allows a unique characterization of each gauge class. The field-strength tensor $F_{\mu\nu}$ is not a candidate since it is gauge covariant. If, however, asymptotic convergence of the Yang-Mills fields obeying the Lorentz condition were to take place, the asymptotic fields $A_\mu^{L,\text{as}}$ would experience, as has been shown in Sec. III, an Abelian gauge transformation under a restricted local gauge transformation of A_μ^L. This would imply that the Abelian field strengths of the asymptotic fields $G_{\mu\nu}^{\text{as}}\equiv A_{\nu|\mu}^{L,\text{as}}-A_{\mu|\nu}^{L,\text{as}}$ are observable up to a global SU_n rotation. Since knowledge of $G_{\mu\nu}^{\text{as}}$ would allow us to infer the Abelian gauge class of $A_\mu^{\text{as},L}$, and since $A_\mu^{\text{as},L}$ has a one-to-one correspondence with A_μ^L, the object $G_{\mu\nu}^{\text{in}}$ (or $G_{\mu\nu}^{\text{out}}$) would give a gauge-invariant unambiguous characterization of the Yang-Mills gauge classes. $G_{\mu\nu}^{\text{as}}$ could therefore be seen as the physical part of the Yang-Mills field. There seems to be no known object, local in terms of A_μ, that could serve this purpose.

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