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Globally stationary but locally static space-times: A gravitational analog of the Aharonov-Bohm effect

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It is well known that gravitational fields may be locally the same but globally distinct due to differences in the topology of their underlying manifolds. Globally stationary but locally static gravitational fields provide an example of gravitational fields which are locally the same but globally distinct in spite of the homeomorphism of their underlying manifolds. Any static metric on a space-time manifold with nonvanishing first Betti number R_1 is shown to generate an R_1-parameter family of such solutions. These fields are seen to provide a gravitational analog of the electromagnetic Aharonov-Bohm effect. The exterior field of a rotating infinite cylinder of matter is discussed as an exactly soluble example.

I. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

The Aharonov-Bohm effect[1] has been extensively studied,[2] and numerous gravitational analogies have been discussed.[3] In both cases nonlocal topological features of space-time come into play, so I shall begin by a brief discussion of the relevant mathematical material. After that, one of the original examples given by Aharonov and Bohm[1] will be somewhat generalized by a discussion of locally electrostatic, but globally magnetostatic fields. The external electromagnetic field of a rotating charged cylinder will be given as an example. Then the gravitational analog will be introduced: locally static, but globally stationary space-times. With the aid of this concept, the exterior field of a rotating cylinder of matter[4] will be analyzed. The apparently paradoxical feature that this field is locally static[5] is then seen to be just the gravitational analog of the previous electromagnetic example.

The use of a beam of quantum-mechanically described charged particles to verify the existence of the electromagnetic effect, and of a classically described light beam to verify the gravitational effect will be considered. It will be argued that both effects are basically classical in nature.

The rest of this section is devoted to the elements of the de Rham cohomology theory of differential forms.[6] The next section takes up the electromagnetic case, and the final section the gravitational case. [The basic results of Secs. II and III were presented in a talk to the American Physical Society in 1969. See Bull. Am. Phys. Soc. 14, 16 (1969).]

In general relativity space-time is represented by a four-dimensional orientable differentiable manifold M provided with a pseudo-Riemannian metric $g_{\mu\nu}$, of Minkowski signature (I take the signature to be $+---$, so the norm of a timelike vector is positive). There are a number of ways of studying the global topological structure of such a differentiable manifold. Since it is a topological manifold, one may use the traditional methods of algebraic or combinatorial topology. One can use some form of cohomology theory, or the dual homology theory.

However, it is possible to take advantage of the differentiable structure and use methods peculiar to differential topology: Morse theory, for example, studies the critical points of C^∞ scalar fields on the manifold.

Since differential forms play a major role in the work, it seems best for present purposes to use the de Rham cohomology theory of exterior differential forms,[6] which involves a study of the relationship between closed and exact differential forms. Fortunately, I shall need very little of this theory.

For the purposes of this paper, it will suffice to think of such a form ω as a totally antisymmetric differentiable tensor field on M, with components $\omega_{[\mu_1 \cdots \mu_p]}$ with respect to some local coordinate chart. The number of indices p defines the rank of the form, often called a p-form. (An n-dimensional differentiable manifold cannot have nonvanishing forms of rank higher than n.) A form is exact if it is the exterior derivative or curl of some form of one rank lower,

$$\omega = d\eta \Longleftrightarrow \omega_{[\mu_1 \cdots \mu_p]} = \partial_{[\mu_p} \eta_{\mu_1 \cdots \mu_{p-1}]} \,. \qquad (1.1)$$

A form is closed if its curl vanishes,

$$d\omega = 0 \Longleftrightarrow \partial_{[\mu_{p+1}} \omega_{\mu_1 \cdots \mu_p]} = 0 \,. \qquad (1.2)$$

According to Poincaré's lemma, every exact form is closed,

$$d^2\eta \equiv d(d\eta) \equiv 0 \Longleftrightarrow \partial_{[\mu_{p+2}}\partial_{\mu_{p+1}}\omega_{\mu_1\cdots\mu_p]} \equiv 0 . \qquad (1.3)$$

(This is just an application of the commutativity of ordinary derivatives to forms.) In physics one often proceeds as if the converse of the Poincaré lemma were always true: that is, as if every closed form were exact. In terms of the tensor components, this would imply that any totally antisymmetric tensor field whose curl vanished could always be written as the curl ("divergence" being just a special case of curl for forms) of some antisymmetric tensor field of one rank lower. Locally, this result is always true in a finite neighborhood of any point of the manifold. However, its global validity for fields defined over the entire manifold M depends on topological properties of that manifold. Since in this paper I shall primarily need to consider the case of one-forms (covariant vector fields), I shall illustrate this point with the example of one-forms. If the manifold in question is such that any simple closed curve within it can be continuously shrunk to a point, then the inverse of Poincaré's lemma is true for one-forms. But suppose that this is not the case. Think of the surface of a cylinder, for example. There, simple closed curves fall into two classes: those that can be continuously shrunk to a point and those that cannot be shrunk to a point because they encircle the entire cylinder. A closed curve that can be continuously shrunk to a point will be called trivial and one that cannot will be called nontrivial. On a manifold with nontrivial closed curves, the converse of Poincaré's lemma does not hold for one-forms.

Let A be a closed one-form,

$$dA = 0 \Longleftrightarrow \partial_{[\mu}A_{\nu]} = 0 . \qquad (1.4)$$

Then the integral of A around any closed curve C,

$$\int_C A \Longleftrightarrow \oint_C A_\mu dx^\mu , \qquad (1.5)$$

will have the same value for any curve C' which can be obtained from C by its continuous deformation. (This follows from the application of Stokes's theorem.) Any two such curves will be called equivalent. If the curve can be continuously shrunk to a point, it follows that this integral must vanish. However, if the curve cannot be so shrunk, then it by no means follows. Indeed, it follows from one of two de Rham theorems[7] that this integral may have any arbitrary value. The value of such an integral around an arbitrary closed curve is called a period of the one-form A (these definitions and results actually can be extended to p-forms, although we shall not need to do so in this paper except for one reference to two-forms). The only requirement on the periods of a closed form is that they be consistent: In case several distinct nonequivalent families of simple closed curves exist on a surface (for example, on a torus, or a sphere with more than one handle), the periods with respect to arbitrary closed curves must be consistently related (see the last paragraph of this section). Another theorem of de Rham[7] states that it is not enough for a form to be closed for it to be exact; in addition, all of its periods must vanish. In the case of interest here, this means integrals of the form (1.5) must vanish for nontrival closed curves C in the manifold. So if there are no nontrivial closed curves on a manifold, the converse of the Poincaré lemma must hold. However, if there are such curves, then a closed one-form with arbitrary (consistent) periods will exist, and unless all these periods vanish, it will *not* be exact.

These results apply as well to a manifold with a boundary as to one which is open. To take an example closely related to the later discussion, consider a plane with a hole cut out. We may consider the boundary of the hole to be attached to the rest of the plane, which thus forms a manifold with a boundary. Then any closed curve enclosing the hole cannot be continuously shrunk to a point, so we must expect that closed one-forms exist which have an arbitrary period about the hole. Indeed, this is easy to demonstrate by constructing such forms using polar coordinates in the plane with the hole. (Since polar coordinates are singular at the origin of the plane, they can only be used globally if the origin of the plane is not part of the manifold. Putting the origin of the polar coordinate system anywhere inside the hole allows their use for the entire manifold with boundary.) If we now choose the two polar components of A as

$$A_r = 0, \quad A_\phi = c/r \quad (r \neq 0) , \qquad (1.6)$$

for example, then A is closed, while the period of A about any closed curve enclosing the origin is given by

$$\int_C A = 2\pi c . \qquad (1.7)$$

Unless $c = 0$, this form is not exact.

Returning to the general theory, consider the set of all closed p-forms on M. Since the sum of two closed p-forms is a closed p-form, this set constitutes an Abelian group with addition as the group operation. Similarly, the set of exact p-forms constitutes an Abelian group under addition. Since every exact p-form is a closed p-form, the group of exact p-forms is a subgroup of the group of closed p-forms. The quotient group of these two groups is called the p-dimensional de Rham cohomology group, which de Rham proved isomorphic to the cohomology groups as ordinarily defined in topology. If it vanishes for a particular p, this means that every closed p-form is exact, so that the converse of the Poincaré lemma holds for p-forms. Its nonvanishing for some p is therefore an index of the degree to which the converse fails to hold for p-forms. The dimension of the group is equal to the pth Betti number R_p of the manifold. If the manifold is compact, this number must be finite. Two p-forms are called cohomologous if they differ by an exact form. Clearly, cohomology of p-forms is an equivalence relation (it is reflexive, symmetric, and transitive), so it divides the set of p-forms into equivalence classes, called cohomology classes. One of these classes is trivial: i.e., it consists of those p-forms which are themselves exact. R_p is the number of nontrivial cohomology classes of closed p-forms. De Rham's two theorems then state that a closed p-form with arbitrary values for R_p independent periods exist on a manifold with pth Betti number R_p, and that a closed p-form will be exact if and only if all these periods vanish.

II. THE ELECTROMAGNETIC CASE

I shall now apply the concepts of the previous section to the global analysis of an electromagnetic field on a space-time manifold M[8]. This field is defined by two two-forms f and h. In the (linear) Maxwell theory h is the dual of f, defined by the star operator,

$$h = {}^*f, \quad {}^*f_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\,\epsilon_{\mu\nu\kappa\lambda} f^{\kappa\lambda}, \tag{2.1}$$

where ϵ is the four-dimensional Levi-Civita tensor density of weight -1, and g is the determinant of the metric tensor $g_{\mu\nu}$ of the manifold. However, for generalized nonlinear electrodynamic theories[9] (2.1) need not hold; indeed, for the general discussion, one need not specify the exact relationship between f and h, but only consider f, which is assumed to be closed and exact,

$$df = 0, \quad f = dA. \tag{2.2}$$

By de Rham's theorems, this implies that the periods of f vanish over all closed two-surfaces of the manifold M. Physically it is just the condition for no magnetic monopoles to exist. The existence of a (single-valued) vector potential is thus equivalent to the assumption that there are no magnetic monopoles.

The vector potential, however, is not unique. Suppose there are two such potentials for some given field f,

$$f = dA = dA' \Longleftrightarrow f_{\mu\nu} = \partial_{[\mu}A_{\nu]} = \partial_{[\mu}A'_{\nu]}\,. \tag{2.3}$$

It follows that their difference $\bar{A}$ is closed,

$$\bar{A} = A - A', \quad d\bar{A} = 0 \quad \partial_{[\mu}\bar{A}_{\nu]} = 0\,. \tag{2.4}$$

By de Rham's theorems, there exist solutions to (2.4) with arbitrary periods over all R_1 independent nontrivial closed curves of M. First, suppose these periods all vanish. Then by the other clause of de Rham's theorems, $\bar{A}$ is exact,

$$\bar{A} = d\chi \Longleftrightarrow \bar{A}_\mu = \partial_\mu \chi\,. \tag{2.5}$$

So A and A' belong to the same cohomology class. We say that A and A' differ only by a gauge transformation generated by χ.

But if some periods of $\bar{A}$ are nonvanishing, then A and A' differ by more than a gauge transformation, i.e., belong to different cohomology classes. Indeed,

$$\int_C A \neq \int_C A'\,, \tag{2.6}$$

for any closed curve C with respect to which the period of A does not vanish. The integral $\int_C A$ around such a curve is gauge invariant, and also independent of any continuous deformation of C (by Stokes's theorem).

Thus, in order to fully (globally) characterize an electromagnetic field on a manifold M, it is not sufficient to give the field f over M. We must also give the periods of A over all R_1 independent classes of nontrivial closed curves which exist in M.

The well-known Aharonov-Bohm effect[1] follows at once from this observation. Generalizing one of their examples, I shall discuss globally magnetostatic (electrostationary) but locally electrostatic fields. These provide a close analogy to the gravitational fields to be considered in the next section.

Suppose the electromagnetic field is time independent in some stationary space-time. This means that there exists a globally timelike (Killing) vector field with respect to which the Lie derivatives of the metric and the electromagnetic fields vanish. One could actually carry out the analysis of the electromagnetic case at this level of generality. But since we have not yet discussed time-like Killing vector fields for space-times (which will be done in the next section), as well as for the sake of easy comparison with standard discussions of the Aharonov-Bohm effect, I shall restrict myself here to the case of a local Minkowski space-time, where the timelike Killing vector field defines an inertial frame. In that case one may make the usual $(3+1)$ decomposition of the field and potential. That is, we may break up f into three-dimensional electric and magnetic vector fields $\vec{E}$ and $\vec{B}$ and decompose A into scalar and vector potentials ϕ and $\vec{A}$. Then

$$\int_C A = \oint_C A_\mu dx^\mu = \oint \phi\, dt - \oint \vec{A}\cdot d\vec{r} = -\oint \vec{A}\cdot d\vec{r}\,. \quad (2.7)$$

Since the fields are time independent, the potentials may also be so chosen, which implies that $\oint \phi dt = \phi \oint dt = 0$. So the (four-dimensional) periods of A are equivalent to the (three-dimensional) periods of $\vec{A}$. Aharonov and Bohm choose $\vec{A}$ curl-free. Since $\vec{B} = \text{curl}\,\vec{A}$, this implies that there is no magnetic field in the region under consideration. In their example, to make the effect more striking, the electric field is also chosen to vanish. But there is no need to make that choice. $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$, so the second term vanishes with the time-independent assumption. But the first term need not. That is, we consider fields which are locally purely electric $[\vec{E} = -\nabla\phi(\vec{r}),\ \vec{B} = 0]$, but for which there are nevertheless global effects of the nonvanishing vector potential $[\oint \vec{A}(\vec{r}) \cdot d\vec{r} \neq 0$ over some closed curves]. We shall call such a field globally magnetostatic, but locally electrostatic. Since magnetostatic fields are due to the effects of stationary electric currents, we may also say globally electrostationary fields, bringing our terminology into closer line with that which is natural for the gravitational case.

To demonstrate that the magnetostatic potential in such a field does indeed produce physical effects, we use a quantum-mechanical ensemble of particles passing through the region in which such a field exists. Sakurai,[10] in his discussion of the Aharonov-Bohm effect, proves the following theorem: Let $\psi(\vec{r})$ be a solution to the nonrelativistic, time-independent Schrödinger equation for spinless charged particles of charge e in a region characterized by a time-independent electrostatic potential $\phi(\vec{r})$. Then the corresponding solution to Schrödinger's equation when a vector potential $A(\vec{r})$ with vanishing curl is added is given by

$$\psi'(\vec{r}) = \psi(\vec{r}) \exp\left[\frac{ie}{\hbar c}\oint_{\vec{r}_0}^{\vec{r}} \vec{A}(\vec{r})\cdot d\vec{r}\right]. \quad (2.8)$$

The line integral may be taken between any two points in the region where $\vec{B} = 0$. By Stokes's theorem, it will be independent of the exact path for any two equivalent paths (i.e., continuously deformable into each other).

Two conclusions follow from this theorem. First of all, suppose a coherent ensemble is separated into parts which then travel along two inequivalent paths, which form a closed path between the point of separation and the point of reunion. Then the wave functions of the two subensembles will suffer a relative phase shift proportional to the period of $\vec{A}$ around the closed path in question. Second, the physical effects produced by this phase shift only depend on the phase factor $\exp[(ie/\hbar c)\oint \vec{A}\cdot d\vec{r}]$. As Wu and Yang put it, "The field strength $f_{\mu\nu}$ underdescribes electromagnetism.... The phase $[(e/\hbar c)\oint A_\mu dx^\mu]$ overdescribes electromagnetism.... What provides a complete description that is neither too much nor too little is the phase factor...".[2]

These results have often been taken to imply that the Aharonov-Bohm effect is basically quantum mechanical in nature. However, it should be noted that the crucial point—that the phase factor is of physical significance—would be true for any wave field, whether that field is classical or quantum mechanical. For example, if there were a classical charged scalar field, classical interference experiments would suffice to demonstrate physical effects of the periods of $\vec{A}$ (see the Appendix). In the gravitational case, as we shall see in the next section, since everything couples to gravitation including light waves, classical optical interference experiments could in principle be used to verify the gravitational analog of the Aharonov-Bohm effect. Therefore, the effect in both cases should be regarded as basically classical, even though quantum-mechanical experiments may be needed to verify its existence in the electromagnetic case.

In order to illustrate the effect, some nontrivial topological configuration must be adopted for the region of space-time with a locally electrostatic but globally electrostationary field. The region outside a toroidally wound solenoid provides such a nontrivial example. If a current flows in such a solenoid, there is a nonvanishing magnetic flux inside the solenoid. This implies that, even though there is no magnetic field outside the solenoid, the line integral of the vector potential around any closed curve encircling the solenoid cannot vanish. Since this toroidal field cannot be solved exactly, Aharonov and Bohm,[1] followed by many others[2] have considered an exactly soluble model: the field of an infinite cylindrical solenoid. The field near such a cylindrical solenoid also provides a good approximation in some ways to the field near a toroidal solenoid.

In order to have a closer analogy with the gravitational example to be discussed in the next section, the original Aharonov-Bohm example may be generalized to the case of an infinite charged cylindrical solenoid. Even more generally, we consider an infinite charged rotating cylinder. The cylinder may be rotating rigidly, or be made of charged cylindrical shells rotating at different rates, so that the angular velocity ω is a function of the radial cylindrical coordinate $\rho: \omega = \omega(\rho)$.

The current density vector J then has only a ϕ-component $J_\phi = \sigma\rho\omega$, where $\sigma(\rho)$ is the charge density of the cylinder, of radius ρ_0. The field produced by this source has only the following nonvanishing components (we assume the linear Maxwell equations to hold here),

$$H_z^{\text{int}} = \frac{4\pi}{c} \int_\rho^{\rho_0} \sigma\omega\rho\, d\rho \quad (0 \leq \rho \leq \rho_0),$$

$$D_\rho^{\text{int}} = \frac{4\pi}{\rho} \int_0^\rho \rho\sigma\, d\rho \quad (0 \leq \rho \leq \rho_0),$$

$$E_\rho^{\text{ext}} = \frac{4\pi}{\rho} \int_0^{\rho_0} \rho\sigma\, d\rho \quad (\rho \geq \rho_0).$$

The exterior field produced by the cylinder is purely electrostatic, even though it is rotating. So no local exterior experiment could detect the rotation. A nonlocal Aharonov-Bohm type of experiment is needed to demonstrate the nonlocal external effects of the rotation.[11] The experiment has actually been performed, with a finite cylinder of course. A magnetized iron whisker,[12] as well as a solenoid,[13] have been used. Appropriate shielding at the ends to contain the magnetic field must be used, or a correction made for the field leakage.

III. THE GRAVITATIONAL CASE

A four-dimensional connected, oriented manifold with a pseudo-Riemannian metric of Minkowski signature, with respect to which it is time oriented, is called a space-time.[14] It follows from the de Rham theorems that two space-times on homeomorphic manifolds M and M', assumed to have nonvanishing first Betti number R_1, may have (contravariant) vector fields v and v' which map onto each other one-one under a (global) diffeomorphism of the two manifolds; yet the corresponding one-forms ω and ω',

$$v \leftrightarrow \omega, \quad v' \leftrightarrow \omega' \Longleftrightarrow v^\mu g_{\mu\nu} = \omega_\nu, \quad v'^\mu g'_{\mu\nu} = \omega'_\nu \tag{3.1}$$

will only map onto each other one-one *locally*. They will not so map globally because their periods differ for some nontrivial closed curves which correspond to each other in the global diffeomorphism between M and M'. It follows that while the two metrics g and g' may also map onto each other one-one locally, they cannot be globally so mapped. If they do correspond locally, the two manifolds may be called locally isometric. Thus, locally but not globally isometric space-times will exist; and the nonglobal nature of the mapping will *not* be due to any topological difference between the manifolds M and M'. A gravitational field corresponds to an equivalence class of space-times which can be mapped onto each other.[15] We have thus shown that globally distinct but locally indistinguishable gravitational fields are possible on homeomorphic manifolds with nonvanishing R_1. Indeed, since the two metric tensors are locally isometric, the Riemann tensor and all invariants built from it and its covariant derivatives will also agree locally. So no local feature of such space-times could distinguish between them. The only way to exhibit their global difference is by means of the differing periods of the locally corresponding but globally differing one-forms.

As an illustration, and a close analog of the generalized Aharonov-Bohm effect discussed in the last section, I shall consider the case of a locally static but globally stationary gravitational field. A gravitational field is called globally stationary in some region of space-time if there exists a global vector field ξ in that region which is a timelike Killing vector with respect to the space-time metric. This may also be expressed by saying that the Lie derivative of the metric with respect to the vector field ξ vanishes;

$$L_\xi g_{\mu\nu} \equiv \xi^\kappa \partial_\kappa g_{\mu\nu} + g_{\kappa\nu}\partial_\mu \xi^\kappa + g_{\mu\kappa}\partial_\nu \xi^\kappa = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,. \tag{3.2}$$

In terms of local coordinates this implies that local charts exist such that $\xi^\mu = \delta_0{}^\mu$, and all $g_{\mu\nu}$ are thus independent of x^0. If the one-form corresponding to ξ is hypersurface orthogonal;

$$\xi_\mu \equiv \lambda \partial_\mu \psi\,, \tag{3.3}$$

where λ and ψ are two scalar fields, then the gravitational field is called static. This implies, in terms of local coordinates, that local charts exist in which g_{0i} vanish, while all other $g_{\mu\nu}$ remain independent of x^0.

Now the possibility arises that, while the vector field ξ may be globally stationary in some region of space-time, it satisfies Eq. (3.3) only locally. That is, in any subregion of the entire region in question, ξ_μ is hypersurface orthogonal, but there exists no global (single-valued) function ψ for the entire region which satisfies (3.3). Such a region of space-time will be called globally stationary but locally static.

I shall now show that the study of such space-times is closely related to that of closed but nonexact one-forms in the region in question. First of all, I recall some results previously established elsewhere.[16] If v is any timelike vector field, and V is the one-form inverse to v, i.e.,

$$V_\mu = (1/\rho^2)\, v_\mu\,, \quad v^\mu V_\mu = 1\,, \quad v_\mu v^\mu = \rho^2\,, \tag{3.4}$$

then one may generalize the well-known hydrodynamical decomposition of a unit timelike (velocity) vector field,

$$\nabla_\mu V_\nu = H_{\mu\nu} + \Omega_{\mu\nu} + (L_v V_\nu) V_\mu - L_v(\ln\rho) V_\mu V_\nu - V_\nu{}' \partial_\mu(\ln\rho) - V_\mu{}' \partial_\nu(\ln\rho) . \quad (3.5)$$

Here $\Omega_{\mu\nu} = (1/\rho)\omega_{\mu\nu}$, where $\omega_{\mu\nu}$ is the usual rotation tensor of the congruence, whose vanishing in some region is the necessary and sufficient condition for the congruence to be (locally) hypersurface orthogonal; $H_{\mu\nu} = (1/\rho)h_{\mu\nu}$, where $h_{\mu\nu}$ is the usual rate of expansion tensor (the trace of which is the expansion scalar and the traceless part of which is the shear tensor), and $'\partial_\mu$ is the projection of the gradient orthogonal to the congruence $[{}'\partial_\mu = (\delta_\mu{}^\nu - \xi^\nu V_\mu)\partial_\nu]$. $L_v V_\mu$ is a generalization of the acceleration vector of the congruence, to which it reduces if $\rho = 1$. It follows that the exterior derivative of the one-form V, i.e., the curl of V_μ is given by

$$dV = \Omega + L_v V \wedge V \Longleftrightarrow \partial_{[\mu} V_{\nu]} = \Omega_{\mu\nu} + (L_v V_{[\mu}) V_{\nu]} . \quad (3.6)$$

So that, if $L_v V = 0$, and the congruence is (locally) hypersurface orthogonal ($\Omega_{\mu\nu} = 0$), the exterior derivative of V vanishes.

Now suppose we take a timelike Killing vector field ξ for v. It is easily shown that Killing's equations (3.2) reduce to

$$L_v V_\mu = 0 , \quad L_v \rho = 0 , \quad H_{\mu\nu} = 0 . \quad (3.7)$$

So, for a Killing vector field

$$dV = \Omega \Leftrightarrow \partial_{[\mu} V_{\nu]} = \Omega_{\mu\nu} . \quad (3.8)$$

Thus, the closure of V is a necessary and sufficient condition for a Killing vector field to be (locally) hypersurface orthogonal. (Naturally, this could also be proved directly, without use of the results from[16] quoted here.) Incidentally, the condition for ξ to be a Killing vector may also be written entirely in terms of V. Inserting (3.7) into (3.5), one sees that a necessary and sufficient condition for ξ to be a Killing vector is

$$\nabla_{(\mu} V_{\nu)} = -V_{(\mu}{}' \partial_{\nu)}(\ln\rho), \quad g^{\mu\nu} V_\mu V_\nu = 1/\rho^2 . \quad (3.9)$$

The question of whether a globally stationary Killing vector field is globally or only locally static thus is equivalent to whether the closed form V is or is not exact.

In local coordinates adapted to the Killing vector field (i.e., for which $\xi^\mu = \delta^\mu{}_0$) $\rho^2 = g_{00}$ and the components of V_μ are $(1, g_{0i}/g_{00})$. Using vectorial notation for $V_i = g_{0i}/g_{00}$, we may write in adapted coordinates (assuming that such a global time coordinate exists[17]),

$$\int_c V = \oint_C V_\mu dx^\mu = \oint_C \vec{\mathrm{V}} \cdot d\vec{\mathrm{x}} . \quad (3.10)$$

The analogy with (2.7) is obvious.

There is an analogy to electromagnetic gauge transformations for locally equivalent but globally inequivalent gravitational fields. Any one-form in the same cohomology class as ω may be used to characterize the periods which distinguish locally equivalent but globally inequivalent gravitational fields. Two such equivalent one-forms differ by an exact one-form, just as do two equivalent electromagnetic potentials. In the case of globally stationary but locally static fields, there is also an analogy to the restricted class of gauge transformations considered in the corresponding electromagnetic case. Since $\xi^\mu V_\mu = 1$ and $L_\xi V = 0$, it is natural to restrict attention to the class of equivalent one-forms (i.e., those in the same cohomology class as V) with the same properties. It then follows that the gauge function χ will also have vanishing Lie derivative with respect to ξ: That is, if W is a one-form such that

$$\xi^\mu W_\mu = 1, \quad L_\xi W = 0 , \quad V - W = d\chi , \quad (3.11)$$

then

$$L_\xi \chi = 0 . \quad (3.12)$$

Suppose we have a static space-time on some manifold M. Then a quotient manifold is obtained by identifying all points on a trajectory of the Killing vector field ξ. Suppose this quotient manifold, or spatial cross section, has nonvanishing first Betti number R_1. Since the space-time is (globally) static, the periods of V must vanish for all R_1 independent cohomology classes of closed curves. But by de Rham's theorems, distinct closed one-forms will exist with arbitrary periods. Any of these may be taken as the V corresponding to ξ. This means that there is an R_1-parameter family of globally stationary but locally static gravitational fields which are locally the same, but globally distinct. Gravitational fields have often been considered which are locally the same but globally distinct because of a difference in the topology of their underlying manifolds.[18] What is unique about this class of globally distinct but locally identical gravitational fields is that the topology of the underlying manifolds is homeomorphic.

Now I shall prove a gravitational analog of the theorem of Sakurai quoted in the last section. It could be formulated in quantum-mechanical terms, as is Sakurai's theorem, but since I am interested in showing that the gravitational analog of the Aharonov-Bohm effect may be tested classically, I shall formulate it in terms suited to a classical light wave. I shall consider the eikonal equation for the phase of such a classical electromagnetic wave field in the eikonal approximation.[19] But similar results would hold for any zero-rest-mass field, and with appropriate modifications they would hold for nonzero-rest-mass fields, including quantum-mechanical wave functions (see the Appendix).

If the field f of a locally approximate plane wave is written in the form $f=\mathrm{Re}[f_0\exp(i\omega S)]$, where S is the eikonal function, ω is the (very large) angular frequency of the solution, and f_0 is the slowly varying amplitude of the wave, then S may be shown to obey the eikonal equation,

$$g^{\mu\nu}\partial_\mu S\partial_\nu S=0\,. \tag{3.13}$$

Suppose the gravitational field to be globally stationary but locally static, with ξ as the timelike Killing vector. Then

$$'g^{\mu\nu}=g^{\mu\nu}-\xi^\mu\xi^\nu/\rho^2 \tag{3.14}$$

is the metric tensor of the hypersurface locally orthogonal to the ξ field. Let W be a scalar field, such that $L_\xi W\equiv\xi^\mu\partial_\mu W=0$, which obeys the equation,

$$'g^{\mu\nu}\partial_\mu W\partial_\nu W+\rho^{-2}=0\,. \tag{3.15}$$

Since the form of this equation is independent of the particular choice of V, its solution will not depend on that choice either. Then

$$S=-\int_{x_0}^{x}V_\mu dx^\mu+W \tag{3.16}$$

is a solution to the eikonal equation (3.13). This may be seen by noting that $\partial_\mu S=-V_\mu+\partial_\mu W$. Substituting this into (3.13) then yields (3.15).

If the amplitude f_0 is then taken to be of the form

$$f_0=k\wedge q\Leftrightarrow f_{0\mu\nu}=k_{[\mu}q_{\nu]}\,,\quad k_\mu=\partial_\mu S\,, \tag{3.17}$$

where q is a one-form which propagates along the null geodesic congruence of rays determined by the solution to the eikonal equation in accord with the equation

$$\dot q_\nu=-\tfrac{1}{2}q_\nu(\nabla_\mu k^\mu)\,,\quad \dot q_\nu\equiv\kappa^\mu\nabla_\mu q_\nu\,, \tag{3.18}$$

then it may be shown that $f=\mathrm{Re}[f_0\exp(i\omega S)]$ obeys Maxwell's equations up to a term of order $1/\omega$. The rays of the wave field are the trajectories of the k^μ field, and it follows from (3.17), (3.16), and $L_\xi W=0$ that

$$k^\mu={}'g^{\mu\nu}\partial_\nu W-\xi^\mu\,. \tag{3.19}$$

Thus, the trajectories are independent of the choice of V, and will be the same no matter what the periods of V.

However, the phase of the wave, ωS integrated along any ray, does depend on V as seen from (3.16). Thus, a coherent beam of light, split into two parts which then travel along two different spatial paths before being reunited, will show interference effects proportional to the relative phase difference

$$\omega\int V\Leftrightarrow\omega\oint V_\mu\,dx^\mu\,.$$

Thus, classically described experiments with light waves can demonstrate the existence of the analog of the Aharonov-Bohm effect in globally stationary but locally static gravitational fields.

It is the universal nature of the gravitational coupling, of course—that is the equivalence principle—which makes a classical experiment show the effect. But the analogy between the electromagnetic and gravitational cases is so close that it would seem strange to insist that the electromagnetic effect is fundamentally quantum mechanical. It seems preferable to regard the effect as basically classical in both cases, although requiring a quantum-mechanical test to demonstrate its existence in the electromagnetic case. After all, from the point of view of classical electromagnetic theory, it is adventitious that no classical charged field exists (see the Appendix).

Analysis of (3.18) shows that the direction of polarization of a linearly polarized wave is parallel transported along the rays, which gives rise to other interesting nonlocal "Aharonov-Bohm" type effects dependent on the Riemann tensor.[3]

I shall briefly consider an example of such a class of gravitational fields which is exactly soluble, and provides a gravitational analog of the exterior electromagnetic field of a rotating charged cylinder considered in the last section. This is the exterior gravitational field of an infinite rotating massive cylinder. Van Stockum[4] was able to smoothly join the exterior metric for such a cylinder, given by Lewis,[4] to an interior metric for an infinite cylinder of rotating perfect dust. (The latter solution had earlier been considered by Lanczos[20] in a cosmological context.) Ehlers and Kundt[5] noticed that the exterior metric could be transformed to the Levi-Civita form of the metric for a static cylindrically symmetric field.[21] However, they did not note that the coordinate transformation required could not be carried out globally. Once this is realized, the result loses its paradoxical character, and the field is seen to provide just the desired gravitational analog.

The exterior field may be joined to a rigidly rotating dust cylinder, as by van Stockum,[4] or to differentially rotating cylinders of dust, as done by Vishveshwara and Winicour,[22] or it may be matched to a rotating hollow cylindrical shell of matter with flat interior, as will be shown elsewhere. The latter example provides an analog to the solenoidal Aharonov-Bohm example, with charged solenoid. I shall not give details of any of the interior solutions, since the main interest here lies in the exterior field.

Any (whole) cylindrically symmetric metric may be put into the form[23]

$$ds^2 = \exp[2(\gamma - \psi)][(dx^0)^2 - (dx^1)^2] - \exp[2(\psi + \mu)](dx^2)^2 - (x^1)^2 \exp(-2\psi)(dx^3)^2, \quad (3.20)$$

$$-\infty < x^0 < +\infty, \quad 0 < x^1 < \infty, \quad -\infty < x^2 < +\infty, \quad 0 \leq x^3 < 2\pi.$$

Here x^0 is a timelike coordinate, x^1 is the analog of the cylindrical coordinate ρ, x^2 is the analog of z, and x^3 the analog of ϕ; all functions depend only on x^0 and x^1. If $T^1{}_1 + T^0{}_0 = 0$, and thus in vacuum, one may show[23] that $x^1 \exp(\mu)$ obeys the two-dimensional wave equation. If the metric is to be time (i.e., x^0) independent, this requires that $\exp(\mu) = A + B/x^1$, where A and B are constants. One may pick $A = 1$ and $B = 0$ for the exterior field, as we shall do. (But it should be noted that this choice is not necessarily the best one for the exterior field when it is to be joined to an interior solution.) With this simplification ($\mu = 0$), the metric takes the form

$$ds^2 = \exp[2(\gamma - \psi)][(dx^0)^2 - (dx^1)^2] - \exp(2\psi)(dx^2)^2 - (x^1)^2 \exp(-2\psi)(dx^3)^2. \quad (3.21)$$

The static exterior solution is then given by

$$\psi = a \ln(x^1/x_0{}^1), \quad \gamma = a^2 \ln(x^1/x_0{}^1) + \ln b, \quad (3.22)$$

where a, b, and $x_0{}^1$ are constants. Consideration of the behavior of a distant test particle or use of the relativistic analog of Gauss's theorem shows that values of a between zero and one correspond to a repulsive source.[24] The values zero and one for a correspond to flat space-times, but x^2 and x^3 exchange roles in Eq. (3.21) if $a \Leftrightarrow 1 - a$. In addition, the circumference of "circles" of constant x^1 "radius" become infinite as $x^1 \to \infty$ and approach zero as $x^1 \to 0$, only if $a < 1$. So it seems necessary to restrict a to values <1 for the positive-mass solutions. In this case, the norm of the Killing vector approaches zero as $x^1 \to 0$; and (unless $a = 0$ or 1) the curvature invariants have a singularity as $x^1 \to 0$ which goes as $(x^1)^{-2(a-1)^2}$ (Ref. 24). So this limit represents some sort of improper null horizon. The metric in the given coordinate system thus covers a manifold homeomorphic to the product of R^3 minus a line times R^1. The first Betti number of this manifold is one, since closed curves encircling the missing line cannot be continuously shrunk to a point. The Killing vector ξ, given in this coordinate system by $\xi^\mu = \delta_0{}^\mu$, corresponds to an exact one-form V, with components $V_\mu = \delta_\mu{}^0$, for this globally static metric. But it may be made to correspond to a closed one-form with arbitrary period. Formally, this may be effected by a coordinate transformation,

$$x^{0\prime} = x^0 - cx^3, \quad c = \text{const}, \quad (3.23)$$

which introduces a cross term g_{03} into the metric, and makes $V_{\mu'} = \delta_{\mu'}{}^0 + c\delta_{\mu'}{}^3$. But, because of the periodic nature of x^3, this is not a proper coordinate transformation on the entire manifold. The resulting family of stationary metrics must therefore be regarded as locally equivalent to the static metric, but globally distinct for each value of c. Matching to various interior solutions would enable the connection between c and the rotation of the source to be established, but I shall not enter into this question here.[25] In the gravitational case, however, in contrast to the electromagnetic, one may simple extend the field as far as the curvature singularity when $x^1 \to 0$, beyond which it is inextensible.

Assuming that a rotating interior source is used, a classical optical experiment performed entirely in the region outside the cylinder will enable the state of rotation of the cylinder to be investigated. If a coherent light beam is divided into two parts, each of which is passed around an opposite side of the cylinder, and they are then reunited and allowed to interfere, the theorem about the eikonal function proved above shows that there will be a phase shift proportional to the period of V, allowing the rotation of the cylinder to be verified without the light beam having entered the region of the cylinder.

The treatment of electromagnetic fields in Sec. II may be combined with the treatment of gravitational fields in this section to obtain combined electrogravitational Aharonov-Bohm effects. The electromagnetic field may be treated as an independent field on a given background (nonflat) stationary space-time, or the combined Einstein-Maxwell field equations may be considered. This generalization is so obvious that I shall not bother to give details. The more interesting possibility of pentadimensional generalizations of these results, for some form of Kaluza-Klein unified gravitational and electromagnetic theory, will be discussed elsewhere[26]; as will be generalizations of the approach followed here to space-times with higher nonvanishing Betti numbers.

Finally, it is clear that this approach could be used to generate a one-parameter family of solutions from the static exterior metric of a toroid of matter, for example. This shows that the question raised by Marder,[18] "Can two physically admissible solutions for systems of finite sources be such that the empty regions of the two space-times are homeomorphic and locally isometric, yet correspond to distinct gravitational fields? If they can, then the metrical properties of an exterior solution for a physically sensible system will not in general be sufficient to determine the gravitational field"

has an affirmative answer.

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APPENDIX

I shall briefly sketch the way a classical charged (complex) scalar field could be used to verify the existence of the Aharonov-Bohm effect. The Lagrangian for such a field ϕ, interacting with the electromagnetic field, is uniquely determined by the requirements of linearity of the free-field equation and invariance of the Lagrangian under gauge transformations of the second kind $\phi \to \phi \exp(-ie\chi)$ when the electromagnetic potentials A undergo the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$ (minimal coupling).[27] If one inserts the decomposition $\phi = \phi_0 \exp(iS)$ into this Lagrangian, where S is the phase of a wave solution, then S transforms by $S \to S - e\chi$ under a gauge transformation. In a region where the electromagnetic fields are time independent and the magnetic field vanishes, curl $\vec{A} = 0$, and we may always write $\vec{A} = \nabla[\int_{\vec{x}_0}^{\vec{x}} \vec{A}(\vec{x}')\cdot d\vec{x}']$, where the integral is taken over any path lying in the region. Let S_0 be a solution for the phase function in the presence of an electrostatic potential but no vector potential $\vec{A}$. Then in the presence of a vector potential with vanishing curl the corresponding solution is $S = S_0 - e\int_{\vec{x}_0}^{\vec{x}} \vec{A}(\vec{x}')\cdot d\vec{x}'$. The phase change corresponding to a closed curve in the region in question is then $\Delta S = -e \oint \vec{A}(\vec{x}')\cdot d\vec{x}'$. An interference experiment using this field would then yield evidence for the Aharonov-Bohm effect.

In the eikonal approximation, S obeys the relativistic Hamilton-Jacobi equation.[28] Since this would be true for fields of arbitrary spin, similar results should be expected for any such classical field.

On the question of whether such a classical limit for charged boson fields obeying the Klein-Gordon equation exists, so that such interference experiments might in principle actually be performed, or whether such interference experiments are forbidden by the charge superselection rule, see the paper by Aharonov and Susskind[29] and the comments by Kalckar.[30]

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*On leave from Boston University.

[1]Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959). The basic idea of regarding $\oint A_\mu dx^\mu$ as the fundamental quantity in electrodynamics seems to have first been stated by P. Weiss, Proc. R. Soc. London A169, 119 (1938). W. Ehrenberg and R. E. Siday stated the fundamental idea of the dependence of the phase on the line integral of $\vec{A}$ which "cannot in general be chosen so as to vanish with the magnetic field," but suggested no experimental verification of the effect.

[2]See. E. L. Feinberg, Usp. Fiz. Nauk. 78, 53 (1962) [Sov. Phys. Usp. 5, 753 (1963)], for a review of the early literature from a viewpoint similar to that adopted here. A few later papers include D. Wisnivesky and Y. Aharonov, Ann. Phys. (N. Y.) 45, 479 (1967); H. Erlichson, Am. J. Phys. 38, 162 (1970); R. L. Ingraham, *ibid.* 40, 1449 (1972); T. H. Boyer, Phys. Rev. D 8, 1667 (1973); T. T. Wu and C. N. Yang, *ibid.* 12, 3845 (1975).

[3]Some papers on this topic: G. Papini, Nuovo Cimento 52B, 136 (1967); J. S. Dowker, *ibid.* 52B, 129 (1967); J. M. Cohen and D. R. Brill, *ibid.* 56B, 209 (1968); K. Kraus, Ann. Phys. (N. Y.) 50, 102 (1969); J. K. Lawrence, D. Leiter, and G. Szamosi, Nuovo Cimento 17B, 113 (1973). See also J. Plebanski, Phys. Rev. 118, 1396 (1960); N. L. Balasz and B. Bertotti, Nuovo Cimento 27, 1087 (1963.

[4]T. Lewis, Proc. R. Soc. London A136, 176 (1932). The interior solution and its matching to the exterior one were given by W. J. van Stockum, Proc. R. S. Edinburgh 57, 135 (1937).

[5]This was first noted by J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York 1962): "Van Stockum has also shown that the rotation cylinder can be regularly joined to Levi-Civita's static (!) cylindrically symmetric exterior field...," p. 84.

[6]See G. de Rham, Rend. Mat. Appl. 20, 105 (1961) for a readable summary of his work. Izu Vaisman, *Cohomology and Differential Forms* (Dekker, New York, 1973), may also be usefully consulted. All that is needed for this paper, in a form that a physicist will find more accessible, will be found in C. W. Misner's article in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), pp. 883–929.

[7]G. de Rham, C. R. Acad. Sci. (Paris) 188, 1651 (1929). De Rham notes that the theorems, without proofs, were stated by Eli Cartan, *ibid.* 187, 196 (1928). De Rham's original proof was valid only for compact manifolds. Since the two theorems are listed in different order by various authors, I shall refer to them collectively. Also see the references in Ref. 6, as well as W. D. V. Hodge, *The Theory and Applications of Harmonic In-*

tegrals (Cambridge University Press, Cambridge, 1952).

[8]See C. Misner and J. A. Wheeler, Ann. Phys. (N. Y.) 2, 525 (1957) for such an analysis of Maxwell's theory, including an exposition of the relevant mathematical background material.

[9]See, e. g., J. Plebanski, *Lectures on Non-Linear Electrodynamics* (NORDITA, Copenhagen, 1970).

[10]J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967), pp. 15–18.

[11]See references in Refs. 1, 2, or 10 for detailed discussion of the *Gedankenexperiment*.

[12]R. G. Chambers, Phys. Rev. Lett. 5, 3 (1960).

[13]G. Mollenstedt and W. Bayh, Naturwissenschaften 49, 81 (1962). For a review of the experimental discussion, see J. Woodilla and H. Schwarz, Am. J. Phys. 39, 111 (1971).

[14]For this definition and others in this paragraph, see R. K. Sachs and H. Wu, *General Relativity for Mathematicians* (Springer, New York, 1977), Vol. 48, Chaps. 0 and 1.

[15]Although this point is often not stressed, it is clearly stated by S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, 1973), p. 56; and R. K. Sachs and H. Wu, Ref. 14, p. 27.

[16]J. Stachel, J. Math. Phys. 21, 1776 (1980).

[17]That is assuming that a scalar field ϕ exists, such that $\xi^\mu \partial_\mu \phi = 1$ throughout the region in question. Such a field is a global time function. (See Sachs and Wu, Ref. 14, p. 258, for the definition of a global time function.) Then $V'_\mu = V_\mu - \partial_\mu \phi$ will obey $\xi^\mu V'_\mu = 0$, $L_\xi V'_\mu = 0$, and $\oint V'_\mu dx^\mu = \oint V_\mu dx^\mu$ around any closed curve. In adapted coordinates, $\oint V'_\mu dx^\mu = \oint \vec{V} \cdot d\vec{x}$.

[18]Various topologies are compatible with local flatness and Minkowski signature, for example. For an application of one such topology to a "conical" space-time, see L. Marder, Proc. R. Soc. London A252, 45 (1959). This example has often been cited as a gravitational analog of the Aharanov-Bohm effect. See, e.g., the papers by Dowker and Kraus cited in Ref. 3.

[19]For a proof of the results quoted, see J. Ehlers, Z. Naturforsch. 22A, 1328 (1967), or in *Relativity, Astrophysics and Cosmology* edited by Werner Israel (Reidel, Dordrecht, 1973), Vol. 38, pp. 99–101.

[20]C. Lanczos, Z. Phys. 21, 73 (1924).

[21]T. Levi-Civita, Rend. Acc. Lincei 28, 101 (1919), reprinted in his *Opere Matematiche*, Vol. quarto 1917-1928 (Zanichelli, Bologna 1960), p. 176.

[22]C. V. Vishveshwara and J. Winicour, J. Math. Phys. 18, 1280 (1977).

[23]L. Marder, Proc. R. Soc. London A246, 133 (1958).

[24]J. Stachel, J. Math. Phys. 7, 1321 (1966).

[25]For a review of the problem of the definition of the angular momentum in general relativity, including a discussion of an interior rotating cylindrical metric and its exterior field see J. Winicour, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York and London, 1980), Vol. 2, p. 71.

[26]I am grateful to Peter Bergmann for pointing out this possibility in private discussion.

[27]See A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (MacMillan, New York, 1964), pp. 142–3.

[28]See J. Stachel and J. Plebanski, J. Math. Phys. 18, 2368 (1977) for a proof that S obeys the relativistic Hamilton-Jacobi equation in order independent of $\hbar$ for various quantum-mechanical cases. Substitution for ω for $1/\hbar$ would yield corresponding classical results in the eikonal approximation.

[29]Y. Aharonov and L. Susskind, Phys. Rev. 155, 1428 (1967).

[30]J. Kalckar, in *Foundations of Quantum Mechanics*, edited by B. d'Espagnat (Academic, New York, 1971).