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Relativistic quantum mechanics of particles with direct interactions

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For an arbitrary number of particles an explicit construction is given for direct interactions satisfying cluster separability (macrolocality). The Hamiltonian is in many ways similar to the nonrelativistic many-body Hamiltonian. Under conditions of physical interest the strength of the N-body interaction decreases rapidly with increasing N.

I. INTRODUCTION

In canonical quantum theories space-time symmetries are implemented by unitary transformations of the Hilbert space of states, and the generator of time evolution (Hamiltonian) governs the dynamics of the system. For relativistic systems we have unitary representations of the Poincaré group; the Hamiltonian is among its generators and the commutation relations require that other generators depend on the dynamics. It is a common practice to assume that the generators of the Euclidean subgroup (translations and rotations) are independent of the dynamics. It follows that the Lorentz boosts must depend on the interactions. This is the so-called "instant form" dynamics.[1] The instant hyperplanes, $t=\text{const}$, of the Minkowski space are invariant under the Euclidean group. Other forms of dynamics are possible and may have physical significance. There are five families of three-dimensional hypersurfaces that are invariant under subgroups of the Poincaré group and intersect every particle world line once.[2] The choice of the invariant hypersurface, on which initial conditions can be specified, specifies the form of the dynamics. Dirac[1] considered the instant, point, and front forms for which the invariant hypersurfaces are respectively $t=\text{const}$; $t^2-\vec{x}^2=\text{const}>0$, $t>0$; and $\vec{n}\cdot\vec{x}+t=\text{const}$. The instant and point forms have four dynamical generators while the front form has only three. The remaining cases are similar to the point form but require six dynamical generators. They do not appear to be of special interest.

The problem is then to introduce interactions into the dynamical generators consistent with the commutation relations.[3] Canonical field theories solve this problem by constructing the generators in terms of the energy-momentum tensor.[4,5] Bakamjian and Thomas[6] have shown that the problem can be solved by introducing interactions in the mass operator. Any such theory is causal in the sense that the states at $t>t_0$ are uniquely determined by those at t_0. Realizing the Lie algebra is, however, not sufficient for an acceptable theory. If a system consists of two parts that are respectively localized in two space-time regions that are completely spacelike with respect to each other then the two subsystems should be dynamically independent. This postulate applied to arbitrarily small regions leads to the locality properties of local field theories.[7] The algebraic approach to quantum field theory[8] suggests the possibility of accepting the weaker requirement that two states localized in the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are completely uncoupled only in the limit of infinite spacelike separation. We will refer to this property as *macrolocality* or *cluster separability.*

Except for two particles,[9] the Bakamjian-Thomas (BT) construction does not satisfy this macrolocality requirement.[10] For three particles cluster separability of the S matrix can be achieved by a BT construction[11] but the Hamiltonian does not become additive for separated subsystems.[12,13] This failure precludes a consistent generalization to more than three particles. Sokolov[14,15] has shown that, for any number of particles, cluster separability can be achieved by a unitary transformation of the BT generators, which we will call the Sokolov tranformation.

The key to the BT construction is always the observation that the Poincaré generators can be writ-

ten as functions of the mass operator and of other operators that may be chosen to be interaction independent. The kinematic generators are then independent of the mass. A BT construction exists in any of Dirac's[1] three forms of dynamics. The instant form is most widely used and is most convenient for the transition to the nonrelativistic limit. The front form[1,2,5] is particularly convenient when a particle beam designates a preferred axis in space. Sokolov's work[14] was done mainly in the point form; to get results in other forms he exploited the unitary equivalence of the different forms.[16,15] We will construct the Sokolov transformations directly in the instant form. The same procedures could be used in any of the forms. It seems, however, that there are no physical conditions under which the point form is especially convenient.

Functions (or functionals) representing vectors in the Hilbert space of states are covariant only under the kinematic subgroup. In the instant form they cannot be Lorentz covariant; in the point form they are Lorentz covariant but they cannot be covariant under translations.[17] On the other hand, Poincaré-covariant state representatives exist in the constraint dynamics of Todorov and Komar,[18–20] where all Poincaré generators are kinematic and the dynamics is introduced in the mass-shell constraints.

The purpose of this paper is to find an explicit construction of Poincaré generators for particles with direct interactions, which have the cluster-separability property, and to find approximations that will be useful for applications. Our construction is presented in Secs. V and VI. In Sec. V we consider the case of three particles and discuss useful approximations to the three-body force. Section VI contains the general recursive construction for arbitrary numbers of particles. We note that $N=3$ is a special case; the full complexity of the problem does not appear until $N\geq 4$, so Sec. VI is not simply a generalization of Sec. V to $N>3$. Our construction combines Sokolov's ideas with the two-Hilbert-space theory of multichannel scattering (Sec. III). The key result in Sec. III is theorem 3.4. We will show that this result allows us to obtain the Sokolov transformations explicitly in terms of wave operators of subsystems. The wave operators can be eliminated from the result, and good approximations are easily obtained. The relevant formal properties of the Poincaré generators will be reviewed in Sec. II. In the Appendix we also give the point- and front-form BT construction in order to indicate that the construction described in detail in Secs. V and VI could also be carried out in these forms. Section IV deals with the combinatorics of particle partitions. It provides a useful notation for the construction in Sec. VI. The treatment of particle creation is sketched in Sec. VII. Our construction retains many familiar features of nonrelativistic many-body quantum mechanics. Thus the obvious applications are in the area of intermediate-energy nuclear physics.[21] We postpone a detailed discussion of the relation of our theory to the covariant constraint dynamics.

II. PROPERTIES OF THE POINCARÉ GENERATORS

Let $\mathscr{H}$ be the Hilbert space of states. The generators of the Poincaré transformations are self-adjoint operators $H,\vec{P}$ for time and space translations, $\vec{J}$ for rotations, and $\vec{K}$ for Lorentz boosts. They satisfy the commutation relations

$$[P_i,P_j]=0,\quad [P_i,H]=0\ , \tag{2.1}$$

$$[J_p,J_q]=i\sum_r \epsilon_{pqr}J_r\ , \tag{2.2}$$

$$[J_r,H]=0,\quad [J_p,P_q]=i\sum_r \epsilon_{pqr}P_r\ , \tag{2.3}$$

$$[J_p,K_q]=i\sum_r \epsilon_{pqr}K_r\ , \tag{2.4}$$

$$[K_p,K_q]=-i\sum_r \epsilon_{pqr}J_r\ , \tag{2.5}$$

$$[K_p,P_q]=i\delta_{pq}H\ , \tag{2.6}$$

and

$$[\vec{K},H]=i\vec{P}\ . \tag{2.7}$$

The operators H, $\vec{P}$, and $\vec{J}$ have the physical significance of energy, momentum, and angular momentum. The operators $\{H,\vec{P}\}=\{P^\mu\}$ transform as a four-vector under Lorentz transformations, and the generators $\vec{J},\vec{K}$ form an antisymmetric tensor $J^{\mu\nu}$. The components of $J^{\mu\nu}$ are related to $\vec{K}$ and $\vec{J}$ by

$$J^{p0}=-J^{0p}=J_{0p}=K_p\ , \tag{2.8}$$

$$J^{pq}=\sum_r J_r\epsilon_{rpq}\ . \tag{2.9}$$

The Pauli-Lubanski[22,23] vector W_μ is defined as a covariant function of P^ν and $J^{\rho\sigma}$,

$$W_\mu=\tfrac{1}{2}\sum_{\nu\rho\sigma} J^{\rho\sigma}P^\nu\epsilon_{\nu\rho\sigma\mu}\ . \tag{2.10}$$

It follows from Eqs. (2.8), (2.9), and (2.10) that

$$W^0=\vec{P}\cdot\vec{J},\ \vec{W}=H\vec{J}+\vec{P}\times\vec{K}\ . \tag{2.11}$$

The mass operator M is defined by

$$M^2=-g_{\mu\nu}P^\mu P^\nu=H^2-\vec{P}^2\ . \tag{2.12}$$

We will assume in the following that the operator M is positive and has a bounded inverse.

For each of Dirac's three forms we show that the dynamical generators can be expressed as functions of the mass operator, the generators of the kinematic subgroup, and other operators that need not involve the interactions. In each case the operators in question can be defined as functions of the generators.

For the instant form we define the Newton-Wigner[24] position operator $\vec{X}$ as a function of the generators by[25]

$$\vec{X}=\tfrac{1}{2}(H^{-1}\vec{K}+\vec{K}H^{-1})-\vec{P}\times\vec{W}[MH(M+H)]^{-1}\ . \tag{2.13}$$

The operator so defined is self-adjoint. It follows from the definition (2.13) and the commutation relations (2.1)–(2.7) that the components of $\vec{X}$ commute with each other, and that $\vec{X}$ and $\vec{P}$ are canonically conjugate,

$$[X_p,X_q]=0,\ \ [X_p,P_q]=i\delta_{pq}\ . \tag{2.14}$$

The operators $\vec{X}$ and $\vec{P}$ commute with the canonical spin $\vec{j}$ defined by

$$\vec{j}=\vec{J}-\vec{X}\times\vec{P} \tag{2.15}$$

and with the mass operator M.

The spin operator $\vec{j}$ is related to the Pauli-Lubanski vector W by a Lorentz transformation

$$L(\vec{Q})W=\{0,M\vec{j}\}\ , \tag{2.16}$$

where $\vec{Q}=\vec{P}/M$ and $L(\vec{Q})$ is the inverse boost defined by

$$L(\vec{Q})\{(1+\vec{Q}^2)^{1/2},\vec{Q}\}=\{1,0,0,0\}\ . \tag{2.17}$$

Explicit expressions for $L(\vec{Q})$ are given by

$$L(\vec{Q})_{ik}=L^{-1}(\vec{Q})_{ik}=\delta_{ik}+\frac{Q_iQ_k}{1+(\vec{Q}^2+1)^{1/2}}\ , \tag{2.18}$$

$$L(\vec{Q})_{00}=-L^{-1}(\vec{Q})_{00}=-(1+\vec{Q}^2)^{1/2}\ , \tag{2.19}$$

and

$$L(\vec{Q})_{r0}=-L^{-1}(\vec{Q})_{r0}=-L(\vec{Q})_{0r}=-Q_r\ . \tag{2.20}$$

From Eq. (2.16) is follows that

$$W^\mu W_\mu=M^2\vec{j}^2\ . \tag{2.21}$$

It follows from Eqs. (2.12), (2.13), and (2.15) that the generators H and $\vec{K}$ can be expressed as functions of the operators $\vec{P}$, $\vec{X}$, $\vec{J}$, and M defined to satisfy the commutation relations (2.14) and

$$[\vec{J},M]=[\vec{X},M]=[\vec{P},M]=0\ . \tag{2.22}$$

These expression are

$$H=(P^2+M^2)^{1/2} \tag{2.23}$$

and

$$\vec{K}=\tfrac{1}{2}(H\vec{X}+\vec{X}H)-\vec{j}\times\vec{P}(M+H)^{-1}\ , \tag{2.24}$$

where $\vec{j}$ is defined by Eq. (2.15).

In this representation the generators of the Euclidean group, $\vec{P}$ and $\vec{J}$, are independent of the mass operator. The instant hyperplane $t=$const is invariant under the Euclidean group. The generators H and $\vec{K}$ are the dynamical generators in this form. The position operator $\vec{X}$ defined by (2.13) depends, in general, on the dynamics.

If all the vectors in $\mathscr{H}$ are eigenvectors of the Casimir operators M and $\vec{j}^2$ then we have an irreducible representation; that is, physically we have the description of a simple particle. Let $\mathscr{H}^{(1)}$ denote a one-particle Hilbert space; the space of states of an N-particle system is then

$$\mathscr{H}=\mathscr{H}_1{}^{(1)}\otimes\mathscr{H}_2{}^{(1)}\otimes\cdots\otimes\mathscr{H}_N{}^{(1)}\ . \tag{2.25}$$

Let $g_i(m,j)$ denote the set of generators of an irreducible Poincaré representation as operators on $\mathscr{H}_i{}^{(1)}$ and I_i' be the identity on the remaining factors. The description of N noninteracting particles is then given by the generators

$$G_0=\sum_{i=1}^{N}g_i(m_i,j_i)\otimes I_i' \tag{2.26}$$

acting on the tensor product (2.25). The mass operator M_0 and the Newton-Wigner operator $\vec{X}_0$ are defined as functions of G_0 according to (2.12) and (2.13).

The precedent of nonrelativistic quantum mechanics suggests the introduction of an interaction by modifying the Hamiltonian $H_0\to H=H_0+V$. The commutation rules (2.6) require that either $\vec{P}$ or $\vec{K}$, or both, must also be interaction dependent.

According to (2.13) the operator $\vec{X}$ will in general depend on the interaction. However Bakamjian and Thomas[6] have pointed out that all commutation relations can be satisfied by

$$\vec{X}=\vec{X}_0,\quad \vec{P}=\vec{P}_0,\quad \vec{J}=\vec{J}_0\ , \tag{2.27}$$

$$\overline{M}=M_0+v,\quad \overline{H}=(\vec{P}^2+\overline{M}^2)^{1/2}\ , \tag{2.28}$$

and

$$\vec{\overline{K}}=\tfrac{1}{2}(\overline{H}\vec{X}_0+\vec{X}_0\overline{H})-\vec{j}\times\vec{P}(\overline{M}+\overline{H})^{-1}\ , \tag{2.29}$$

provided the interaction operator v commutes with $\vec{X}$, $\vec{P}$, and $\vec{J}$. The operator M is, of course, Lorentz invariant, but M_0 and v are not. In the same spirit we may generate a point-form[26,27] or front-form dynamics. Details are given in the Appendix. In the front form the dynamical generators are $P_-=H-\vec{n}\cdot\vec{P}$ and $\vec{F}=\vec{K}-\vec{n}\times\vec{J}$. If we wish to construct a front-form dynamics by adding an interaction term to P_- then the commutation rules (A26) require that $\vec{F}$ also be interaction dependent.

We still need a mathematical formulation of the cluster-separability requirement. Let a denote a partition of the N particles into clusters a_i, $i=1,\ldots,n_a$. The states of the cluster a_i are vectors in a Hilbert space and it follows from (2.25) that

$$\mathscr{H}=\bigotimes_{i=1}^{n_a}\mathscr{H}_{ai} \tag{2.30}$$

for every partition. Let $U_{ai}(d,\Lambda)$ be a unitary representation of the Poincaré group for the cluster a_i on the space $\mathscr{H}_{ai}$. The four-vector d specifies a translation and Λ labels a Lorentz tranformation. We will also use the notation $U_{ai}(d,\Lambda)$ for the operator $U_{ai}(d,\Lambda)\otimes I'(a_i)$ on $\mathscr{H}$,

$$U_{ai}(d,\Lambda)\otimes I'(a_i)\to U_{ai}(d,\Lambda)\ , \tag{2.31}$$

where $I'(a_i)$ is the identity on the space $\otimes_{j\neq i}\mathscr{H}_{aj}$. The representation $U_a(d,\Lambda)$ describing the noninteracting clusters of the partition a is then

$$U_a(d,\Lambda)=\prod_{i=1}^{n_a}U_{ai}(d,\Lambda)\ . \tag{2.32}$$

For operators $\mathscr{O}$ that are functions of single-particle Poincaré generators and interaction operators we use the notation $\mathscr{O}_a$ to denote the values of these functions when the interactions between particles in different clusters of a are set to zero. Let G denote the set of Poincaré generators. We say the generators G for N particles satisfy the cluster condition (C1) if for every partition a with at least two clusters

$$G_a=\sum_{i=1}^{n_a}G(a_i)\ , \tag{C1}$$

where $G(a_i)$ are the generators of the representation $U_{ai}(d,\Lambda)$.

Intuitively the cluster separability is the property that disjoint subsystems behave independently when they are separated beyond the range of the interactions. However, condition (C1) is an algebraic condition on the operators which does not involve the range of interactions. The physical requirement that spatial separation of the clusters implies vanishing interaction between them is realized by

$$\operatorname*{s\text{-}lim}_{\min|\vec{\delta}_i-\vec{\delta}_j|\to\infty}[U(d,\Lambda)-U_a(d,\Lambda)]\times T_a(\vec{\delta}_1\cdots\vec{\delta}_{n_a})=0\ , \tag{C2}$$

where T_a is the product of simple cluster translation operators,

$$T_a(\vec{\delta}_1\cdots\vec{\delta}_{n_a})=\prod_{i=1}^{n_a}U_{a_i}(\delta_i,1)\ . \tag{2.33}$$

Our task is to construct explicitly a representation such that (C1) and (C2) are satisfied and that multichannel scattering operators exist.

III. MULTIPARTICLE SCATTERING THEORY[28]

A scattering state $|\psi(t)\rangle$ is a solution of the time-dependent Schrödinger equation,

$$|\psi(t)\rangle=e^{-iHt}|\psi(0)\rangle\ , \tag{3.1}$$

which becomes equal to a state $|\phi(t)\rangle$ of noninteracting particles (or bound fragments) in the remote past,

$$\lim_{t\to-\infty}||\,|\psi(t)\rangle-|\phi(t)\rangle\,||=0\ . \tag{3.2}$$

In order to formulate this asymptotic condition in more detail we need the mass eigenfunction of the fragments appearing in the initial and final states. Let a be a partition of the N particles such that for each cluster a_i the mass operator M_{ai} has at least one point eigenvalue. The corresponding eigenvectors $|\phi_{ai}\rangle$,

$$\begin{aligned}&M_{ai}|\phi_{ai}\rangle=|\phi_{ai}\rangle m_{ai}\ ,\\ &\vec{j}_{ai}{}^2|\phi_{ai}\rangle=|\phi_{ai}\rangle s_{ai}(s_{ai}+1)\ ,\end{aligned} \tag{3.3}$$

define a channel α. Any eigenvector $|\phi_{ai}\rangle$ is a superposition,

$$|\phi_{ai}\rangle=\int d^3p\sum_{\mu}\phi_{ai\,\vec{p}\,\mu}\chi_{ai}(\vec{p},\mu)\qquad(-s_{ai}\le\mu\le s_{ai})\ , \tag{3.4}$$

of improper eigenvectors $\phi_{ai\vec{p}\mu}$,

$$\vec{P}_{ai}\phi_{ai\vec{p}\mu}=\phi_{ai\vec{p}\mu}\vec{p}\ ,$$
$$j_{ai3}\phi_{ai\vec{p}\mu}=\phi_{ai\vec{p}\mu}\mu\ . \tag{3.5}$$

If the mass operator M_{ai} of at least one cluster has more than one point eigenvalue, then there is more than one channel associated with the partition a. The square-integrable functions $\chi_{ai}(\vec{p},\mu)$ span a Hilbert space $\mathscr{H}_{fai}$ which is the representation space of the irreducible representation $g(m_{ai},s_{ai})$. Equation (3.4) can be written more abstractly as an operator equation

$$|\phi_{ai}\rangle=\Phi_{ai}|\chi_{ai}\rangle\ , \tag{3.6}$$

where $|\chi_{ai}\rangle\in\mathscr{H}_{fai}$. The operator Φ_{ai} so defined maps $\mathscr{H}_{fai}$ into $\mathscr{H}_{ai}$, $\Phi_{ai}\mathscr{H}_{fai}\subset\mathscr{H}_{ai}$. By construction it has the intertwining property

$$G(a_i)\Phi_{ai}=\Phi_{ai}g_{ai}(m_{ai},s_{ai})\ . \tag{3.7}$$

If we define

$$\mathscr{H}_{f\alpha}=\bigotimes_{i=1}^{n_a}\mathscr{H}_{fai}\ , \tag{3.8}$$

$$G_{f\alpha}=\sum_{i=1}^{n_a}g_{ai}(m_{ai},s_{ai})\ , \tag{3.9}$$

$$\Phi_{\alpha}=\bigotimes_{i=1}^{n_a}\Phi_{ai}\ , \tag{3.10}$$

it follows from (C1) and (3.7) that

$$G_a\Phi_{\alpha}=\Phi_{\alpha}G_{f\alpha}\ . \tag{3.11}$$

Let $\mathscr{H}_f$ be the direct sum of the channel spaces $\mathscr{H}_{f\alpha}$,

$$\mathscr{H}_f=\bigoplus_{\alpha}\mathscr{H}_{f\alpha}\ , \tag{3.12}$$

and define

$$\Phi_{\alpha}\mathscr{H}_f=\Phi_{\alpha}\mathscr{H}_{f\alpha}\ , \tag{3.13}$$

$$G_{f\alpha}\mathscr{H}_f=G_{f\alpha}\mathscr{H}_{f\alpha}\ , \tag{3.14}$$

$$G_f=\sum_{\alpha}G_{f\alpha}\ , \tag{3.15}$$

$$\Phi=\sum_{\alpha}\Phi_{\alpha}\ . \tag{3.16}$$

The asymptotic condition (3.2) can now be written in the form[11]

$$\lim_{t\to-\infty}||\,|\psi(t)\rangle-\Phi e^{-iH_ft}|\chi\rangle||=0\ , \tag{3.17}$$

where $|\chi\rangle\in\mathscr{H}_f$. If a scattering state $|\psi(t)\rangle$ satisfying (3.17) exists for every $|\chi\rangle$, then there exists a wave operator Ω_- that maps the initial state $|\chi\rangle\in\mathscr{H}_f$ into the scattering state $|\psi(0)\rangle\in\mathscr{H}$ at the time $t=0$,

$$|\psi(0)\rangle=\Omega_-|\chi\rangle\ . \tag{3.18}$$

Wave operators $\Omega_{\pm}(H,\Phi,H_f)$ are defined by

$$\Omega_{\pm}(H,\Phi,H_f)=\underset{t\to\pm\infty}{\text{s-lim}}\ e^{iHt}\Phi e^{-iH_ft}\ . \tag{3.19}$$

The scattering matrix $S_{\alpha\beta}$ is

$$S_{\beta\alpha}=\langle\chi_{\beta}|S|\chi_{\alpha}\rangle=\lim_{t\to+\infty}\langle\chi_{\beta}|e^{iH_ft}\Phi^{\dagger}|\psi_{\alpha}(t)\rangle\ , \tag{3.20}$$

and the scattering operator is thus

$$S=\Omega_+^{\dagger}\Omega_-\ . \tag{3.21}$$

The wave operators Ω_+ and Ω_- are said to be asymptotically complete if they have the same range,

$$\Omega_+\Omega_+^{\dagger}=\Omega_-\Omega_-^{\dagger}\ . \tag{3.22}$$

It is convenient to include the one-particle channels, $n_{\alpha}=1$, in the set of all channels. We can then expect that the wave operators $\Omega_{\pm}$ are unitary maps of $\mathscr{H}_f$ onto $\mathscr{H}$,

$$\Omega_+\Omega_+^{\dagger}=\Omega_-\Omega_-^{\dagger}=1\ . \tag{3.23}$$

The Hilbert spaces $\mathscr{H}_f$ and $\mathscr{H}$ are direct integrals[29] over the total momentum,

$$\mathscr{H}_f=\int_{\oplus}d^3P\hat{\mathscr{H}}_f(\vec{P}) \tag{3.24}$$

and

$$\mathscr{H}=\int_{\oplus}d^3P\hat{\mathscr{H}}(\vec{P})\ . \tag{3.25}$$

Any vector $|\chi\rangle\in\mathscr{H}_f$ is represented by a vector-valued function $|\hat{\chi}(\vec{P})\rangle\in\hat{\mathscr{H}}_f$ such that

$$||\,|\chi\rangle||^2=\int d^3P||\,|\hat{\chi}(\vec{P})\rangle||^2\ . \tag{3.26}$$

Any translationally invariant operator with domain and range in $\mathscr{H}$, $\mathscr{O}\in L(\mathscr{H},\mathscr{H})$, has the representation

$$(\vec{P}'|\mathscr{O}|\vec{P})=\delta(\vec{P}'-\vec{P})\hat{\mathscr{O}}(\vec{P})\ , \tag{3.27}$$

where $\hat{\mathscr{O}}(\vec{P})\in L(\hat{\mathscr{H}},\hat{\mathscr{H}})$. Similarly for a translationally invariant injection operator, e.g., $\Phi\in L(\mathscr{H}_f,\mathscr{H})$, we have

$$(\vec{P}|\Phi|\vec{P}_f)=\delta(\vec{P}-\vec{P}_f)\hat{\Phi}(\vec{P})\ , \tag{3.28}$$

where $\hat{\Phi}(\vec{P})\in L(\mathscr{H}_f(\vec{P}),\mathscr{H}(\vec{P}))$. It follows that for $\mathscr{V}\in L(\mathscr{H}_f,\mathscr{H})$ defined by

$$\mathscr{V}=H\Phi-\Phi H_f \tag{3.29}$$

we have

$$\hat{\mathscr{V}}(\vec{P})=\hat{H}(\vec{P})\hat{\Phi}(\vec{P})-\hat{\Phi}(\vec{P})\hat{H}_f(\vec{P}) . \tag{3.30}$$

Euclidean-invariant unitary transformations of $\mathscr{H}$ may change the $\vec{P}$ dependence of $\hat{\mathscr{V}}(\vec{P})$ and hence the $\vec{P}$ dependence of the operator $\hat{\mathscr{O}}(\vec{P})$ associated with the translationally invariant operator.

Sufficient conditions for the existence of the wave operators $\Omega_{\pm}(\hat{H},\hat{\Phi},\hat{H}_f)$ are well known.[30] In the following we will always assume that the interactions are "short-range." Short-range interactions can be defined by the condition that

$$\lim_{t\to\infty} t^{3/2}||\hat{\mathscr{V}}(\vec{P})e^{\mp i\hat{H}_f t}|\hat{\chi}\rangle||=0 \tag{3.31}$$

for all vectors $|\hat{\chi}\rangle$ in a suitable dense set. For short-range interactions so defined the wave operators $\Omega_{\pm}(\hat{H}(\vec{P}),\hat{\Phi}(\vec{P}),\hat{H}_f(\vec{P}))$ exist. In the following we will always assume that the short-range condition (3.31) is satisfied for almost all $\vec{P}$.

Since $\hat{\Phi}(\vec{P})$ is by definition bounded uniformly in $\vec{P}$ there exists for any $|\chi\rangle\in\mathscr{H}_f$ a constant C, independent of t and $\vec{P}$, such that

$$||[e^{\pm i\hat{H}(\vec{P})t}\hat{\Phi}(\vec{P})e^{\mp i\hat{H}_f(\vec{P})t}-\hat{\Omega}_{\pm}(\vec{P})]\,|\hat{\chi}(\vec{P})\rangle||^2<C||\,|\hat{\chi}(\vec{P})\rangle||^2 . \tag{3.32}$$

It follows therefore from the dominated convergence theorem[31] that

$$\lim_{t\to\pm\infty}\int d^3P||[e^{i\hat{H}(\vec{P})t}\hat{\Phi}(\vec{P})e^{-i\hat{H}_f(\vec{P})t}-\hat{\Omega}_{\pm}(\vec{P})]||^2=0 , \tag{3.33}$$

and hence

$$(\vec{P}\,|\,\Omega_{\pm}(H,\Phi,H_f)\,|\,\vec{P}_f)=\delta(\vec{P}-\vec{P}_f)\hat{\Omega}_{\pm}(\vec{P}) . \tag{3.34}$$

We have thus established the existence of $\Omega_{\pm}(H,\Phi,H_f)$. The wave operators $\Omega_{\pm}$ satisfy the intertwining relations

$$H\Omega_{\pm}=\Omega_{\pm}H_f \tag{3.35}$$

and

$$\vec{P}\Omega_{\pm}=\Omega_{\pm}\vec{P}_f . \tag{3.36}$$

It follows that

$$M\Omega_{\pm}=\Omega_{\pm}M_f . \tag{3.37}$$

Intuitively it seems reasonable to expect $\Omega_{\pm}(H,\Phi,H_f)=\Omega_{\pm}(M,\Phi,M_f)$. If either H and Φ or M and Φ satisfy the short-range condition (3.31), then the wave operators $\Omega_{\pm}(\hat{H},\hat{\Phi},\hat{H}_f)$ and $\Omega_{\pm}(\hat{M},\hat{\Phi},\hat{M}_f)$ both exist and they are equal,[32]

$$\Omega_{\pm}(\hat{H},\hat{\Phi},\hat{H}_f)=\Omega_{\pm}(\hat{M},\hat{\Phi},\hat{M}_f)=\hat{\Omega}_{\pm} . \tag{3.38}$$

It follows that $\Omega_{\pm}(H,\Phi,H_f)$ and $\Omega_{\pm}(M,\Phi,M_f)$ exist and that

$$\Omega_{\pm}(H,\Phi,H_f)=\Omega_{\pm}(M,\Phi,M_f)=\Omega_{\pm} . \tag{3.39}$$

Equation (3.39) provides the justification for including interactions in the mass operator in the BT construction. A similar relation holds between M and P_- [see (A35)].

The injection operator Φ is used only to formulate the initial and final asymptotic conditions (3.17) and the definition of the wave operators (3.19). It is therefore not necessary to construct Φ exactly as was done above. Any other operator Φ' with the same asymptotic form would be equivalent.

Lemma 3.1. The relation

$$\text{s-lim}_{t\to\pm\infty}(\Phi-\Phi')e^{-iH_f t}=0 \tag{3.40}$$

is necessary and sufficient for the identity of the wave operators

$$\Omega_{\pm}(H,\Phi,H_f)=\Omega_{\pm}(H,\Phi',H_f) . \tag{3.41}$$

Lemma 3.2. Let $A\in L(\mathscr{H},\mathscr{H})$ be any unitary operator. Then

$$\Omega_{\pm}(AHA^{-1},A\Phi,H_f)=A\Omega_{\pm}(H,\Phi,H_f) \tag{3.42}$$

and

$$S(AHA^{-1},A\Phi,H_f)=S(H,\Phi,H_f) . \tag{3.43}$$

The proof of these two lemmas follows immediately from the definitions (3.19) and (3.21).

Lemma 3.3. Let $\Omega_{\pm}(H,\Phi,H_f)$ and $\Omega_{\pm}(\overline{H},\overline{\Phi},H_f)$ be two sets of wave operators satisfying the completeness relation (3.23), such that the S operators are identical,

$$S=\Omega_+^{\dagger}\Omega_-=\overline{\Omega}_+^{\dagger}\overline{\Omega}_-=\overline{S} , \tag{3.44}$$

then

$$\overline{\Omega}_-\Omega_-^{\dagger}=\overline{\Omega}_+\Omega_+^{\dagger} . \tag{3.45}$$

Equation (3.45) is an obvious consequence of (3.44) and (3.23). It follows from lemma 3.3 and Eq. (3.35) that

$$\Omega_{\pm}(\overline{H},\overline{\Phi},H_f)=B\Omega_{\pm}(H,\Phi,H_f)=\Omega_{\pm}(\overline{H},B\Phi,H_f) , \tag{3.46}$$

where

$$B=\overline{\Omega}_{\pm}\Omega_{\pm}^{\dagger}\ . \tag{3.47}$$

Hence by lemma 3.1

$$\text{s-lim}_{t\to\infty}(B\Phi-\Phi)e^{\pm iH_f t}=0\ . \tag{3.48}$$

The injection operator Φ constructed from the mass eigenfunctions (3.3) is Euclidean invariant,

$$\vec{\mathrm{P}}\Phi=\Phi P_f,\quad \vec{\mathrm{J}}\Phi=\Phi\vec{\mathrm{J}}_f\ . \tag{3.49}$$

The Euclidean invariance of the wave operators,

$$\vec{\mathrm{P}}\Omega_{\pm}=\Omega_{\pm}\vec{\mathrm{P}}_f,\quad \vec{\mathrm{J}}\Omega_{\pm}=\Omega_{\pm}\vec{\mathrm{J}}_f \tag{3.50}$$

follows from (3.49) and the definition (3.19), but the Lorentz invariance

$$\vec{\mathrm{K}}\Omega_{\pm}=\Omega_{\pm}\vec{\mathrm{K}}_f \tag{3.51}$$

or

$$[\vec{\mathrm{K}}_f,S]=0 \tag{3.52}$$

does not follow[9] and requires further investigation.

Theorem 3.1. The wave operators $\Omega_{\pm}$ are Lorentz invariant if and only if Φ is asymptotically Lorentz invariant, i.e.,

$$\lim_{t\to\infty}||(\vec{\mathrm{K}}\Phi-\Phi\vec{\mathrm{K}}_f)e^{\pm iH_f t}|\chi\rangle||=0 \tag{3.53}$$

for all $|\chi\rangle$ in a dense set $D\subset\mathscr{H}_f$.

The proof follows easily from

$$e^{iHt}\vec{\mathrm{K}}e^{-iHt}=\vec{\mathrm{K}}+\vec{\mathrm{P}}t \tag{3.54}$$

and the definition (3.19).

Theorem 3.2. If the wave operators exist and are complete then the S operator is Lorentz invariant if and only if

$$\Omega_{+}\vec{\mathrm{K}}_f\Omega_{+}^{\dagger}=\Omega_{-}\vec{\mathrm{K}}_f\Omega_{-}^{\dagger}\ . \tag{3.55}$$

Equation (3.52) follows from (3.21), (3.23), and (3.55). Equation (3.55) follows from (3.52), (3.21), and (3.23). From

$$\vec{\mathrm{K}}=\Omega_{\pm}\vec{\mathrm{K}}_f\Omega_{\pm}^{\dagger} \tag{3.56}$$

and (3.23) follows (3.51).

Theorem 3.3. The intertwining relation

$$\vec{\mathrm{X}}\Phi'=\Phi'\vec{\mathrm{X}}_f \tag{3.57}$$

for some Φ' satisfying

$$\Omega_{\pm}(H,\Phi,H_f)=\Omega_{\pm}(H,\Phi',H_f) \tag{3.58}$$

is necessary and sufficient for the Lorentz invariance of the wave operators.

Proof: From Eqs. (3.57), (3.58), and (3.39) it follows that

$$\vec{\mathrm{X}}\Omega_{\pm}=\Omega_{\pm}\vec{\mathrm{X}}_f\ . \tag{3.59}$$

From (3.59), (3.37), (3.50), and (2.13) follows (3.51). The Hilbert spaces $\hat{\mathscr{H}}(\vec{\mathrm{P}})$ are related by unitary transformations which are subject to some arbitrary choice. If we choose

$$\hat{\mathscr{H}}(\vec{\mathrm{P}}+\vec{\mathrm{q}})=e^{-i\vec{\mathrm{X}}\cdot\vec{\mathrm{q}}}\hat{\mathscr{H}}(\vec{\mathrm{P}})\ ,$$

then $M(\vec{\mathrm{P}})$ is independent of $\vec{\mathrm{P}}$, and $\hat{\Omega}_{\pm}$ is independent of $\vec{\mathrm{P}}$ if the wave operators are Lorentz invariant. Therefore Φ' defined by

$$(\vec{\mathrm{P}}\,|\,\Phi'\,|\,P_f)=\delta(\vec{\mathrm{P}}-\vec{\mathrm{P}}_f)\hat{\Phi}(0) \tag{3.60}$$

satisfies both (3.57) and (3.58).

If the wave operators are Lorentz invariant and complete, then it is always possible to construct a scattering-equivalent Bakamjian-Thomas representation.

Theorem 3.4. If $\vec{\mathrm{K}}\Omega_{\pm}=\Omega_{\pm}\vec{\mathrm{K}}_f$ and $\Omega_{+}\Omega_{+}^{\dagger}=\Omega_{-}\Omega_{-}^{\dagger}=1$, then there exists a Euclidean-invariant unitary operator B such that

$$\vec{\mathrm{X}}_0B=B\vec{\mathrm{X}}\ . \tag{3.61}$$

Proof: The proof is by construction. Choose a representation in which

$$\hat{\mathscr{H}}_f(\vec{\mathrm{P}}+\vec{\mathrm{q}})=e^{-i\vec{\mathrm{X}}_f\cdot\vec{\mathrm{q}}}\hat{\mathscr{H}}_f(\vec{\mathrm{P}}) \tag{3.62}$$

and

$$\hat{\mathscr{H}}(\vec{\mathrm{P}}+\vec{\mathrm{q}})=e^{-i\vec{\mathrm{X}}_0\cdot\vec{\mathrm{q}}}\hat{\mathscr{H}}(\vec{\mathrm{P}})\ . \tag{3.63}$$

From the Lorentz invariance of S it follows that $\hat{S}$ is independent of $\vec{\mathrm{P}}$,

$$\begin{aligned}\hat{S}(\vec{\mathrm{P}})&=\hat{\Omega}_{+}^{\dagger}(\vec{\mathrm{P}})\hat{\Omega}_{-}(\vec{\mathrm{P}})\\&=\hat{S}(0)=\hat{\Omega}_{+}^{\dagger}(0)\hat{\Omega}_{-}(0)\equiv\hat{S}\ ,\end{aligned} \tag{3.64}$$

but for $\vec{\mathrm{X}}\neq\vec{\mathrm{X}}_0$ $\hat{\Omega}_{\pm}(\vec{\mathrm{P}})$ is not independent of $\vec{\mathrm{P}}$. Define $\overline{\Omega}_{\pm}$ by

$$(\vec{\mathrm{P}}'\,|\,\overline{\Omega}_{\pm}\,|\,\vec{\mathrm{P}})=\delta(\vec{\mathrm{P}}'-\vec{\mathrm{P}})\hat{\Omega}_{\pm}(0)\ . \tag{3.65}$$

It follows from lemma 3.3 and (3.64) that

$$\overline{\Omega}_{+}\Omega_{+}^{\dagger}=\overline{\Omega}_{-}\Omega_{-}^{\dagger}\ . \tag{3.66}$$

The unitary operator

$$B=\overline{\Omega}_{+}\Omega_{+}^{\dagger}=\overline{\Omega}_{-}\Omega_{-}^{\dagger} \tag{3.67}$$

satisfies (3.61).

It follows from (3.65) that

$$\overline{\Omega}_{\pm}=\Omega_{\pm}(\overline{M},\overline{\Phi},M_f) \tag{3.68}$$

with

$$(\vec{P}\,|\,\bar{\Phi}\,|\,\vec{P}_f)=\delta(\vec{P}-\vec{P}_f)\hat{\Phi}(0)\ , \tag{3.69}$$

and

$$(\vec{P}\,|\,\bar{M}\,|\,\vec{P}_f)=\delta(\vec{P}-\vec{P}_f)\hat{M}(0)\ . \tag{3.70}$$

From (3.67) we get

$$\bar{M}=BMB^{-1} \tag{3.71}$$

and from lemma 3.2

$$\Omega_{\pm}(\bar{M},\bar{\Phi},M_f)=\Omega_{\pm}(\bar{M},B\Phi,M_f)\ . \tag{3.72}$$

In this section we worked in a representation in which the injection operator Φ is Euclidean invariant. In the front- and point-form the improper eigenfunctions used to construct Φ will be covariant under different kinematic subgroups. Poincaré invariance of the wave operators is equivalent to the asymptotic covariance of Φ with respect to the remaining dynamical generators. In each case the existence of the wave operators guarantees asymptotic covariance of Φ with respect to one dynamical generator, H in the point form and P_- in the front form. The asymptotic covariance with respect to the remaining dynamical generators must be proved.

In order to formulate the clustering properties of the scattering operators we need several definitions. A partition b is said to be compatible with the channel α if b does not divide any of the bound fragments in channel α. In analogy to (2.70) we define the cluster-separation operator T_{fb} by

$$T_{fb}\mathscr{H}_{f\alpha}=\begin{cases}\displaystyle\prod_{i=1}^{n_b} U_{fb_i}(\vec{\delta}_i,0)\mathscr{H}_{f\alpha} & \text{if } b \text{ is compatible with } \alpha\ ,\\ 0 & \text{otherwise}\ ,\end{cases} \tag{3.73}$$

$$\Phi_b\mathscr{H}_{f\alpha}=\begin{cases}\Phi\mathscr{H}_{f\alpha} & \text{if } b \text{ is compatible with } \alpha\ ,\\ 0 & \text{otherwise}\ .\end{cases} \tag{3.74}$$

The cluster-separability requirements for the wave operators and for the S operator can be written in the form

$$\operatorname*{s-lim}_{\min|\vec{\delta}_i-\vec{\delta}_j|\to\infty}(\Omega_{\pm}-\Omega_{a\pm})T_{fa}(\vec{\delta}_i\cdots\vec{\delta}_{n_a})=0 \tag{C3}$$

and

$$\operatorname*{s-lim}_{\min|\vec{\delta}_i-\vec{\delta}_j|\to\infty}(S-S_a)T_{fa}(\vec{\delta}_1\cdots\vec{\delta}_{n_a})=0\ . \tag{C4}$$

The condition (C4) follows from (C3), but the converse is not true. From (C2) it follows that

$$\operatorname*{s-lim}_{\min|\vec{\delta}_i-\vec{\delta}_j|}(e^{iHt}-e^{iH_at})T_a(\vec{\delta}_1\cdots\vec{\delta}_{n_a})=0\ . \tag{3.75}$$

If this limit converges uniformly in t, then (C3) is a consequence of (3.75). On the other hand the validity of (C2) follows from (C3).

Our main task is to construct representations of the Poincaré-Lie algebra for an arbitrary number of particles such that invariant wave operators exist and the cluster conditions (C1) and (C3) are satisfied. We will do this recursively. For two particles the Bakamjian-Thomas construction satisfies all requirements,[9,11] but $\vec{X}=\vec{X}_0$ is not necessary; any other construction of Poincaré-invariant two-body wave operators can serve as a starting point of the recursion. For three particles a BT construction satisfying (C4) has been given,[11] but it is not possible to satisfy (C3) in this fashion.[12,13] Following Sokolov's ideas we can find a unitary transformation which transforms the BT representation into a representation satisfying (C1). For short-range interactions the conditions (C2) and (C3) are then also satisfied. The details of the construction for three particles will be given in Sec. V. The general inductive construction for N particles is described in Sec. VI.

IV. THE LATTICE STRUCTURE OF CLUSTER PARTITIONS

In this section we introduce the lattice structure[33–35] of the set $\mathscr{P}$ of partitions of N particles. This structure will provide a useful bookkeeping device for constructing Poincaré generators consistent with (C1). We introduce the following partial ordering, $\supset$, on $\mathscr{P}$:

$a \supset b$ if and only if each pair of particles in the same cluster of b is in the same cluster of a . (4.1)

Unions and intersections of two partitions a and b, written $a \cup b$ and $a \cap b$, respectively, are the unique least upper and greatest lower bound with respect to $\supset$.[36,37] We use the notation N and 0 to denote the upper and lower bounds of $\mathscr{P}$ with respect to $\supset$. The following examples for $N=5$ serve as an illustration:

$$\begin{aligned} &N=(12345)\ , \\ &0=(1)(2)(3)(4)(5)\ , \\ &(12)(345)\supset(12)(3)(45)\ , \\ &(1)(2)(345)\cup(1)(23)(45)=(1)(2345)\ , \\ &(1)(2)(345)\cap(1)(23)(45)=(1)(2)(3)(45)\ . \end{aligned} \tag{4.2}$$

In this paper the clusters of a partition correspond to mutually noninteracting subsystems. If the clusters of the partition a denote noninteracting subsystems and we turn off all interactions between particles in different clusters of b, then the particles that remain mutually interacting are those in the same clusters of $a \cap b$.

The ζ and Möbius functions[35,37] for the partition lattice are integer-valued functions on $\mathscr{P}\times\mathscr{P}$ defined by

$$\Delta(a,b)=\begin{cases} 1 & \text{if } a\supset b\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.3}$$

$$\Delta^{-1}(a,b)=\begin{cases} (-)^{n_a}\displaystyle\prod_{i=1}^{n_a}(-)^{n_{bi}}(n_{bi}-1)! & \text{if } a\supset b\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.4}$$

where n_a is the number of clusters of a and n_{bi} is the number of clusters of b in the ith cluster of a when $a\supset b$. These functions have the properties[36,37]

$$\sum_{c\in\mathscr{P}}\Delta(a,c)\Delta^{-1}(c,b)=\sum_{c\in\mathscr{P}}\Delta^{-1}(a,c)\Delta(c,b)=\delta_{a,b}\ , \tag{4.5}$$

$$\Delta(a\cap b,c)=\Delta(a,c)\Delta(b,c)\ , \tag{4.6}$$

$$\Delta(a,b\cup c)=\Delta(a,b)\Delta(a,c)\ , \tag{4.7}$$

$$C_a\equiv-\Delta^{-1}(N,a)\bar{\delta}_{Na}=(-)^{n_a}(n_a-1)!\bar{\delta}_{Na}\ , \qquad \bar{\delta}_{Na}=1-\delta_{Na}\ , \tag{4.8}$$

$$\sum_a C_a\Delta(a,b)=\bar{\delta}_{Nb}\ . \tag{4.9}$$

Let $\mathscr{P}'$ denote the set

$$\mathscr{P}'=\{a\in\mathscr{P}\mid a\neq N\}\ . \tag{4.10}$$

The fully linked part $[\mathscr{O}]_N$, of an operator $\mathscr{O}$ is defined by

$$[\mathscr{O}]_N=\mathscr{O}-\sum_a C_a\mathscr{O}_a=\mathscr{O}+\sum_a\bar{\delta}_{N,a}\Delta^{-1}(N,a)\mathscr{O}_a\ . \tag{4.11}$$

$[\mathscr{O}]_N$ is the part of the operator that vanishes when *all* of the interactions involving *any* one particle vanish. This definition is consistent with more standard notions of linked operators.[38]

V. THREE-BODY SYSTEMS

We assume that for each two-body subsystem the Poincaré generators are known. We have for every partition a

$$G(a)=\sum_{i=1}^{n_a}G(a_i)\ . \tag{5.1}$$

For each partition a the generator $G(a)$ is simply the sum of the generators for the mutually noninteracting clusters of a, which are known from the BT construction for $N=2$. The generators $G(a)$ satisfy the cluster requirement

$$(G(a))_b=G(a\cap b)\ . \tag{5.2}$$

If a designates an interacting pair plus a spectator and the interaction is turned off, then $G(a)$ becomes the sum of three single-particle generators since (C1) is satisfied for $N=2$. The equation is trivial for $a=0$. The wave operators

$$\Omega_{a\pm}=\Omega_\pm(H(a),\Phi_a,H_f) \tag{5.3}$$

exist, and satisfy the completeness relation

$$\Omega_{a\pm}\Omega_{a\pm}^\dagger=1 \tag{5.4}$$

and Poincaré invariance

$$G(a)\Omega_{a\pm}=\Omega_{a\pm}G_f \tag{5.5}$$

provided the subsystem operators satisfy the same conditions.

The first step toward a fully interacting three-body system is a BT construction for an interacting pair with a noninteracting spectator. To this end we must find a mass operator $\overline{M}(a)$ that com-

mutes with $\vec{X}_0$. According to the proof of theorem 3.4 there exists a unitary operator $B(a)$, given by (3.67), satisfying

$$B(a)\vec{X}(a)=\vec{X}_0 B(a) , \tag{5.6}$$

and a mass operator $\overline{M}(a)$, given by (3.70), which satisfies

$$B(a)M(a)=\overline{M}(a)B(a) \tag{5.7}$$

and

$$\overline{M}(0)=M_0 .$$

It is clear that

$$(B(a))_b=B(a\cap b) \tag{5.8}$$

holds as a special feature of the three-body problem since $B(a)\to 1$ if the interaction is turned off. Obviously

$$B(0)=1 . \tag{5.9}$$

It is now easy to define a BT mass operator $\overline{M}$ for the fully interacting system,

$$\begin{aligned}\overline{M}&=\sum_a C_a\overline{M}(a)+[\overline{M}]_3\\&=\overline{M}(12,3)+\overline{M}(23,1)+\overline{M}(31,2)\\&\quad -2M_0+[\overline{M}]_3\\&=M_0+\overline{V}_{12}+\overline{V}_{23}+\overline{V}_{31}+\overline{V}_{123} ,\end{aligned} \tag{5.10}$$

where

$$\overline{V}_{ij}\equiv\overline{M}(ij,k)-M_0,\quad \overline{V}_{123}\equiv[\overline{M}]_3 . \tag{5.11}$$

The fully linked part, $\overline{V}_{123}$ must commute with $\vec{J}$, $\vec{P}$, and $\vec{X}_0$ but is otherwise arbitrary; it can be set to zero. Since $\overline{M}$ manifestly commutes with $\vec{X}_0$ we can now construct the BT representations

$$\begin{aligned}&\overline{H}(a)=[\vec{P}_0^{\,2}+\overline{M}(a)^2]^{1/2} ,\\&\vec{\overline{K}}(a)=\tfrac{1}{2}[\vec{X}_0\overline{H}(a)+\overline{H}(a)\vec{X}_0]\\&\qquad -\frac{\vec{j}\times\vec{P}}{\overline{M}(a)+\overline{H}(a)} ,\\&\overline{H}=(\vec{P}^2+\overline{M}^2)^{1/2} ,\end{aligned} \tag{5.12}$$

and

$$\vec{\overline{K}}=\tfrac{1}{2}(\vec{X}_0\overline{H}+\overline{H}\vec{X}_0)-\frac{\vec{j}\times\vec{P}}{\overline{M}+\overline{H}} . \tag{5.13}$$

It follows from these definitions and Eqs. (5.2), (5.6), (5.7), (5.10), (5.12), and (5.13) that

$$\overline{H}_a=\overline{H}(a)=B(a)H(a)B(a)^\dagger , \tag{5.14}$$

$$\vec{\overline{K}}_a=\vec{\overline{K}}(a)=B(a)\vec{K}(a)B(a)^\dagger . \tag{5.15}$$

These equations express the failure of the generators $\overline{H}$ and $\vec{\overline{K}}$ to satisfy the cluster condition (C1). A unitary transformation of the generators preserves their commutation relations. Therefore, if we can construct a unitary operator B such that

$$B_a=B(a) , \tag{5.16}$$

then

$$H=B^\dagger\overline{H}B , \tag{5.17}$$

$$\vec{K}=B^\dagger\vec{\overline{K}}B \tag{5.18}$$

satisfies (C1).

Such an operator B can be constructed as follows. Let $\beta(a)$ be the Cayley transform of $B(a)$,

$$i\beta(a)=\frac{B(a)-1}{B(a)+1} , \tag{5.19}$$

and define β and B by

$$\begin{aligned}\beta&=\sum_a C_a\beta(a)+[\beta]_N\\&=\beta(12,3)+\beta(23,1)+\beta(31,2)+[\beta]_N\end{aligned} \tag{5.20}$$

and

$$B=\frac{1+i\beta}{1-i\beta} , \tag{5.21}$$

where $[\beta]_N$ is an arbitrary Euclidean-invariant self-adjoint operator and $[\beta]_N=0$ is a possible choice. Clearly the unitary operator B so defined satisfies (5.16) since $\beta(ij,k)\to 0$ as $V_{ij}\to 0$ and the generators H, $\vec{K}$, $\vec{P}$, and $\vec{J}$ define a representation of the Poincaré group consistent with the clustering condition (C1).

In order to verify the Lorentz invariance of $\Omega_\pm(H,\Phi,H_f)$ note that

$$\vec{K}\Omega_\pm(H,\Phi,H_f)=B^{-1}\vec{\overline{K}}\Omega_\pm(\overline{H},B\Phi,H_f) .$$

The Lorentz invariance of $\Omega_\pm(H,\Phi,H_f)$ is therefore assured if

$$\Omega_\pm(\overline{H},B\Phi,H_f)=\Omega_\pm(\overline{H},\Phi,H_f) , \tag{5.22}$$

By lemma 3.1 the condition (5.22) is equivalent to

$$\operatorname*{s-lim}_{t\to\pm\infty}(B\Phi-\overline{\Phi})e^{-iH_f t}=0 . \tag{5.23}$$

For any partition $a\neq N$ the corresponding relation

$$\operatorname*{s-lim}_{t\to\pm\infty}[B(a)\Phi_a-\overline{\Phi}_a]e^{-iH_f t}=0 \tag{5.24}$$

follows from the definition of $B(a)$. Since the definition (3.74) of Φ_b can be expressed in the form

$$\Phi_b=\sum_a \Delta(b,a(\alpha))\Phi_\alpha\ , \tag{5.25}$$

it is easy to verify using (4.9), (4.11), and (5.16) that

$$\begin{aligned}\Phi&=\sum_a C_a\Phi_a+[\Phi]_N\ ,\\ \overline{\Phi}&=\sum_a C_a\overline{\Phi}_a+[\Phi]_N\ ,\end{aligned} \tag{5.26}$$

and

$$B\Phi-\overline{\Phi}=\sum_a C_a[B(a)\Phi_a-\overline{\Phi}_a]+[B\Phi-\overline{\Phi}]_N\ .$$

Thus, since (5.24) is already satisfied we need only

$$\operatorname*{s-lim}_{t\to\pm\infty}[B\Phi-\overline{\Phi}]_N e^{-H_f t}=0 \tag{5.27}$$

in order to satisfy (5.23). Equation (5.27) holds under rather weak assumptions which are related to the sufficient conditions (3.31) for the existence of the three-body wave operators.

The Hamiltonian (5.17) has the form

$$\begin{aligned}H&=B\left[\sum_a C_a B(a)^\dagger H(a)B(a)\right]B^\dagger\\ &=\sum_a C_a H(a)+[H]_N\ .\end{aligned} \tag{5.28}$$

It is clear by comparing these expressions that the entire effect of the $B(a)$'s and B is to contribute a three-body force $[H]_N$. With our construction this force vanishes in the three-body rest frame, $\vec{P}=0$, since, by construction, $B(a)=1$ for $\vec{P}=0$.

For the explicit construction of $B(a)$,

$$B(a)=\overline{\Omega}_\pm(a)\Omega_\pm^\dagger(a)\ , \tag{5.29}$$

we take the partition (12,3) as an illustration. In the notation of Sec. III we have the subsystem wave operators

$$(\vec{P}_{12}\,|\,\Omega_{12\pm}\,|\,\vec{P}'_{12})=\delta(\vec{P}_{12}-\vec{P}'_{12})\hat{\Omega}_{12\pm}\ , \tag{5.30}$$

and hence

$$(\vec{p}_3,\vec{P}\,|\,\Omega_\pm(a)\,|\,\vec{P}',\vec{p}'_3)=\delta(\vec{p}_3-p'_3)(\vec{P}_{12}\,|\,\Omega_{12\pm}\,|\,\vec{P}'_{12})=\delta(\vec{P}'-\vec{P})\delta(\vec{p}_3-\vec{p}'_3)\hat{\Omega}_{12\pm}\ . \tag{5.31}$$

The wave operator $\overline{\Omega}_\pm(a)$, on the other hand, can be defined in a $\vec{q}_3,\vec{P}$ representation by

$$\begin{aligned}&(\vec{q}_3,\vec{P}\,|\,\Omega_\pm(a)\,|\,\vec{P},\vec{q}'_3)\\ &\qquad=\delta(\vec{q}_3-\vec{q}'_3)\delta(\vec{P}-\vec{P}')\hat{\Omega}_{12\pm}\ ,\end{aligned} \tag{5.32}$$

where

$$q_3=L\left[\frac{\vec{P}}{M_0}\right]p_3\ . \tag{5.33}$$

$L(\vec{Q})$ is given by (2.16)–(2.20) and p_3 is the four-vector $\{\vec{p}_3,(p_3{}^2+m_3{}^2)^{1/2}\}$. Note that

$$\begin{aligned}M_0&=(H_0{}^2-\vec{P}^2)^{1/2}\\ &=[\vec{q}_3{}^2+(M_{12}^0)^2]^{1/2}+(\vec{q}_3{}^2+m_3{}^2)^{1/2}\ .\end{aligned} \tag{5.34}$$

It follows from the definitions that $\Omega_\pm(a)$ does not commute with $\vec{q}_3$, and that $\overline{\Omega}_\pm(a)$ does not commute with $\vec{p}_3$.

In order to obtain a representation of $B(a)$ from (5.29) we must transform $\overline{\Omega}_\pm(a)$ given by (5.32) to the standard $\vec{p}_3,\vec{P}$ representation. According to (5.33) we have

$$\vec{p}_3=\vec{q}_3+\vec{P}\phi(\hat{M}_{12}^0,\vec{q}_3,\vec{P})\ , \tag{5.35}$$

where

$$\phi(\hat{M}_{12}^0,\vec{q}_3,\vec{P})=\frac{1}{M_0}\left[\frac{\vec{P}\cdot\vec{q}_3}{M_0+H_0}+(\vec{q}_3{}^2+m_3{}^2)^{1/2}\right]\ . \tag{5.36}$$

It follows that the unitary matrix $F(M_{12}^0)$ that transforms the $\vec{q}_3,\vec{P}$ representation into the $\vec{p}_3,\vec{P}$ representation is given by

$$(\vec{p}_3\vec{P}\,|\,F(\hat{M}_{12}^0)\,|\,\vec{P}'\vec{q}_3)=\delta(\vec{p}_3-\vec{q}_3-\vec{P}\phi)\mathscr{J}^{1/2}\ , \tag{5.37}$$

where the Jacobian $\mathscr{J}$ is

$$\begin{aligned}\mathscr{J}&=\det\left[\delta_{ik}+P_i\frac{\partial\phi(\hat{M}_{12}^0,\vec{q}_3,\vec{P})}{\partial q_{3k}}\right]\\ &=1+\sum_i P_i\frac{\partial\phi}{\partial q_{3i}}\ .\end{aligned} \tag{5.38}$$

It follows from (5.31), (5.32), and (5.37) that

$$\begin{aligned}&(\vec{p}_3,\vec{P}\,|\,B(a)\,|\,\vec{P}',\vec{p}'_3)\\ &\quad=(\vec{p}_3,\vec{P}\,|\,F(\hat{M}_{12}^0)\hat{\Omega}_{12\pm}F^\dagger(\hat{M}_{f12})\hat{\Omega}_{12\pm}^\dagger\,|\,\vec{P}',\vec{p}'_3)\\ &\quad=(\vec{p}_3,\vec{P}\,|\,F(\hat{M}_{12}^0)F^\dagger(\hat{M}_{12})\,|\,\vec{P}',\vec{p}'_3)\ .\end{aligned} \tag{5.39}$$

The last step follows from the intertwining properties and the completeness of $\Omega_{12\pm}$. The operator

$\hat{M}_{12}$ is the two-body mass operator including the interaction. It is important that wave operators need not be calculated to obtain $B(a)$. Clearly $B(a)-1$ vanishes if the interaction vanishes. From (5.39) we may expect that $||B(a)-1|| \ll 1$ if $||\bar{v}_{12}M_0^{-1}|| \ll 1$. Let f and f_0 be the Cayley transforms of $F(M_{12})$ and $F(M_{12}^0)$,

$$f=-i[F(\hat{M}_{12})-1][F(\hat{M}_{12})+1]^{-1}\,, \tag{5.40}$$

and

$$f_0=-i[F(\hat{M}_{12}^0)-1][F(\hat{M}_{12}^0)+1]^{-1}\,. \tag{5.41}$$

It seems reasonable to approximate $B(a)$ by expansion in powers of $f-f_0$. From

$$\begin{aligned} B(a)&=\frac{1+\mathrm{if}_0}{1-\mathrm{if}_0}\frac{1-\mathrm{if}}{1+\mathrm{if}} \\ &=1-\frac{2i}{1-\mathrm{if}_0}(f-f_0)\frac{1}{1+\mathrm{if}} \\ &\simeq 1-\frac{2i}{1-\mathrm{if}_0}(f-f_0)\frac{1}{1+\mathrm{if}_0} \end{aligned} \tag{5.42}$$

it follows that the lowest-order term of $\beta(a)$ in this expansion is given by

$$\beta(a)\simeq\tfrac{1}{2}[F(\hat{M}_{12}^0)+1](f-f_0)\tfrac{1}{2}[F^{\dagger}(\hat{M}_{12}^0)+1]\,. \tag{5.43}$$

From the definition (5.37) we can derive an operator form by expansion in powers of ϕ. To second order in ϕ we find

$$\begin{aligned} f=&\tfrac{1}{4}\{\vec{\mathrm{P}}\cdot\vec{\mathrm{y}}_3,\phi\} \\ &+\tfrac{1}{16}\{[i\vec{\mathrm{P}}\cdot\vec{\mathrm{y}}_3,\phi],\{\vec{\mathrm{P}}\cdot\vec{\mathrm{y}}_3,\phi\}\}\,, \end{aligned} \tag{5.44}$$

with

$$\vec{\mathrm{y}}_3=i\frac{\partial}{\partial\vec{\mathrm{q}}_3}\,. \tag{5.45}$$

To first order in $||\bar{v}_{12}M_0^{-1}||$ and $\vec{\mathrm{P}}^2$ we find

$$\begin{aligned} f-f_0\simeq&\frac{1}{4}\left\{\vec{\mathrm{P}}\cdot\vec{\mathrm{y}}_3,\frac{1}{M_0}\bar{v}_{12}\frac{1}{M_0}\right\}+\frac{1}{8}\left\{\vec{\mathrm{P}}\cdot\vec{\mathrm{y}}_3,\vec{\mathrm{P}}\cdot\vec{\mathrm{q}}_3\left\{\frac{1}{M_0},\frac{1}{M_0}\bar{v}_{12}\frac{1}{M_0}\right\}\right\} \\ &+\frac{1}{16}\left\{\left\{\vec{\mathrm{P}}\cdot y_3,\omega_3\frac{1}{M_0}\bar{v}_{12}\frac{1}{M_0}\right\},\frac{\vec{\mathrm{P}}\cdot\vec{\mathrm{q}}_3}{M_0}\left[\frac{1}{\omega_3}-\frac{1}{\omega_{12}^0}\right]\right\} \\ &+\frac{1}{16}\left\{\left\{\vec{\mathrm{P}}\cdot\vec{\mathrm{y}}_3,\frac{\omega_3}{M_0}\right\},\vec{\mathrm{P}}\cdot\vec{\mathrm{q}}_3\left[\frac{1}{M_0}\bar{v}_{12}\frac{1}{M_0}\left[\frac{1}{\omega_3}-\frac{1}{\omega_{12}^0}\right]-\frac{1}{M_0}\frac{1}{\omega_{12}^0}\bar{v}_{12}\frac{1}{\omega_{12}^0}\right]\right\}\,, \end{aligned} \tag{5.46}$$

where

$$\omega_{12}^0=[\vec{\mathrm{q}}_3^2+(M_{12}^0)^2]^{1/2}\,, \tag{5.47}$$

and

$$\omega_3=(q_3{}^2+m_3{}^2)^{1/2}\,. \tag{5.48}$$

In that approximation the three-body interaction in the Hamiltonian is

$$\begin{aligned} [H]_N\simeq 2i\{&[\beta(12,3),(V_{23}+V_{31})] \\ &+[\beta(23,1),(V_{31}+V_{12})] \\ &+[\beta(31,2),(V_{12}+V_{23})]\}\,, \end{aligned} \tag{5.49}$$

where

$$V_{ij}=H(ij,k)-H_0\,. \tag{5.50}$$

The three-body interaction constructed in this section vanishes for $\vec{\mathrm{P}}=0$. It is, of course, easy to add an arbitrary three-body interaction by

$$\bar{M}\to\bar{M}+\delta\bar{M}\,, \tag{5.51}$$

where $\delta\bar{M}$ commutes with $\vec{\mathrm{J}}$, $\vec{\mathrm{P}}$, and $\vec{\mathrm{X}}_0$, and vanishes sufficiently rapidly if any one particle is removed.

VI. CONSTRUCTION OF POINCARÉ GENERATORS FOR AN ARBITRARY FIXED NUMBER OF PARTICLES

In this section we present the general recursive construction of the generators H and $\vec{\mathrm{K}}$ for N particles. The steps are similar to those of Sec. V except that the operators $B(a)$ constructed as before do not satisfy Eq. (5.8). Additional transformations are needed to construct Euclidean-invariant operators $A(a)$ that satisfy

$$A(a)_b = A(a \cap b) , \tag{6.1}$$

as well as

$$A(a)\vec{X}(a) = \vec{X}_0 A(a) . \tag{6.2}$$

We assume that for each subsystem of $k < N$ particles the Poincaré generators are known and that complete invariant wave operators exist. For every $a \in \mathscr{P}'$ we can construct the set of generators $G(a)$ in the form

$$G(a) = \sum_{i=1}^{n_a} G(a_i) . \tag{6.3}$$

They satisfy Eq. (5.2) by assumption. Wave operators

$$\Omega_{\pm}(a) = \Omega_{\pm}(H(a), \Phi_a, H_f) \tag{6.4}$$

exist. They are complete, (5.4), and invariant, (5.5). As in the three-body case the proof of theorem 3.4 gives a mass operator $\overline{M}(a)$ and a unitary operator $B(a)$ that satisfies Eqs. (5.6) and (5.7) for every $a \in \mathscr{P}'$. If the operators $B(a)$ also satisfied Eq. (5.8), then we could now repeat the steps used in the three-body construction. Since this is not the case we construct Euclidean-invariant unitary operators $D(a)$ such that the operators $A(a)$,

$$A(a) = D(a)B(a) , \tag{6.5}$$

satisfy Eqs. (6.1) and (6.2).

We construct $A(a)$ by induction in the number of clusters of a, starting from $n_a = N-1$, proceeding to $n_a = 2$. For $n_a = N-1$ we take

$$A(a) = B(a) , \tag{6.6}$$

$$D(a) = 1 . \tag{6.7}$$

Equation (6.1) is satisfied in this case because only one pair of particles interacts in the partition a. When the interaction is turned off $B(a) \to B(0) = A(0) = 1$.

For the purpose of induction, we assume that we have constructed the operators $A(a)$ consistent with (6.1) and (6.2) for all a with $n_a > m$. For $n_a = m$ we define

$$D(a) = \frac{1 + i\mu(a)}{1 - i\mu(a)} , \tag{6.8}$$

where

$$\mu(a) = -\sum_b \bar{\delta}_{a,b} \Delta^{-1}(a,b) \mu(a,b) \tag{6.9}$$

and

$$\mu(a,b) = i \frac{1 - A(b)B(a)_b^{\dagger}}{1 + A(b)B(a)_b^{\dagger}} . \tag{6.10}$$

It follows from Eq. (4.4) and the relations

$$a \supset b \Rightarrow n_a \leq n_b , \tag{6.11}$$

$$a \supset b, \quad n_a = n_b \Rightarrow a = b \tag{6.12}$$

that $\bar{\delta}_{a,b} \Delta^{-1}(a,b)$ vanishes for $n_b \leq n_a$. Therefore the operator $\mu(a,b)$ is needed for the definition of $\mu(a)$ only for $n_b > n_a$. For $n_b > n_a$ all the operators $A(b)$ on the right-hand side of (6.10) are known by assumption. Thus the operators $A(a)$ can be determined recursively for all partitions $a \in \mathscr{P}'$. The unitarity and the Euclidean invariance of the operators $A(a)$ so defined follow from the corresponding properties of the operators $B(a)$.

For $n_a = N-1$, Eq. (6.2) follows from (5.6); for all other values of n_a we establish (6.2) recursively. From Eq. (5.6) it follows that

$$B(a)_b \vec{X}(a \cap b) B(a)_b^{\dagger} = \vec{X}_0 . \tag{6.13}$$

Assume that (6.2) is satisfied for all partitions b for which $n_b > n_a$. For $a \supset b$ with $n_b > n_a$ it follows that

$$[A(b)B(a)_b^{\dagger}, \vec{X}_0] = 0 , \tag{6.14}$$

and therefore

$$[D(a), \vec{X}_0] = 0 , \tag{6.15}$$

by (6.8)–(6.10). Equation (6.2) follows from (5.6) and (6.5).

It remains to verify (6.1). Evidently Eq. (6.1) holds for $n_a = N-1$. For smaller values of n_a we proceed again by induction. Assume (6.1) is satisfied for all partitions b for which $n_b > m$. It will be sufficient to show that for $n_a = m$

$$D(a)_b = A(a \cap b) B(a)_b^{\dagger} \tag{6.16}$$

for any $a \not\subset b$. (The case $b \supset a$ is trivial.) From the definitions (6.8) and (6.9) it follows that

$$D(a)_b = [1 + i\mu(a)_b][1 - i\mu(a)_b]^{-1} \tag{6.17}$$

with

$$\mu(a)_b = -\sum_c \bar{\delta}_{a,c} \Delta^{-1}(a,c) \mu(a,c)_b \tag{6.18}$$

and

$$\mu(a,c)_b = i \frac{1 - A(c \cap b) B(a)_{c \cap b}^{\dagger}}{1 + A(c \cap b) B(a)_{c \cap b}^{\dagger}} = \mu(a, c \cap b) . \tag{6.19}$$

In (6.19) we used the assumption that for $n_c > m$ $A(c)_b = A(c \cap b)$. The operator can be explicitly evaluated using Eqs. (4.4)–(4.8):

$$\mu(a)_b = -\sum_c \bar{\delta}_{a,c}\Delta^{-1}(a,c)\mu(a,c\cap b) = -\sum_{c,d,e} \bar{\delta}_{a,c}\Delta^{-1}(a,c)\Delta(c\cap b,d)\Delta^{-1}(d,e)\mu(a,e)$$
$$= -\sum_{c,d,e} \bar{\delta}_{a,c}\Delta^{-1}(a,c)\Delta(c,d)\Delta(b,d)\Delta^{-1}(d,e)\mu(a,e)$$
$$= \sum_{c,d,e} (\delta_{a,c}-1)\Delta^{-1}(a,c)\Delta(c,d)\Delta(b,d)\Delta^{-1}(d,e)\mu(a,e)$$
$$= \sum_{d,e} \Delta(a\cap b,d)\Delta^{-1}(d,e)\mu(a,e) \quad (a \not\subset b)$$
$$= \mu(a,a\cap b) = i\frac{1-A(a\cap b)B(a)^\dagger_{a\cap b}}{1+A(a\cap b)B(a)^\dagger_{a\cap b}} = i\frac{1-A(a\cap b)B(a)^\dagger_b}{1+A(a\cap b)B(a)^\dagger_b} . \tag{6.20}$$

Equation (6.16) follows from (6.20) and the properties of the Cayley transforms. Equation (6.1) immediately follows for $n_a = m$ and hence for all values of n_a by induction. This completes the construction of the operators $A(a)$.

To carry out the N-body BT construction we define

$$\tilde{M}(a) = A(a)M(a)A(a)^\dagger \tag{6.21}$$

and

$$\tilde{M} = \sum_a C_a \tilde{M}(a) + [\tilde{M}]_N . \tag{6.22}$$

The term $[\tilde{M}]_N$ must commute with $\vec{X}_0$, $\vec{P}$, and $\vec{J}$ but is otherwise arbitrary. A possible choice is $[\tilde{M}]_N = 0$. It follows from (6.2) that

$$[\vec{X}_0, \tilde{M}(a)] = [\vec{X}_0, \tilde{M}] = 0 . \tag{6.23}$$

We have therefore the BT representations

$$\tilde{H}(a) = [\vec{P}^2 + \tilde{M}(a)^2]^{1/2} , \tag{6.24}$$

$$\vec{\tilde{K}} = \tfrac{1}{2}(\vec{X}_0\tilde{H} + \tilde{H}\vec{X}_0) - \frac{\vec{j}\times\vec{P}}{\tilde{M}(a)+\tilde{H}(a)} , \tag{6.25}$$

and

$$\tilde{H} = (\vec{P}^2 + \tilde{M}^2)^{1/2} , \tag{6.26}$$

$$\vec{\tilde{K}} = \tfrac{1}{2}(\vec{X}_0\tilde{H} + \tilde{H}\vec{X}_0) - \frac{\vec{j}\times\vec{P}}{\tilde{M}+\tilde{H}} . \tag{6.27}$$

Using Eqs. (4.5), (4.6), and (4.8) we can easily verify that $\tilde{M}_a = \tilde{M}(a)$,

$$\tilde{M}_a = \sum_b C_b(\tilde{M}(b))_a = \sum_b C_b(A(b)M(b)A(b)^\dagger)_a = \sum_b C_b A(a\cap b)M(a\cap b)A(a\cap b)^\dagger$$
$$= -\sum_{b,c,d} \bar{\delta}_{N,b}\Delta^{-1}(N,b)\Delta(a\cap b,c)\Delta^{-1}(c,d)\tilde{M}(d)$$
$$= -\sum_{b,c,d} \bar{\delta}_{N,b}\Delta^{-1}(N,b)\Delta(a,c)\Delta(b,c)\Delta^{-1}(c,d)\tilde{M}(d)$$
$$= \sum_{b,c,d} (\delta_{N,b}-1)\Delta^{-1}(N,b)\Delta(a,c)\Delta(b,c)\Delta^{-1}(c,d)\tilde{M}(d)$$
$$= \sum_{c,d} \Delta(a,c)\Delta^{-1}(c,d)\tilde{M}(d) = \tilde{M}(a) . \tag{6.28}$$

It follows from Eqs. (6.26), (6.27), (6.28), (6.21), and (6.2) that

$$\tilde{H}_a = \tilde{H}(a) = A(a)H(a)A(a)^\dagger , \tag{6.29}$$

and

$$\vec{\tilde{K}}_a = \vec{\tilde{K}}(a) = A(a)\vec{K}(a)A(a)^\dagger . \tag{6.30}$$

If we can construct a unitary, Euclidean-invariant A satisfying

$$A_a = A(a) , \tag{6.31}$$

then (6.29) and (6.30) imply that the generators H and $\vec{K}$ defined by

$$H = A^\dagger \tilde{H} A , \tag{6.32}$$

$$\vec{K} = A^\dagger \vec{\tilde{K}} A \tag{6.33}$$

will satisfy (C1). Such an operator A is given by

$$A=\frac{1+i\alpha}{1-i\alpha}\ , \tag{6.34}$$

where

$$\alpha=\sum_a C_a\alpha(a)+[\alpha]_N \tag{6.35}$$

and

$$\alpha(a)=i\frac{1-A(a)}{1+A(a)}\ . \tag{6.36}$$

The fully linked term $[\alpha]_N$ is an arbitrary self-adjoint operator. If $[\alpha]_N$ is not Euclidean invariant, then we must also transform $\vec{\mathrm{P}}$ and $\vec{\mathrm{J}}$, i.e.,

$$\vec{\mathrm{P}}=A^{\dagger}\vec{\mathrm{P}}_0A\ ,$$
$$\vec{\mathrm{J}}=A^{\dagger}\vec{\mathrm{J}}_0A\ .$$

This takes us out of the instant form, but we still have a Poincaré representation consistent with the clustering requirements. It can be verified by a calculation similar to (6.28) that

$$\alpha_a=\alpha(a)\ . \tag{6.37}$$

Equation (6.31) follows from (6.34) and (6.37).

As in the case of $N=3$ we can prove that the wave operators are Lorentz invariant provided

$$\mathrm{s}-\lim_{t\to\pm\infty}(A\Phi-\bar{\Phi})_N e^{-iH_f t}=0\ . \tag{6.38}$$

The proof is identical to the proof in the three-body case if we replace all B's by A's.

We have now for arbitrary N the result

$$\begin{aligned} H&=\sum_a C_aH(a)+[H]_N\ ,\\ \vec{\mathrm{K}}&=\sum_a C_a\vec{\mathrm{K}}(a)+[\vec{\mathrm{K}}]_N\ . \end{aligned} \tag{6.39}$$

This means that the entire effect of the construction is the introduction of a suitable N-body force required to maintain the Poincaré commutation relations.

An approximate construction of $\alpha(a)$ can be carried out in the same manner as in Sec. V. To first order in the two-body interaction $\alpha(a)$ vanishes for $n_a<N-1$ and hence $[H]_N$ vanishes in that approximation. The leading term of the N-body interaction is of the order $||\bar{v}_{12}M_0{}^{-1}||^{N-2}$ relative to the two-body interaction.

VII. PARTICLE CREATION

The construction of the previous section can be extended to models that do not conserve particle number.[39] For that purpose it is simplest to treat all particles as distinguishable. The resulting operators will have the desired symmetry for identical particles. The number of degrees of freedom of a finite system is given by the maximum number N of particles that may be present. The Hilbert space will be a direct sum of n-particle Hilbert spaces associated with all proper subsystems. The restriction to finite N is unphysical unless N is limited by a conservation law such as baryon-number conservation. For real systems it is necessary to consider the limits $N\to\infty$. We will see that while local field theories describe systems with infinitely many degrees of freedom, relativistic systems with infinitely many degrees of freedom are not necessarily local field theories. As an illustration we have for $N=3$

$$\begin{aligned} \mathscr{H}=&\mathscr{H}_1\oplus\mathscr{H}_2\oplus\mathscr{H}_3\oplus(\mathscr{H}_1\otimes\mathscr{H}_2)\oplus(\mathscr{H}_2\otimes\mathscr{H}_3)\\ &\oplus(\mathscr{H}_3\otimes\mathscr{H}_1)\oplus(\mathscr{H}_1\otimes\mathscr{H}_2\otimes\mathscr{H}_3)\ . \end{aligned} \tag{7.1}$$

Let S_N denote the set of all subsets of the set of N particles, excluding the empty subset. The Hilbert space of states is then

$$\mathscr{H}=\bigoplus_{s\in S_N}\mathscr{H}_s\ , \tag{7.2}$$

where

$$\mathscr{H}_s=\bigotimes_{i\in s}\mathscr{H}_i\ . \tag{7.3}$$

Vectors in the space (7.2) are (2^N-1)-component objects,[40]

$$|\psi\rangle=\bigoplus_{s\in S_N}|\psi_s\rangle\ , \tag{7.4}$$

where $\langle\psi_s|\psi_s\rangle$ is the probability that the state $|\psi\rangle$ has elementary-particle content associated with the subset s. The operators on $\mathscr{H}$ will be $(2^N-1)\times(2^N-1)$ matrices of operators from $\mathscr{H}_s$ to $\mathscr{H}_{s'}$. Interactions that do not conserve particle number necessarily have off-diagonal matrix elements, $s'\neq s$.

Partitions of our system can be defined unambiguously in terms of partitions of the N-body sector. Every particle is assigned to a cluster a_i of the partition a. For each partition $a\in\mathscr{P}'$ the Hilbert space $\mathscr{H}$ naturally factors into a product of n_a terms, with each term a sum of $2^{n_{ai}}-1$ components. Here n_{ai} is the number of particles in the ith cluster of a. In the three-body example we have for the partition $a=(12,3)$

$$\begin{aligned} \mathscr{H}&=\mathscr{H}_3\otimes(\mathscr{H}_1\oplus\mathscr{H}_2\oplus(\mathscr{H}_1\otimes\mathscr{H}_2))\\ &=\mathscr{H}_3\otimes\mathscr{H}(12)\ . \end{aligned} \tag{7.5}$$

The general case is

$$\mathscr{H}=\bigotimes_{i=1}^{n_a}\mathscr{H}(a_i)\ , \tag{7.6}$$

where

$$\mathscr{H}(a_i)=\bigoplus_{s\in S_{ai}}\left(\bigotimes_{i\in s}\mathscr{H}_i\right) \tag{7.7}$$

and S_{ai} is the set of all nonempty subsets of particles in the ith cluster of a. The space $\mathscr{H}(a_i)$ includes all subsets of particles from the ith cluster of a.

If an operator $\mathcal{O}\in L(\mathscr{H},\mathscr{H})$ is a function of single-particle Poincaré generators and of the interactions, then we can define $\mathcal{O}_a$ as the result of turning off all interactions involving particles in different clusters of a. With this definition all the combinatorial results of Sec. IV hold, and Poincaré generators satisfying the separability requirement (C1) are constructed by the recursive procedure developed in Secs. V and VI.

If M_{ai} is the mass operator of the cluster a_i and each M_{ai} for $i=1,\ldots,n_a$ has a nonempty point spectrum, then there is at least one scattering channel associated with the partition a. An asymptotic Hilbert space $\mathscr{H}_f$ and an injection operator $\Phi\in L(\mathscr{H}_f,\mathscr{H})$ can be constructed as in Sec. III in terms of the eigenstates of the $|\phi_{\alpha i}\rangle$ of M_{ai}. The only difference is that the states $|\phi_{\alpha i}\rangle$ are not eigenstates of the number operator. The construction of wave operators and Poincaré generators proceeds as before. Equation (3.31) provides sufficient conditions for the existence of wave operators. The main conceptual difficulty with variable particle numbers stems from the fact that the truncation of the interactions required to limit the particle numbers produces macroscopic nonlocal effects which are quite unphysical. These truncations are needed for the recursive construction of Poincaré generators that satisfy (C1). We expect that the construction converges as the number of degrees of freedom increases and that (C2) can be satisfied in the limit of infinitely many degrees of freedom. A detailed investigation of this limit is beyond the scope of this paper. We confine ourselves to a discussion of the main features of three examples.

A. The relativistic Lee model

We have three different elementary particles (labeled N, V, and θ following Lee[41]) with a vertex interaction $N\theta\leftrightarrow V$. The Hilbert space of states is

$$\mathscr{H}=\bigoplus_{s_N s_V s_\theta}\left(\left(\bigotimes_{i\in s_N}\mathscr{H}_{Ni}\right)\otimes\left(\bigotimes_{j\in s_V}\mathscr{H}_{Vj}\right)\otimes\left(\bigotimes_{k\in s_\theta}\mathscr{H}_{\theta_k}\right)\right)\ . \tag{7.8}$$

The interaction conserves the numbers $N=N_N+N_V$ and $K=N_\theta+N_V$ where N_x is the number of particles of type x. With these conservation laws the Hilbert space breaks up into a direct sum of invariant sectors indexed by N and K. Any independent subsystem which can be obtained by spatial isolation can be characterized by values of N and K. As an example consider a system with at most three particles, one N, one V, and one θ: the state space is

$$=\mathscr{H}_N\oplus\mathscr{H}_V\oplus\mathscr{H}_\theta\oplus\mathscr{H}_{NV}\oplus\mathscr{H}_{V\theta}\oplus\mathscr{H}_{NV\theta}\ . \tag{7.9}$$

All interactions with ranges outside this space are turned off. The values of the conserved quantities N and K associated with each subspace are given in Table I. The subspaces in which no interactions are artificially deleted are $(N,K)=(1,0)$, $(0,1)$, and $(1,1)$, that is $\mathscr{H}_N$, $\mathscr{H}_\theta$, and $\mathscr{H}_V\oplus\mathscr{H}_{N\theta}$. In this truncation all other subspaces, $N+K\geq 3$, are trivial; that is, there are no interactions. The recursive construction proceeds by successively increasing the number of degrees of freedom. Irrelevant subspaces can be avoided by projecting always on a specific N,K sector. For finite values of N and K the recursion converges in a finite number of steps. The (1,1) sector has been treated in great detail by Dormale.[42]

The nucleon (N), the Δ isobar, and the pion form systems of the same type as long as the N-N interaction is disregarded and only the $N\pi$-Δ vertex interaction is considered.[21] The fact that the isobar is unstable presents no difficulties.

TABLE I. Particle content of invariant sections of the Lee model.

Subspace	N	K
N	1	0
θ	0	1
V	1	1
NV	2	1
$N\theta$	1	1
θV	1	2
$NV\theta$	2	2

B. The nuclear isobar model

It has been proposed to treat Δ isobars as explicit degrees of freedom in nuclei and to introduce transition potentials connecting NN states with $N\Delta$ and $\Delta\Delta$ states.[43] The baryon number is conserved; the space

$$\mathscr{H}=\bigotimes_{i=1}^{N}(\mathscr{H}_{Ni}\oplus\mathscr{H}_{\Delta i}) \tag{7.10}$$

is invariant. Because of the N-Δ mass difference of this type nonrelativistic models are not Galilean invariant. As a consequence the multinucleon S matrix does not satisfy cluster separability. However, assuming stable Δ's these models can obviously be designed to satisfy both Poincaré invariance and cluster separability following the methods of Secs. V and VI.

A new complication arises when the decay mode $\Delta\to N\pi$ is added.[21] The case of baryon number 1 is already covered by subsection A. For larger baryon numbers the combination of interactions can produce an infinite number of pions by the production mechanism

$$NN\to N\Delta\to NN\pi\to N\Delta\pi\to NN\pi\pi\to\cdots . \tag{7.11}$$

For the recursive construction we must restrict the number N_π of possible pions. For any finite value of N_π the pion interactions depend on the presence or absence of other pions which may be at macroscopic distances. It is possible to satisfy (C1) but macrolocality (C2) can only be achieved approximately. A detailed investigation of the convergence for $N_\pi\to\infty$ is beyond the scope of this paper.

C. Extended nucleons coupled to pions[44]

Similar features occur in models which couple extended bare nucleons N_0 to pions by a vertex interaction $N_0\leftrightarrow N_0\pi$ which may be introduced in the mass operator of the BT construction. For the recursive construction it is again necessary to limit the number of possible pions. As long as the number of possible pions is finite there remains a difference between an absent pion and a pion removed to a large distance. This difference should disappear as the number of possible pions increases to infinity. The convergence needs further investigation. In the limit such models have infinitely many pion degrees of freedom and they are fully relativistic. They are, however, not local field theories as is evident from the fact that no antinucleons are required and from the absence of the divergences associated with point sources.

VIII. DISCUSSION

The principal result of this paper is the explicit construction of relativistic direct interactions for an arbitrary fixed number of particles. A comparison with the nonrelativistic many-body theory serves to put this result in perspective. Nucleons are a concrete illustration.

The *nonrelativistic* theory is invariant under translations, rotations, and Galilei boosts. The generators of these transformations are independent of the interactions and the Hamiltonian is

$$H=\sum_i T_i+\tfrac{1}{2}\sum_{i,j}V_{ij}+\frac{1}{3!}\sum_{i,j,k}V_{ijk}+\frac{1}{4!}\sum_{klmn}V_{klmn}+\cdots , \tag{8.1}$$

where T_i is the kinetic energy of the ith particle and each of the interaction terms is invariant under space translations, rotations, and Galilei boosts. The short-range character of the interactions guarantees the existence of scattering operators and the dynamical independence of systems with macroscopic separation (macrolocality). There are no general consistency requirements that relate different many-body interactions to each other. For a physically interesting theory we require that the N-body interactions for $N>2$ are relatively small.

In the *relativistic* theory we have again a Hamiltonian of the form (8.1). In the instant form each term is invariant under translations and rotations. The Lie algebra of the Poincaré group requires that the Lorentz boosts $\vec{\mathrm{K}}$ depend on the interactions,

$$\vec{\mathrm{K}}=\sum_i\vec{\mathrm{K}}_i+\tfrac{1}{2}\sum_{i,j}\vec{\mathrm{K}}_{ij}+\frac{1}{3!}\sum_{i,j,k}\vec{\mathrm{K}}_{ijk}+\cdots . \tag{8.2}$$

The commutation relations imply relations between all the interaction terms of H and $\vec{\mathrm{K}}$. The short-range character of the interactions again guarantees macrolocality and the existence of scattering operators. The arbitrariness of acceptable two-body interactions is the same as in the nonrelativistic theory. Lorentz invariance require the presence of many-body interactions but a large amount of arbi-

trary choice remains. Arbitrary N-body interactions $[\tilde{M}]_N$ can be added to the BT mass operator $\tilde{M}$ for each N at the N-body level of the recursive construction. The operators $[\alpha]_N$ occurring in the construction of the Sokolov transformation are also arbitrary.

The predictive power of such models depends strongly on the assumption that arbitrary N-body interactions become small with increasing N. Under the assumption $[\alpha]_N=[\bar{M}]_N=0$ we have shown that the three-body force is small if the strength of the two-body interaction is small compared to the rest of the two-body cluster. The latter assumption is reasonable for baryon-baryon and baryon-meson interactions. Under the same conditions the N-body force was shown to decrease rapidly with N. The explicit construction in Secs. V and VI was done in the instant form. A similar construction could be performed in any of Dirac's three forms.

Nothing in the general theory described here either requires or prohibits particle creation. However, the macrolocality condition requires in general infinitely many degrees of freedom. In order to illustrate the features peculiar to particle creation we have sketched the treatment of several models. The subsystems needed for the recursive construction are obtained by turning off part of the interactions. They are not necessarily realizable by spatial separation and may thus be quite artificial. A full treatment of particle-creating models is beyond the scope of the present paper.

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APPENDIX

For the point-form construction we may define $\vec{Q}$ and $\vec{R}$ by scaling $\vec{P}$ and $\vec{X}$ by M,

$$\vec{Q}=\vec{P}M^{-1},\quad \vec{R}=M\vec{X}\ . \tag{A1}$$

The operators $\vec{Q}$ and $\vec{R}$ so defined are canonically conjugate and the generators of the homogeneous Lorentz group, $\vec{J}$ and $\vec{K}$, expressed as functions of $\vec{Q}$, $\vec{R}$, and $\vec{j}$,

$$\vec{J}=\vec{R}\times\vec{Q}+\vec{j}\ , \tag{A2}$$

$$\vec{K}=\tfrac{1}{2}(\mathsf{E}\vec{R}+\vec{R}\mathsf{E})-\vec{j}\times\vec{Q}(1+\mathsf{E})^{-1}\ ,$$

$$\mathsf{E}\equiv(1+\vec{Q}^2)^{1/2} \tag{A3}$$

are independent of the mass operator. The point-form dynamical generators are then

$$H=M\ \text{ and }\ \vec{P}=M\vec{Q}\ . \tag{A4}$$

The BT dynamics is then specified by

$$\vec{Q}=\vec{Q}_0,\ \ \vec{K}=\vec{K}_0,\ \ \vec{J}=\vec{J}_0,\ \ \bar{M}=M_0+v \tag{A5}$$

and

$$H=\bar{M}(1+\vec{Q}^2)^{1/2},\ \ \vec{P}=\bar{M}\vec{Q}\ . \tag{A6}$$

In this case v must be Lorentz invariant and commute with $\vec{Q}$ but it is not invariant under translations.

Similar relations obtain in the front-form dynamics. The invariant hyperplane is in this case the null plane $\vec{n}\cdot\vec{x}+t=\text{const}$, where $\vec{n}$ is a unit vector usually taken in the direction of the 3 axis, $\vec{n}=(0,0,1)$. The kinematic generators are those generators that leave the null plane invariant, i.e., $\vec{n}\cdot\vec{K}$, $\vec{n}\cdot\vec{J}$,

$$P_+=H+\vec{n}\cdot\vec{P},\ \ \vec{P}=\vec{P}-\vec{n}(\vec{n}\cdot\vec{P}) \tag{A7}$$

and

$$\vec{E}=\vec{K}_\perp+\vec{n}\times\vec{J}\ . \tag{A8}$$

where $\vec{K}$ is the projection of $\vec{K}$ into the plane perpendicular to $\vec{n}$. These generators satisfy the Lie algebra

$$[\vec{n}\cdot\vec{K},\vec{n}\cdot\vec{J}]=[\vec{n}\cdot\vec{J},P_+]=0\ , \tag{A9}$$

$$[\vec{E},P_+]=[\vec{E},E_r]=0\ , \tag{A10}$$

$$[\vec{n}\cdot\vec{K},P_+]=iP_+\ , \tag{A11}$$

$$[\vec{n}\cdot\vec{K},\vec{E}]=i\vec{E}\ , \tag{A12}$$

$$[\vec{n}\cdot\vec{J},\vec{E}]=-i(\vec{n}\times\vec{E})\ , \tag{A13}$$

$$[E_r,P_{\perp s}]=i(\delta_{rs}-n_rn_s)P_+\ , \tag{A14}$$

$$[\vec{n}\cdot\vec{K},\vec{P}_\perp]=0,\ \ [P_+,\vec{P}_\perp]=0\ , \tag{A15}$$

$$[\vec{n}\cdot\vec{J},\vec{P}_\perp]=-i(\vec{n}\times\vec{P}_\perp)\ . \tag{A16}$$

The dynamical generators

$$P_-=H-\vec{n}\cdot\vec{P} \tag{A17}$$

and

$$\vec{F}=\vec{K}_\perp-\vec{n}\times\vec{J} \tag{A18}$$

commute with each other,

$$[\vec{F},P_-]=[\vec{F},F_r]=0\ . \tag{A19}$$

The remaining commutation relations are

$$[\vec{n}\cdot\vec{K},P_-]=iP_- \ , \tag{A20}$$

$$[\vec{n}\cdot\vec{K},\vec{F}]=-i\vec{F} \ , \tag{A21}$$

$$[\vec{n}\cdot\vec{J},\vec{F}]=-i(\vec{n}\times\vec{F}) \ , \tag{A22}$$

$$[E_r,F_s]=-2i\left[(\delta_{rs}-n_rn_s)\vec{n}\cdot\vec{K} + \sum_p \epsilon_{rsp}n_p\vec{n}\cdot\vec{J}\right] , \tag{A23}$$

$$[\vec{E},P_-]=2i\vec{P}_\perp, \quad [P_+,\vec{F}]=-2i\vec{P}_\perp \ , \tag{A24}$$

$$[P_+,P_-]=0, \quad [\vec{P}_\perp,P_-]=0 \ , \tag{A25}$$

$$[P_{\perp r},F_s]=-i(\delta_{rs}-n_rn_s)P_- \ . \tag{A26}$$

The Pauli-Lubanski vector W has the following form as a function of the "front-form" generators:

$$W_+ = W^0+\vec{n}\cdot\vec{W}=\vec{n}\cdot\vec{J}\vec{P}_+-(\vec{E}\times\vec{P}_\perp)\cdot\vec{n} \ , \tag{A27}$$

$$\vec{W}_\perp=\tfrac{1}{2}[(\vec{n}\times\vec{F})P_+ -P_-(\vec{n}\times\vec{E})] -\vec{n}\times\vec{P}_\perp\vec{n}\cdot\vec{K} \ . \tag{A28}$$

A spin vector $\vec{s}$ defined by

$$\vec{s}\cdot\vec{n}=\vec{n}\cdot\vec{J}-(\vec{E}\times\vec{P})\cdot n/\vec{P}_+=W_+/P_+ \ , \tag{A29}$$

$$\vec{s}_\perp=M^{-1}(\vec{W}_\perp-\vec{P}_\perp W_+/P_+) \tag{A30}$$

satisfies the commutation relations

$$[s_p,s_q]=i\sum_k \epsilon_{pqk}s_k \ , \tag{A31}$$

and

$$\vec{s}^{\,2}=M^{-2}(\vec{W}^2-W^{0^2})=\vec{j}^{\,2} \ . \tag{A32}$$

The spin vectors $\vec{s}$ and $\vec{j}$ differ from each other by a rotation. The mass operator M is related to P_+, P_-, and $\vec{P}_\perp$ by

$$M^2=P_+P_- -\vec{P}_\perp^{\,2} \ . \tag{A33}$$

The dynamical $\vec{F}$ and P_- can be written as functions of M, $\vec{s}$, and the generators that leave the null plane invariant:

$$\vec{F}=\tfrac{1}{2}\left[P_-\frac{\vec{E}}{P_+}+\frac{\vec{E}}{P_+}P_-\right]+\vec{P}\left[\frac{1}{P_+}\vec{n}\cdot\vec{K}+\vec{n}\cdot\vec{K}\frac{1}{P_+}\right] -\frac{2M}{P_+}(\vec{n}\times\vec{s}_\perp)-2\frac{\vec{n}\times\vec{P}_\perp}{P_+}\vec{n}\cdot\vec{s} \ , \tag{A34}$$

$$P_-=\frac{1}{P_+}(M^2+\vec{P}_\perp^{\,2}) \ . \tag{A35}$$

In the corresponding front-form construction the dynamical generators $\vec{F}$ and P_- are given by (A34) and (A35) with

$$M\to M_0+v, \quad \vec{s}_\perp=(\vec{s}_\perp)_0 \tag{A36}$$

and

$$\{\vec{E},\vec{P}_\perp,P_+,\vec{n}\cdot\vec{K},\vec{n}\cdot\vec{J}\}=\{\vec{E},\vec{P}_\perp,P_+,\vec{n}\cdot\vec{K},\vec{n}\cdot\vec{J}\}_0 \ . \tag{A37}$$

In this case v must commute with $\vec{s}$ and with the kinematic generators.

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