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Anisotropic fluid spheres in general relativity

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We present various analytic solutions for anisotropic fluid spheres in general relativity. First we consider generalizations of the $P=\alpha\rho$ solution to the case where pressure is anisotropic, and study the effects of anisotropy on the structure of neutron stars. Next we study radiating anisotropic fluid spheres and present three classes of analytic solutions. We also study slowly rotating anisotropic fluid spheres and present two analytic solutions corresponding to the nonradiating case. One of these solutions corresponds to uniform rotation, while the other corresponds to differential rotation. We also present differential equations to be solved for slowly rotating and radiating anisotropic fluid spheres.

I. INTRODUCTION

In some of my previous papers I have discussed spherically symmetric fluid spheres with isotropic pressure and constructed several analytic solutions,[1–5] which could be used to represent some of the compact objects observed in nature. However, recent theoretical works on more realistic equations of state and stellar models indicate that some of these objects could have anisotropic pressure, at least in portions of them. The purpose of this paper is to discuss the mathematical structure of the field equations and to present various analytic, anisotropic stellar models.[6] In Sec. II, we will study static anisotropic fluid spheres, which were also studied by Cosenza *et al.*[7] and by Bowers and Liang.[8] In particular, we will discuss generalizations of the solution for $P=\alpha\rho$ to the case where pressure is anisotropic, and study the effects of anisotropy on the neutron-star mass limit and structure. In Sec. III, we will consider radiating anisotropic fluid spheres and present several classes of analytic solutions. In Sec. IV, we will discuss slowly rotating anisotropic fluid spheres with and without radiation. We will present two solutions for the nonradiating case, where one of them corresponds to uniform rotation, while the other corresponds to differential rotation. Boundary conditions will be discussed in Sec. V, and finally Sec. VI will be the conclusion.

Anisotropy could be introduced by the existence of a solid core, by the presence of type-*P* superfluid, by the complexity of the interactions, or by the existence of an external field.[9,10] Actually, recently it has been suggested that cooling of neutron stars might be accompanied by a phase transition from one anisotropic superfluid to another with significantly different properties. Such phase transitions might have significant influence on the evolution of the star.[11]

Besides these, the energy-momentum tensor for anisotropic fluids could also be obtained when the fluid is composed of two perfect fluids, with the following energy-momentum tensor:

$$T^{\mu\nu}=(P_1+\rho_1)U^\mu U^\nu-P_1 g^{\mu\nu}+(P_2+\rho_2)W^\mu W^\nu-P_2 g^{\mu\nu}\ , \tag{1.1}$$

where

$$U^\mu U_\mu=1,\quad W^\mu W_\mu=1\ .$$

This tensor can be cast into the standard form for anisotropic fluids by the following transformations[12]:

$$U^\mu\to U^{*\mu}=\cos\alpha\, U^\mu+\left[\frac{P_2+\rho_2}{P_1+\rho_1}\right]^{1/2}\sin\alpha\, W^\mu\ ,$$
$$W^\mu\to W^{*\mu}=-\left[\frac{P_1+\rho_1}{P_2+\rho_2}\right]^{1/2}\sin\alpha\, U^\mu+\cos\alpha\, W^\mu\ . \tag{1.2}$$

Notice that (1.2) leaves the quadratic form

$$(P_1+\rho_1)U^\mu U^\nu+(P_2+\rho_2)W^\mu W^\nu$$

invariant. Thus,

$$T^{\mu\nu}(U,W)=T^{\mu\nu}(U^*,W^*)\ .$$

Now we shall rotate U and W such that one becomes timelike, while the other is spacelike. This implies

$$U^{*\mu}W^*_\mu=0\ . \tag{1.3}$$

From (1.3) and (1.2) we obtain

$$\tan(2\alpha)=\frac{[(P_1+\rho_1)(P_2+\rho_2)]^{1/2}}{P_1+\rho_1-P_2-\rho_2}2W^\mu U_\mu \ . \tag{1.4}$$

Thus U^*_μ is a timelike vector and W^*_μ is a spacelike vector. Now we define the following quantities:

$$V^\mu = U^{*\mu}/(U^{*\alpha}U^*_\alpha)^{1/2} \ , \tag{1.5}$$

$$\chi^\mu = W^{*\mu}/(-W^{*\alpha}W^*_\alpha)^{1/2} \ , \tag{1.6}$$

$$\rho = T^{\mu\nu}V_\mu V_\nu = (P_1+\rho_1)U^{*\alpha}U^*_\alpha-(P_1+P_2) \ , \tag{1.7}$$

$$\sigma = T^{\mu\nu}\chi_\mu\chi_\nu = (P_1+P_2)-(P_2+\rho_2)W^{*\alpha}W^*_\alpha \ , \tag{1.8}$$

$$\pi = P_1+P_2 \ . \tag{1.9}$$

Thus the energy-momentum tensor can now be given as

$$T^{\mu\nu}=(\rho+\pi)V^\mu V^\nu-\pi g^{\mu\nu}+(\sigma-\pi)\chi^\mu\chi^\nu \ , \tag{1.10}$$

where

$$V^\mu V_\mu = 1 = -\chi^\mu\chi_\mu, \ \ \chi^\mu V_\mu = 0 \ .$$

A direct computation shows that

$$\rho=\tfrac{1}{2}(\rho_1-P_1+\rho_2-P_2)+\tfrac{1}{2}\{(P_1+\rho_1+P_2+\rho_2)^2+4(P_1+\rho_1)(P_2+\rho_2)[(U^\mu W_\mu)^2-1]\}^{1/2} \ , \tag{1.11}$$

$$\sigma=-\tfrac{1}{2}(\rho_1-P_1+\rho_2-P_2)+\tfrac{1}{2}[(P_1+\rho_1-P_2-\rho_2)^2+4(U_\mu W^\mu)^2(P_1+\rho_1)(P_2+\rho_2)]^{1/2} \ . \tag{1.12}$$

In comoving coordinates we may choose

$$V^1=V^2=V^3=0, \ \ V^0V_0=1 \ ,$$

and (1.13)

$$\chi^0=\chi^2=\chi^3=0, \ \ \chi^1\chi_1=-1 \ .$$

Thus the components of (1.10) become

$$T_0{}^0=\rho, \ \ T_1{}^1=-\sigma, \ \ T_2{}^2=T_3{}^3=-\pi \ . \tag{1.14}$$

Hence, σ is the pressure along the radial direction, while π is the tangential pressure on the $r=$constant surface. The energy-momentum tensor given in (1.10) is the standard form for anisotropic fluids.

In general, a fluid with two perfect fluid components, with the energy-momentum tensor given in (1.1), is expected to reach equilibrium through dissipative mechanisms. In that case $U^\mu \to W^\mu$ and $T^{\mu\nu}$ becomes

$$T^{\mu\nu}=(P_1+P_2+\rho_1+\rho_2)U^\mu U^\nu-(P_1+P_2)g^{\mu\nu} \ , \tag{1.15}$$

which is the energy-momentum tensor for a perfect fluid. However, in some cases the two perfect fluids in (1.1) could be decoupled from each other or at least could be weakly interacting. In that case reaching equilibrium might take a significantly long time to make anisotropic pressure important in the evolution of the object. Such a case could actually be realized in neutron stars. It is well known that it is not possible to have pure neutron star, since neutrons are unstable against β decay. Hence, in order to stabilize neutrons, neutron matter should be contaminated with sufficient protons and electrons (sometimes with hyperons and heavy mesons). The partial density of protons and electrons obviously depends on the density of the neutrons. For example, when the neutron density is 10^{15} g/cc, the partial density of protons and electrons must become 10^{13} g/cc.[13]

Let us take the two fluids in (1.1) such that they have differing four-velocities, so that we may write

$$U^\mu W_\mu = 1+\frac{a}{4} \ . \tag{1.16}$$

In this picture we let U^μ represent the four-velocity of neutrons while W^μ represents that of the protons and electrons. We also choose our coordinates such that we are comoving with the neutrons. Thus,

$$U^0=\frac{1}{(g^{00})^{1/2}}, \ \ U^1=U^2=U^3=0 \ ,$$

$$\text{also let } W^2=W^3=0 \ . \tag{1.17}$$

From $U^\mu U_\mu=1$, $W^\mu W_\mu=1$, and (1.16) we can obtain

$$W^0=\left[1+\frac{a}{4}\right]\frac{1}{(g^{00})^{1/2}} \ ,$$

$$(W^1)^2=-\frac{a}{2g_{11}}-\frac{a^2}{16g_{11}} \ . \tag{1.18}$$

With these we get

$$P_r=\sigma$$
$$=P_1+P_2+\left[\frac{a}{2}+\frac{a^2}{16}\right]\frac{(P_1+\rho_1)(P_2+\rho_2)}{(P_1+P_2+\rho_1+p_2)}\ , \quad (1.19)$$

$$P_\perp=\pi=P_1+P_2\ , \quad (1.20)$$

$$\rho=\rho_1+\rho_2+\left[\frac{a}{2}+\frac{a^2}{16}\right]\frac{(P_1+\rho_1)(P_2+\rho_2)}{(P_1+P_2+\rho_1+\rho_2)}\ , \quad (1.21)$$

where we have assumed

$$\left[2a+\frac{a^2}{4}\right]\frac{(P_1+\rho_1)(P_2+\rho_2)}{(P_1+P_2+\rho_1+\rho_2)^2}\ll 1\ . \quad (1.22)$$

We take $P=\rho$ as roughly the equation of state for both gases, for the typical densities expected in neutron stars. We also take $\rho_1\sim 10^{15}$ g/cc for neutrons and $\rho_2\sim 10^{13}$ g/cc for protons and electrons.[13] With these values we obtain ~ 0.02 for the ratio

$$\frac{(P_1+\rho_1)(P_2+\rho_2)}{(P_1+P_2+\rho_2+\rho_1)^2}\ .$$

Also for $P=\rho$ the $|g_{11}|$ component of the metric is a constant and equal to $+2$. (For the solution see the next section.) Thus, from (1.17)–(1.22) we see that, in principle, a small amount of difference in the radial velocities of the neutron and the proton-electron components of the fluid could cause a significant amount of anisotropy. Existence of such differences in the four-velocities is probably normal during the early phases of the formation of neutron stars. However, whether this difference will last long enough or not is subject to further research, which is beyond the scope of this paper. This point seems promising in bringing further explanation to the observed properties of pulsars such as glitches. Anisotropic pressure due to two perfect fluids, which are weakly interacting, is also an interesting problem in cosmology. Anisotropic fluid energy-momentum tensor (1.10) does also appear in relation to string fluids, which are easier to realize in cosmology.[14]

From the above discussion we easily see that anisotropic fluid spheres could have a wide range of applications in nature. Besides, the presence of an additional degree of freedom makes it easier to obtain analytic solutions by mathematical means.

II. STATIC ANISOTROPIC FLUIDS

Let us consider a static distribution of matter, which is spherically symmetric with the metric

$$ds^2=A^2(r)dt^2-B^2(r)dr^2 -r^2d\theta^2-r^2\sin^2\theta\, d\phi^2\ . \quad (2.1)$$

The corresponding field equations are

$$-8\pi T_1{}^1=\frac{1}{B^2}\left[\frac{2A'}{Ar}+\frac{1}{r^2}\right]-\frac{1}{r^2}\ , \quad (2.2)$$

$$-8\pi T_2{}^2=-8\pi T_3{}^3 =\frac{1}{B^2}\left[\frac{A''}{A}-\frac{A'B'}{AB}+\frac{1}{r}\left[\frac{A'}{A}-\frac{B'}{B}\right]\right]\ , \quad (2.3)$$

$$8\pi T_0{}^0=\frac{1}{B^2}\left[\frac{2B'}{Br}-\frac{1}{r^2}\right]+\frac{1}{r^2}\ . \quad (2.4)$$

For anisotropic fluids the energy-momentum tensor is given by (1.10), and in comoving coordinates its components are

$$T_0{}^0=\rho,\quad -T_1{}^1=P_r,\quad T_2{}^2=T_3{}^3=-P_\perp\ . \quad (2.5)$$

For isotropic fluids $P_r=P_\perp=P$, and one has to supply an equation of state to the above set of coupled differential equations. However, to obtain analytic solutions one usually assumes a relation between A,B,r, and their derivatives such that the above system is integrable[1,15]; of course at the end one has to check this solution for physical reasonableness. For anisotropic fluid spheres we need two relations to be supplemented to the set (2.2)–(2.4). These relations may be taken as $\rho=\rho(P_r,P_\perp)$ and $P_r=f(r)P_\perp$, where $f(r)$ is a measure of the anisotropy.

These two relations may in principle come from statistical physics. However, to obtain analytic solutions this way would almost be impossible. On the other hand, the necessity of two relations makes it easier to obtain analytic solutions for anisotropic fluids. Using $P_r=f(r)P_\perp$ and Eqs. (2.2) and (2.3) we obtain

$$B'=\frac{f\frac{A''}{A}+f\frac{A'}{Ar}-\frac{1}{r^2}-\frac{2A'}{Ar}}{f\left[\frac{A'}{A}+\frac{1}{r}\right]}B+\frac{B^3}{fr^2\left[\frac{A'}{A}+\frac{1}{r}\right]}\ . \quad (2.6)$$

For a given $A(r)$ and $f(r)$ this is a Bernoulli equation, which could be reduced to quadratures immediately. Notice that (2.6) can be solved for $f(r)$ giving

$$f(r)=\frac{\frac{B^3}{r^2}-\frac{B}{r^2}-\frac{2A'B}{Ar}}{B'\left[\frac{A'}{A}+\frac{1}{r}\right]-B\left[\frac{A''}{A}+\frac{A'}{Ar}\right]} . \tag{2.7}$$

As seen, instead of A and f we could actually guess both of the metric coefficients. Thus any metric given as in (2.1) is a solution to the Einstein field equations for an anisotropic fluid energy-momentum tensor, where the amount of anisotropy is given by (2.7).

In order to study the effects of anisotropy we may take $A(r)$ from one of the isotropic solutions and introduce a certain amount of anisotropy through $f(r)$ and find B from (2.6). Later we check this model for physical reasonableness. To elucidate the role of anisotropy on neutron-star mass limit calculations we will consider the generalizations of the isotropic solution corresponding to the isothermal equation of state given as $P=\alpha\rho$. The solution of the field equations is now given as[16]

$$P(r)=\frac{\alpha^2}{2\pi D r^2} , \tag{2.8}$$

$$\rho(r)=\frac{\alpha}{2\pi D r^2} , \tag{2.9}$$

$$A(r)=C_1 r^{2\alpha/(\alpha+1)} , \tag{2.10}$$

$$B^2(r)=\frac{D}{(1+\alpha)^2} , \tag{2.11}$$

where C_1 is an integration constant and $D=(1+\alpha)^2+4\alpha$. This equation of state, even though it does not yield a solution with finite radius, is important in neutron-star mass limit calculations, since it corresponds to the asymptotic form of the equation of state at high densities.[17] Since $P(r)$ does not vanish at finite radius the solution should be connected to an envelope over which pressure drops to zero.

In order to study the effect of anisotropy we take $A(r)$ to be the same as given in (2.10) and take $B(r)$ to be a constant different from (2.11), such as

$$B=B_0{}^2=\frac{D}{(1+\alpha)^2}+g_0 , \tag{2.12}$$

where g_0 is the difference in B from the isotropic case. This model still does not have a finite radius, also $f(r)$ is equal to a constant given as

$$f(r)=\frac{B_0{}^3-\left[1+\frac{4\alpha}{\alpha+1}\right]B_0}{-\frac{4\alpha^2}{(\alpha+1)^2}B_0} . \tag{2.13}$$

When we evaluate the core mass we find it to be

$$m_c=\frac{1}{2}\frac{1}{(8\pi\rho_c)^{1/2}}\times\left[1-\frac{1}{(\alpha^2+6\alpha+1)/(1+\alpha)^2+g_0}\right]^{3/2} , \tag{2.14}$$

where ρ_c is the density at the boundary of the core. For various values of g_0 we obtain the following core masses given in Table I.

Next we consider the following anisotropic solution:

$$\frac{1}{B^2}=1-\frac{8\pi a_0}{(n+3)}r^{n+2} , \tag{2.15}$$

$$A=C_1 r^{2\alpha/(\alpha+1)} . \tag{2.16}$$

Note that A is the same as given in (2.10). For this solution, pressure and density distributions are given as

$$8\pi P_r=\frac{4\alpha}{(\alpha+1)}\frac{1}{r^2}-\frac{8\pi a_0(5\alpha+1)}{(n+3)(\alpha+1)}r^n , \tag{2.17}$$

$$8\pi P_\perp=\frac{4\alpha^2}{(\alpha+1)}\frac{1}{r^2}-\frac{4\pi a_0}{(n+3)(\alpha+1)^2}(14\alpha^2+3n\alpha^2+4n\alpha+8\alpha+n+2)r^n , \tag{2.18}$$

$$\rho=a_0 r^n , \tag{2.19}$$

where a_0, α, n are constants. For monotonic de-

TABLE I. Core masses, where $g_0=0$ corresponds to the mass with isotropic pressure, $\alpha=1$, $\rho_c=5\times10^{14}$ g/cc, and $D/(1+\alpha)^2=2$.

g_0						
0	1	2	3	4	...	∞
$m/m_\odot$						
0.8	1.3	1.5	1.7	1.8	...	2.3

creasing pressure and density we require $n<0$, also for positive mass we need $n>-3$. This solution reduces to the isotropic case for $n=-2$ and $a_0=\alpha/2\pi D$. Even though the pressure diverges at the origin the fluid sphere has a finite radius given by

$$a_0=\frac{4\alpha(n+3)}{8\pi(5\alpha+1)}\frac{1}{R^{n+2}} . \quad (2.20)$$

We evaluate the mass from

$$M=4\pi\int_0^R \rho r^2 dr , \quad (2.21)$$

which gives

$$M=\frac{4\pi a_0}{(n+3)}R^{n+3} . \quad (2.22)$$

Using (2.20) we obtain the following mass-radius relation:

$$M=\left[\frac{2\alpha}{5\alpha+1}\right]R . \quad (2.23)$$

As seen the largest possible value for M/R is 0.40, which is less compact then the Schwarzschild solution which gives 0.44.[18]

In order to obtain a larger M/R value we try

$$A(r)=C_2r^m \quad (2.24)$$

with the same B as given in (2.15). This solution has the following pressure and density distributions:

$$8\pi P_r=\frac{2m}{r^2}-\frac{8\pi a_0}{(n+3)}(2m+1)r^n , \quad (2.25)$$

$$8\pi P_\perp=\frac{m^2}{r^2}-\frac{4\pi a_0}{(n+3)}\times[2m^2+(n+2)(m+1)]r^n , \quad (2.26)$$

$$\rho=a_0r^n , \quad (2.27)$$

where C_2,a_0,n,m are constants. Again the pressure and density diverge at the origin but they are both monotonic decreasing functions of r, while the radius is

$$R=\left[\frac{2m(n+3)}{8\pi a_0(2m+1)}\right]^{1/(n+2)} . \quad (2.28)$$

The mass of the fluid is given as

$$M=\frac{4\pi a_0}{(n+3)}R^{3+n} \quad (2.29)$$

or

$$\frac{M}{R}=\frac{m}{2m+1} , \quad (2.30)$$

which gives $2M/R\to 1$ for large m. This is more compact than the Schwarzschild interior solution and hence allows one to put more mass in a given region in space. It also allows us to explain redshifts larger than 2.[18]

For stability against radial oscillations we may require

$$\frac{dP_r}{d\rho}>0 .$$

Then, we obtain

$$\rho>\frac{(n+3)}{4\pi}\left[\frac{|n|}{2}\right]^{|n|/(2-|n|)}\frac{M}{R^3} .$$

For the sake of completeness we will conclude this section with a mathematical property of anisotropic fluid spheres. Notice that we can write (2.6) as

$$f\left[C+\frac{1}{r}\right]B'=fBC'+\left[f\left[C^2+\frac{C}{r}\right]B-\frac{B}{r^2}-\frac{2CB}{r}+\frac{B^3}{r^2}\right] , \quad (2.31)$$

where $C=A'/A$. This can also be written as

$$f\left[C+\frac{1}{r}\right]dB-(fB)dC-\left[fB\left[C^2+\frac{C}{r}\right]-\frac{B}{r^2}-\frac{2CB}{r}+\frac{B^3}{r^2}\right]dr=0 , \quad (2.32)$$

which is a Pfaffian differential equation in three dimensions, in general given as

$$F_1(B,C,r)dB+F_2(B,C,r)dC+F_3(B,C,r)dr=0 . \quad (2.33)$$

Solution of this equation is a surface in B,C,r space, and once such a surface is found then we know that any curve on that surface, which could be given as $B=B(r)$ and $C=C(r)$, will also be a solution to the field equations. Existence of such a surface has a necessary and sufficient condition of the form

$$\vec{X}\cdot\vec{\partial}\times\vec{X}=0 , \quad (2.34)$$

where $\vec{X}$ is the vector field defined as[19]

$$\vec{X}=(F_1,F_2,F_3) .$$

For our case Eq. (2.32) is integrable if $f(r)$ satisfies the following relation:

$$f(r)=\frac{1}{r}\left[\frac{1-B^2}{2C+C^2 r}\right] . \tag{2.35}$$

Once such a solution to Eq. (2.32) is found for an $f(r)$ satisfying (2.35) it will amount to finding a surface in the mathematical B,C,r space that will contain all the solutions [with such $f(r)$ of course] to the field equations. We have mentioned that such a surface does not exist for static isotropic fluid spheres.

III. RADIATING ANISOTROPIC FLUID SPHERES

For this case the energy-momentum tensor is given as[2]

$$T^{\mu\nu}=(\rho+P_\perp)U^\mu U^\nu-P_\perp g^{\mu\nu} +(P_r-P_\perp)\chi^\mu\chi^\nu+\sigma W^\mu W^\nu , \tag{3.1}$$

where $U^\mu U_\mu=1$, $\chi^\mu\chi_\mu=-1$, $W^\mu W_\mu=0$, and $\chi^\mu U_\mu=0$. In comoving coordinates we take

$$U^0U_0=1,\ \chi^1\chi_1=-1,\ W^0\neq 0,\ W^1\neq 0 ,$$

and all the other components are zero. Thus the components of $T^{\mu\nu}$ become

$$T_1{}^1=-P_r+\sigma W_1W^1,\ T_2{}^2=T_3{}^3=-P_\perp ,\quad T_0{}^0=\rho+\sigma W_0W^0,\ T_1{}^0=\sigma W_1W^0 . \tag{3.2}$$

The field equations are given as

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=-8\pi T^{\mu\nu} ,$$

$$8\pi T_1{}^1=-e^{-\alpha}\left[\frac{\beta'^2}{4}+\frac{\beta'\gamma'}{2}+\frac{\beta'+\gamma'}{r}+\frac{1}{r^2}\right]+\frac{e^{-\beta}}{r^2}+e^{-\gamma}\left[\ddot\beta+\tfrac{3}{4}\dot\beta^2-\frac{\dot\beta\dot\gamma}{2}\right] , \tag{3.3}$$

$$8\pi T_2{}^2=8\pi T_3{}^3=-e^{-\alpha}\left[\frac{\beta''}{2}+\frac{\gamma''}{2}+\frac{\beta'^2}{4}+\frac{\gamma'^2}{4}-\frac{\alpha'\beta'}{4}+\frac{\beta'\gamma'}{4}-\frac{\alpha'\gamma'}{4}-\frac{\alpha'}{2r}+\frac{\beta'}{r}+\frac{\gamma'}{2r}\right]$$
$$+e^{-\gamma}\left[\frac{\ddot\alpha}{2}+\frac{\ddot\beta}{2}+\frac{\dot\alpha^2}{4}+\frac{\dot\beta^2}{4}+\frac{\dot\alpha\dot\beta}{4}-\frac{\dot\alpha\dot\gamma}{4}-\frac{\dot\beta\dot\gamma}{4}\right] , \tag{3.4}$$

$$8\pi T_0{}^0=-e^{-\alpha}\left[\beta''+\frac{3}{4}\beta'^2-\frac{\alpha'\beta'}{2}+\frac{3\beta'}{r}-\frac{\alpha'}{r}+\frac{1}{r^2}\right]+\frac{e^{-\beta}}{r^2}+e^{-\gamma}\left[\frac{\dot\alpha\dot\beta}{2}+\frac{\dot\beta^2}{4}\right] , \tag{3.5}$$

$$8\pi T_1{}^0=-e^{-\gamma}\left[\dot\beta'-\frac{\dot\beta\gamma'}{2}+(\dot\beta-\dot\alpha)\frac{\beta'}{2}+(\dot\beta-\dot\alpha)\frac{1}{r}\right] , \tag{3.6}$$

where the metric is

$$ds^2=-e^\alpha dr^2-r^2e^\beta(d\theta^2+\sin^2\theta\, d\phi^2)+e^\gamma dt^2 . \tag{3.7}$$

From (3.2)–(3.7) we obtain the physical variables in terms of the metric coefficients:

$$P_\perp=-T_2{}^2=-T_3{}^3,\ P_r=-T_1{}^1+e^{(\gamma-\alpha)/2}T_1{}^0 , \tag{3.8}$$

$$\rho=T_0{}^0+T_1{}^1+P_r,\ \sigma=-e^{-(3\alpha-\gamma)/2}(W^1)^{-2}T_1{}^0 . \tag{3.9}$$

Now the analogous equation to (2.25) in Bayin[2] becomes

$$T_1{}^1-\theta T_2{}^2=e^{(\gamma-\alpha)/2}T_1{}^0 , \tag{3.10}$$

where

$$P_r(r,t)=\theta(r,t)P_\perp(r,t) .$$

We make the following substitutions for the metric coefficients:

$$e^\alpha=h(r)^2m(t)^2,\ e^\beta=l(r)^2m(t)^2,\ e^\gamma=f(r)^2g(t)^2,\ \theta=n(r)k(t) . \tag{3.11}$$

With these (3.10) becomes

$$-\frac{1}{h^2}\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}-nk\left[\frac{l''}{l}+\frac{f''}{f}-\frac{h'l'}{hl}+\frac{l'f'}{lf}-\frac{h'f'}{hf}-\frac{h'}{hr}+\frac{2l'}{lr}+\frac{f'}{fr}\right]\right]+\frac{1}{l^2r^2}$$
$$=\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{n}{f^2}\frac{m^2k}{g^2}-\frac{m^2}{f^2g^2}\right]+\frac{2\dot{m}f'}{f^2hg}\,. \tag{3.12}$$

This could be written as

$$-nk\left[\frac{l'}{l}+\frac{f'}{f}+\frac{1}{r}\right]h'=h\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}-nk\left[\frac{l''}{l}+\frac{f''}{f}+\frac{l'f'}{lf}+\frac{2l'}{lr}+\frac{f'}{fr}\right]\right]-\frac{h^3}{l^2r^2}$$
$$+\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{m^2}{g^2}\right]\left[\frac{nk}{f^2}-\frac{1}{f^2}\right]h^3+\frac{2\dot{m}f'h^2}{f^2g}\,. \tag{3.13}$$

For isotropic radiating fluid spheres we have seen that to solve (3.13) we need two auxiliary relations, one of which corresponds to an equation of state and the other one corresponds to the law of energy generation.[2,3] However, to obtain analytic solutions we assumed two relations between the metric coefficients such that the above system is integrable and later we evaluated the physical parameters and checked the model for physical reasonableness. We also mentioned that if we assume

$$\frac{l'}{l}+\frac{f'}{f}+\frac{1}{r}=0,\ \text{which means}\ l=\frac{C_0}{fr}\,, \tag{3.14}$$

we can express $h(r)$ immediately in terms of $l(r)$ without a quadrature.[2,3] Now again for anisotropic fluids if we assume (3.14) we can solve for $h(r)$ immediately as follows:

$$-\frac{1}{h^2}\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}-nk\left[\frac{l''}{l}+\frac{f''}{f}+\frac{l'f'}{lf}+\frac{2l'}{lr}+\frac{f'}{fr}\right]\right]+\frac{1}{l^2r^2}$$
$$=\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{m^2}{g^2}\right]\left[\frac{nk}{f^2}-\frac{1}{f^2}\right]+\left[\frac{2\dot{m}}{g}\right]\left[\frac{f'}{f^2h}\right]. \tag{3.15}$$

For anisotropic radiating fluids we have e^{α}, e^{β}, e^{γ}, P_r, $P_{\perp}$, ρ, and σ as our unknowns and we have four equations (3.3)–(3.6) to be solved simultaneously. Hence, we need three more relations to be supplied. One of these we take to be Eq. (3.14). Now we have the following possibilities.

Case I. Solution I. For this case we let $k(t)=1$. Thus, even though the model is time dependent and radiating, anisotropy does not change with time. Now, the left-hand side of (3.15) is entirely a function of r. Hence, for (3.15) to be meaningful the right-hand side should be separable in r and t. This can be achieved if we assume

$$n=1+\frac{f'}{h}\,. \tag{3.16}$$

Now the equations become

$$\frac{1}{l^2r^2}h^2-\frac{S_0f'}{f^2}h-\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}-n\left[\frac{l''}{l}+\frac{f''}{f}+\frac{l'f'}{lf}+\frac{2l'}{lr}+\frac{f'}{fr}\right]\right]=0 \tag{3.17}$$

and

$$\left[2\frac{\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{m^2}{g^2}\right]+\frac{2\dot{m}}{g}=S_0\,, \tag{3.18}$$

where S_0 is a separation constant. So far we have assumed (3.14) and (3.16), for our third assumption we would take $f(r)$. Actually we could express the remaining variables in terms of $f(r)$ without quadratures as follows:

$$h(f)=\frac{1}{f'}\left\{\frac{S_0}{2(n+1)}\pm\frac{1}{2(n+1)}\left[S_0{}^2-\frac{4}{C_0{}^2}f^4(n+1)\right]^{1/2}\right\}, \tag{3.19}$$

$$l(f)=\frac{C_0}{fr}\ , \tag{3.20}$$

and

$$n(f)=1-\frac{C_0{}^2f'^2}{8f^4}(4-4S_0)\pm\frac{C_0{}^2f'}{8f^4}\left[f'^2(4-4S_0)^2-128\frac{f^4}{C_0^2}\right]^{1/2}\ . \tag{3.21}$$

Time dependence of the metric will come from (3.18), which has to be solved such that the radius of the star is independent of time. This is so since we are using comoving coordinates. From the boundary condition we require $P_r(R,t)=0$. Using $P_r(r,t)=n(r)k(t)P_\perp(r,t)$, we see that if we do not require $P_\perp$ to simultaneously vanish at the surface with P_r, then $n(R)=0$ defines the radius of the star and the time dependence of the metric comes from the solution of (3.18) with an additional equation supplied. However, if we also require $P_\perp(R,t)=0$, then from the field equations which give $P_\perp$ as

$$-8\pi P_\perp(r,t)=-\frac{1}{h^2m^2}\left[\frac{l''}{l}+\frac{f''}{f}-\frac{h'l'}{hl}+\frac{l'f'}{lf}-\frac{h'f'}{hf}-\frac{h'}{hr}+\frac{2l'}{lr}+\frac{f'}{fr}\right]+\frac{1}{f^2g^2}\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}}{m}\frac{\dot{g}}{g}\right], \tag{3.22}$$

we see that we can have a radius which is independent of time if

$$\frac{m^2}{g^2}\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}}{m}\frac{\dot{g}}{g}\right]=\bar{C}_0\ , \tag{3.23}$$

where $\bar{C}_0$ is a constant. Remember that m and g also have to satisfy (3.18). However, even though these two equations determine $\bar{C}_0$ as

$$\bar{C}_0=S_0+2\pm2\sqrt{S_0+1}\ ,$$

they do not determine m or g, so we will still need an additional assumption on the time dependence of the metric. Note that for isotropic radiating fluid spheres m and g have to satisfy

$$\dot{m}=S_0g\ .$$

This automatically satisfies (3.23) so the boundary conditions do not put any restriction on m or g. This is contrary to the previous opinion.[2,3,20]

Case I. Solution II. The next solution we would obtain consistent with the $k(t)=1$ case is with the following relation:

$$\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{m^2}{g^2}\right]=\frac{2\dot{m}}{g}\ . \tag{3.24}$$

This allows us to separate (3.15) as

$$-\frac{1}{h^2}\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}-n\left[\frac{l''}{l}+\frac{f''}{f}+\frac{l'f'}{lf}+\frac{2l'}{lr}+\frac{f'}{fr}\right]\right]+\frac{1}{l^2r^2}=\left[(n-1)\frac{1}{f^2}+\frac{f'}{f^2h}\right]S_0, \tag{3.25}$$

$$\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{m^2}{g^2}\right]=\frac{2\dot{m}}{g}=S_0\ , \tag{3.26}$$

where again S_0 is a separation constant. Solution of (3.26) gives

$$\left[1-\frac{4}{S_0}\right]\frac{\dot{m}}{m}=0\ .$$

This fixes the value of the separation constant S_0 as 4. From (3.25) and (3.14) we can obtain h as a function of f and n as

$$h=\frac{4\frac{f'^2}{f^2}\pm\left\{4\frac{f'^2}{f^2}+4\frac{f'^2}{f^2}(n+1)\left[\frac{f^2}{C_0^2}+4(1-n)\frac{1}{f^2}\right]\right\}^{1/2}}{2\left[\frac{f^2}{C_0^2}+4(1-n)\frac{1}{f^2}\right]} \tag{3.27}$$

and

$$l=\frac{C_0}{fr}\,. \tag{3.28}$$

So far the only assumption we have made on the r-dependent part of the metric is (3.14). Thus, f and n are left arbitrary in the solution.

Time dependence of the metric will come from

$$2\dot{m}/g=4 \tag{3.29}$$

with an additional relation. Note that if we require $P_r(R,t)=P_\perp(R,t)=0$, then $\bar{C}_0$ in (3.23) is determined as 4.

Case II. In this case we allow the anisotropy to evolve with time, so that $k=k(t)$. Now Eq. (3.15) can be rearranged to give

$$-\frac{1}{h^2}\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}\right]+\frac{1}{l^2r^2}=-k\frac{n}{h^2}\left[\frac{l''}{l}+\frac{f''}{f}+\frac{l'f'}{lf}+\frac{2l'}{lr}+\frac{f'}{fr}\right]$$
$$+\left[\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}\right]\left[\frac{m^2}{g^2}\right]\left[\frac{nk}{f^2}-\frac{1}{f^2}\right]+\frac{2\dot{m}}{g}\left[\frac{f'}{f^2h}\right]. \tag{3.30}$$

We can arrange the separability of this equation by assuming

$$\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}=0 \tag{3.31}$$

and

$$-\frac{n}{h^2}\left[\frac{l''}{l}+\frac{f''}{f}+\frac{l'f'}{lf}+\frac{2l'}{lr}+\frac{f'}{fr}\right]=\frac{f'}{f^2h}\,. \tag{3.32}$$

With these Eq. (3.30) becomes

$$-\frac{1}{h^2}\left[\frac{l'^2}{l^2}+\frac{2l'f'}{lf}+\frac{2l'}{lr}+\frac{2f'}{fr}+\frac{1}{r^2}\right]+\frac{1}{l^2r^2}=\left[\frac{f'}{f^2h}\right]\left[k+\frac{2\dot{m}}{g}\right]. \tag{3.33}$$

Thus

$$k+\frac{2\dot{m}}{g}=S_0\,, \tag{3.34}$$

which gives the time dependence of the anisotropy, where S_0 is a separation constant. We could obtain k, l, and n in terms of f as

$$h(r)=\frac{S_0C_0^2f'\mp f'C_0(S_0^2C_0-4f^4)^{1/2}}{2f^4}\,, \tag{3.35}$$

$$l(r)=\frac{C_0}{fr}\,, \tag{3.36}$$

$$n(r)=\frac{-S_0C_0^2\pm C_0(S_0^2C_0-4f^4)^{1/2}}{2f^4}\,, \tag{3.37}$$

and

$$k(t)=S_0-\frac{2\dot{m}}{g}\,, \tag{3.38}$$

where $m(t)$ and $g(t)$ should come from an additional assumed relation, which is simultaneously solved with

$$\frac{2\ddot{m}}{m}+\frac{\dot{m}^2}{m^2}-\frac{2\dot{m}\dot{g}}{mg}=0\,. \tag{3.39}$$

IV. SLOWLY ROTATING ANISOTROPIC FLUID SPHERES WITH AND WITHOUT RADIATION

It is well known that the metric for slowly rotating fluid spheres could be given as[4,5]

$$ds^2 = f(r)^2 g(t)^2 dt^2 - h(r)^2 m(t)^2 dr^2 - l(r)^2 m(t)^2 (r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + 2l(r)^2 m(t)^2 \Omega(r,t) r^2 \sin^2\theta\, d\phi\, dt \; . \quad (4.1)$$

Here we took the metric separable for the sake of simplicity. To first order in $\Omega(r,t)$ only the following components of the Ricci tensor are nonzero:

$$R_2{}^2 = R_3{}^3, \;\; R_0{}^0, \;\; R_1{}^1, \;\; R_0{}^3, \text{ and } R_1{}^3 \; . \quad (4.2)$$

The first four components to the order we are considering do not involve $\Omega(r,t)$, and their solution gives $g_{ik}^{(0)}$, where

$$g_{ik} = g_{ik}^{(0)} + h_{ik} \; .$$

This determines the pressure and the density distributions, which are not perturbed to first order in Ω.[4,5] The remaining field equations which are solved for Ω, which represents the dragging of inertial frames,[4] are given as

$$R_{03} = -8\pi(T_{03} - \tfrac{1}{2} g_{03} T) \; , \quad (4.3)$$

$$R_{13} = -8\pi T_{13} \; . \quad (4.4)$$

We take the energy-momentum tensor to be the anisotropic fluid energy-momentum tensor plus the energy-momentum tensor for a radially expanding null fluid, which could be either for photons or neutrinos.[3] Hence,

$$T^{\mu\nu} = (\rho + P_\perp) U^\mu U^\nu - P_\perp g^{\mu\nu} + (P_r - P_\perp)\chi^\mu \chi^\nu + \sigma W^\mu W^\nu \; , \quad (4.5)$$

where

$$U^\mu U_\mu = 1, \;\; \chi^\mu \chi_\mu = -1, \;\; \chi^\mu U_\mu = 0, \;\; W^\mu W_\mu = 0 \; .$$

We choose

$$U^0 \neq 0, \;\; U^1 = U^2 = 0, \;\; U^3 = \frac{\omega}{fg} \; ,$$

where $\omega = d\phi/dt$,

$$\chi^0 = \chi^2 = \chi^3 = 0, \;\; \chi^1 \neq 0$$

and

$$W^0 = \frac{1}{fg}, \;\; W^1 \neq 0, \;\; W^3 = \frac{\omega}{fg} \; , \quad (4.6)$$

and σ represents the radiation density. With these (4.3) becomes

$$R_{03} = -8\pi[(\rho + P_\perp) l^2 m^2 r^2 \sin^2\theta(\Omega - \omega) + \tfrac{1}{2} l^2 m^2 r^2 \sin^2\theta\, \Omega (P_r - \rho) + \sigma W_0 W_3] \quad (4.7)$$

and

$$R_{13} = 0 \; , \quad (4.8)$$

where R_{03} and R_{13} are

$$R_{03} = -\sin^2\theta \left[\left[\frac{f'l'lr^2}{fh^2} + \frac{4l'lr}{h^2} + \frac{l'^2 r^2}{h^2} + \frac{f'l^2 r}{fh^2} + \frac{l^2}{h^2} - \frac{h'l'lr^2}{h^3} - \frac{h'l^2 r}{h^3} + \frac{ll''r^2}{h^2} + \frac{l^2 m\dot{m}\dot{g}r^2}{f^2 g^3} - \frac{2l^2 \dot{m}^2 r^2}{f^2 g^2} - \frac{l^2 \ddot{m} m r^2}{f^2 g^2} - 1 \right] \Omega + \left[-\frac{h'l^2 r^2}{2h^3} - \frac{1}{2}\frac{f'l^2 r^2}{fh^2} + \frac{2ll'r^2}{h^2} + \frac{2l^2 r}{h^2} \right] \Omega' + \frac{1}{2}\frac{l^2 r^2}{h^2}\Omega'' \right] , \quad (4.9)$$

$$R_{13} = -\tfrac{1}{2}\sin^2\theta \left[\frac{m^2 l^2 r^2}{f^2 g^2} \right] \left[\Omega' \left[\frac{3\dot{m}}{m} - \frac{\dot{g}}{g} \right] + \dot{\Omega}' \right] . \quad (4.10)$$

Finally the field equations to be solved for $\Omega(r,t)$ can be given as

$$\Omega'' + \left[-\frac{h'}{h} - \frac{f'}{f} + \frac{4l'}{l} + \frac{4}{r} \right] \Omega' = 16\pi h^2 m^2 (\rho + P_\perp + \sigma)(\Omega - \omega) \quad (4.11)$$

and

$$\Omega' \left[\frac{3\dot{m}}{m} - \frac{\dot{g}}{g} \right] + \dot{\Omega}' = 0 \; . \quad (4.12)$$

As seen, (4.12) will give the time-dependent part of

$\Omega(r,t)$ while (4.11) will determine the radial dependent part. Equation (4.12) can be solved immediately to give

$$\Omega(r,t)=C(r)\frac{g(t)}{m^3(t)} . \tag{4.13}$$

When (4.13) is substituted into (4.11) it will give a differential equation to be solved for $C(r)$ for a given rotation function $\omega(r)$.

For nonradiating and stationary solutions (4.12) is satisfied identically while (4.11) reduces to

$$\Omega''(r)+\left[-\frac{h'}{h}-\frac{f'}{f}+\frac{4l'}{l}+\frac{4}{r}\right]\Omega'(r)$$
$$=16\pi h^2(\rho+P_\perp)[\Omega(r)-\omega(r)], \tag{4.14}$$

where we have set $m=g=1$, $\sigma=0$. Now we will consider the solution of (4.14) for the anisotropic solution given in Sec. II as

$$A=C_1 r^{2\alpha/(\alpha+1)} , \tag{4.15}$$

$$B=B_0 , \tag{4.16}$$

$$8\pi\rho=\frac{1}{r^2}\left[1-\frac{1}{B_0{}^2}\right] , \tag{4.17}$$

$$8\pi P_\perp=\frac{1}{B_0{}^2}\left[\frac{4\alpha^2}{(\alpha+1)^2}\right]\frac{1}{r^2} , \tag{4.18}$$

$$8\pi P_r=\left[\frac{1}{B_0{}^2}\left[\frac{5\alpha+1}{\alpha+1}\right]-1\right]\frac{1}{r^2} . \tag{4.19}$$

Equation (4.14) in Schwarzschild canonical coordinates would be given as

$$\left[\frac{1}{r}-\frac{B'}{4B}-\frac{A'}{4A}\right]\Omega'(r)+\tfrac{1}{4}\Omega''(r)$$
$$=4\pi B^2(\rho+P_\perp)[\Omega(r)-\omega(r)] , \tag{4.20}$$

where the metric is given as in (2.1). For the above solution (4.20) becomes

$$r^2\Omega''+\left[\frac{4+2\alpha}{1+\alpha}\right]r\Omega'-8\left[\frac{\alpha}{\alpha+1}+\frac{g_0}{4}\right]\Omega$$
$$=8\left[\frac{\alpha}{\alpha+1}+\frac{g_0}{4}\right]\omega(r) . \tag{4.21}$$

For uniform rotation we take $\omega(r)=\omega_0$ (constant) and the solution of (4.21) becomes

$$\Omega(r)=\omega_0+\frac{1}{r^k}\left[\frac{C_0}{(1-A_1)}r^{A_1-1}+C_1\right] , \tag{4.22}$$

where

$$k^2-\left[\frac{\alpha+3}{\alpha+1}\right]k-8\left[\frac{\alpha}{\alpha+1}+\frac{g_0}{4}\right]=0 \tag{4.23}$$

and

$$A_1=2k-\frac{2}{1+\alpha}\neq 1 . \tag{4.24}$$

For

$$A_1=1,\quad \Omega(r)=\omega_0+\frac{1}{r^k}(C_0\ln r+C_1) ,$$

where $k=\frac{1}{2}+1/(1+\alpha)$, and

$$\alpha^2(\tfrac{33}{4}-2g_0)+\alpha(\tfrac{19}{2}+4g_0)+(\tfrac{9}{4}+2g_0)=0 .$$

Finally, we will present a solution to (4.21) corresponding to differential rotation as

$$\Omega(r)=C_1 r^a+C_2 r^{a+1}+e^r r^a , \tag{4.25}$$

where

$$\omega(r)=\frac{1}{8}\left[\frac{\alpha}{\alpha+1}+\frac{g_0}{4}\right]^{-1} e^r r^{a+2} \tag{4.26}$$

and $\alpha=(-2a-4)/(2a+2)$ so for $\alpha>0$ we need $-2<a<-1$, and

$$g_0=4\left[a(a+1)+\frac{8\alpha}{\alpha+1}\right] .$$

V. BOUNDARY CONDITIONS

We have discussed the boundary conditions relevant to each section in my previous papers in detail.[1–5] However, for the sake of completeness we will give the exterior metrics for each case we discussed. The boundary of the star is defined by

$P_r(R,t)=0$, and as we have mentioned because we are in comoving coordinates, R should be independent of time. At the boundary, static anisotropic fluid spheres have to be matched with the Schwarzschild exterior solution, which is given as

$$ds^2=\left[1-\frac{2M}{r}\right]dt^2-\frac{1}{\left[1-\frac{2M}{r}\right]}dr^2-r^2d\theta^2$$
$$-r^2\sin^2\theta\, d\phi^2\ , \qquad (5.1)$$

where $r \geq R$. Metrics corresponding to the radiating anisotropic fluid spheres have to be matched with Vaidya's[2,3] radiating solution, which is given as

$$ds^2=\left[1-\frac{2M}{r'}\right]dt^2-2\left[\frac{2M}{r'}\right]dt\,dr'$$
$$-\left[1+\frac{2M}{r'}\right]dr'^2-r'^2(d\theta^2+\sin^2\theta\, d\phi^2)\ , \qquad (5.2)$$

where $r \geq R$. Of course we have to make a change in the radial marker of our metric (3.7) to put it into the same form with (5.2) as[2]

$$r'=l(r)m(t)r\ . \qquad (5.3)$$

In this metric M is a function of the retarded time $t-r'$.

For slowly rotating and radiating anisotropic fluid spheres the exterior solution is simply the solution given by Murenbeeld and Trollope[21,5] which is

$$ds^2=\frac{\bar{r}^2\gamma^2}{4m^2a^2}\left[1-\frac{2m}{\bar{r}}\right]dt^2+2d\bar{r}\,dt\left[\frac{\gamma}{a}+\left[1-\frac{4m^2}{\bar{r}^2}\right]\frac{\bar{r}^2\gamma}{4m^2a}\right]$$
$$-d\bar{r}^2\left[-1+\frac{2m}{\bar{r}}-\frac{\bar{r}^2}{4m^2}\left[1-\frac{2m}{\bar{r}}\right]\left[\frac{2m}{\bar{r}}+1\right]^2\right]-\bar{r}^2d\theta^2-\bar{r}^2\sin^2\theta\, d\phi^2+2\sin^2\theta\gamma\, d\phi\, dt\ , \qquad (5.4)$$

where a is a constant, m is a function of retarded time $\bar{t}-\bar{r}$, and γ is an integrating factor obtained from,[5]

$$\gamma(\bar{r},\bar{t})dt=\frac{2ma}{\bar{r}}d\bar{t}-\left[\frac{2ma}{\bar{r}}+a\right]d\bar{r}\ . \qquad (5.5)$$

Again the interior metric we have (4.1) can be put into the form (5.4) most easily by the transformation

$$\bar{r}=l(r)m(t)r\ . \qquad (5.6)$$

Finally, for the slowly rotating stationary anisotropic fluids the exterior metric is the Kerr metric to first order in angular velocity, which is given as

$$ds^2=\left[1+\frac{m}{2r}\right]^4(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2)$$
$$-\frac{(r-m/2)^2}{(r+m/2)^2}dt^2+\frac{4ma\sin^2\theta}{r(1+m/2r)^2}d\phi\, dt\ . \qquad (5.7)$$

VI. CONCLUSIONS

We have discussed that anisotropy in stars could appear in a number of ways. One of these is the possibility of a star being composed of two perfect fluids, with the energy-momentum tensor given as in (1.1). We have said that such a case could actually be realized in neutron stars. Also, with small differences in the radial component of the velocities we could in principle introduce a significant amount of anisotropy.

In Sec. II we considered various generalizations of the $P=\alpha\rho$ solution to the anisotropic case. For the models we have discussed a small amount of anisotropy affects the mass by a small amount. We have also shown that through anisotropy we could approach the $2M/R\to 1$ limit as much as we want. This implies that surface red-shift could be as large as possible. We have also seen that the core mass for one of the solutions we have considered could be increased roughly by 300% with respect to the isotropic case, if we allow large amounts of anisotropy.[22,23]

It seems that there is quite a lot to be done on anisotropic fluid spheres in general relativity. Even though it seems possible to affect the mass limit and the surface red-shift significantly through anisotropy, first is has to be established physically how much anisotropy we could generate in neutron stars and how long this anisotropy would last. Also stability is another problem that should be investigated in detail. The models we considered correspond to equilibrium.

In the next section we considered radiating anisotropic fluid spheres and constructed several classes of analytic solutions. We would like to point out that these solutions are obtained through mathematical assumptions, which do not have any *a priori* physical basis. They just allow us to solve the differential equations. At the end one has to check the physical parameters for physical reasonableness. We also considered slowly rotating anisotropic fluid spheres and gave two analytic solutions for the nonradiating, stationary case. One of these solutions corresponds to uniform rotation while the other corresponds to differential rotation. We have also given the differential equations to be solved for slowly rotating and radiating anisotropic fluid spheres.

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