Class of solutions of Einstein's field equations for static fluid spheres

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In this paper we solve the field equations of general relativity for a static, spherically symmetric material distribution and present a class of new analytic solutions describing perfect fluid spheres. In general, the pressure and density diverge at the center, while their ratio remains finite. Each solution has a maximum mass which is less than $(\sqrt{2}-1)/(2\sqrt{2}-1)$ times the radius of the sphere. The solution is a generalization of Tolman's I, IV, and V solutions and the de Sitter solution. As a special case, another class of new analytic solutions is derived which has an equation of state. The existence of a class of solutions describing gaseous distributions has also been established.

I. INTRODUCTION

After the formulation of the general theory of relativity the first methodical attempt to solve the field equations for the description of the interior of material distributions was made by Tolman.[1] Tolman considered the case of static spherically symmetric distributions of fluid and showed that in curvature coordinates the problem consists in solving three field equations in four unknown variables, that is, two field variables and two physical variables, viz., the density and the pressure of the distribution. In this way one is always free to choose an additional relation between the field variables or the physical parameters to get a solution of the field equations. Tolman succeeded in obtaining a single differential equation in the two field variables, and by considering eight different relations between the field variables he derived eight exact solutions out of which five were not known by then. Another notable success was achieved by Wyman[2] who generalized Tolman's V, VI, and VIII solutions separately. Apart from this, by simplifying Tolman's differential equation and assuming various integrability conditions some authors[3–9] were able to give new analytic solutions. Successful attempts[9–11] have been made to solve the field equations in isotropic coordinates as well. But the physical conditions for the existence of real material distributions always put a limit to such attempts.

Another set of attempts consists in solving the field equations by assuming a relation between the physical variables, viz., an equation of state describing the physical nature of the distribution under consideration. Tolman observed that this method is almost unworkable because of the complicated nonlinear nature of the expressions involved. However, some authors have tried successfully to integrate Tolman's equation numerically by assuming realistic equations of state corresponding to known fluid distributions. Oppenheimer and Volkoff[12] considered the equation of state for a cold Fermi gas and obtained a solution describing equilibrium configurations of neutronic matter. Tooper[13] integrated Tolman's equation numerically for equilibrium configurations of matter which everywhere follows a γ law of equation of state. The equation has also been solved[14] to discuss the stability of the equilibrium configurations of cold catalyzed matter which follows the Harrison-Wheeler equation of state.

In spite of all such successful efforts, the importance of physically meaningful analytic solutions remains. In the present paper a fresh attempt has been made to explore the possibility of the existence of new analytic solutions which may be used in describing physical situations.

II. FIELD EQUATIONS AND METHOD OF OBTAINING ANALYTIC SOLUTIONS

We consider the spherically symmetric form of the line element in curvature coordinates

$$ds^2=-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+e^{\nu}dt^2, \tag{2.1}$$

where the field variables λ and ν are functions of r alone. The field equations of general relativity are (in geometric units $G=c=1$)

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=-8\pi T_{ij}. \tag{2.2}$$

The energy-momentum tensor for a perfect fluid distribution is defined by

$$T_{ij}=(\rho+p)U_iU_j-pg_{ij}, \tag{2.3}$$

ρ being the density, p the pressure of the distribution, and U_i is the velocity four-vector satisfying the relation

$$g_{ij}U^iU^j=1.$$

In comoving coordinates U^i is given by

$$U_i=(0,0,0,e^{\nu/2}).$$

We find that, for the metric form (2.1), the field equations reduce to the following:

$$8\pi p=e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}, \tag{2.4a}$$

$$8\pi p=e^{-\lambda}\left[\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right], \tag{2.4b}$$

$$8\pi\rho=e^{-\lambda}\left[\frac{\lambda'}{r}-\frac{1}{r^2}\right]+\frac{1}{r^2}, \tag{2.4c}$$

where a prime denotes differentiation with respect to r. These equations constitute the physical problem of determining a mechanical equilibrium of a spherically symmetric distribution of matter. Within the distribution, the density and pressure satisfy the conditions $\rho > 0$, $p > 0$, and $p' < 0$. The last condition on the pressure gradient signifies the hydrostatic equilibrium. The boundary of the distribution is given by the hypersurface $r=r_0$ where the pressure assumes the value zero. In the empty space surrounding the fluid sphere, the field is described by Schwarzschild's exterior solution so that on the boundary the two solutions must match.

The condition that g_{44} must not be negative anywhere within the space occupied by matter, on the boundary, or in the empty space puts a limitation upon the maximum value of the mass-radius ratio. For the external field of a fluid sphere, $g_{44}=1-2m_0/r_0$ where m_0 is the mass and r_0 is the radius of the sphere as measured by an external observer. Thus the value of m_0/r_0 must not exceed $\frac{1}{2}$.

In the above we have three equations to be solved for four unknown quantities, so we can always choose one additional relation connecting the latter. Of course, the choice of the additional relation is restricted in such a way as to lead to a solution describing physical situations only. From Eqs. (2.4a) and (2.4b) Tolman obtained the following differential equation in ν and λ:

$$\frac{d}{dr}\left[\frac{e^{-\lambda}-1}{r^2}\right]+\frac{d}{dr}\left[\frac{e^{-\lambda}\nu'}{2r}\right]+e^{-\lambda-\nu}\frac{d}{dr}\left[\frac{e^{\nu}\nu'}{2r}\right]=0. \tag{2.5}$$

It is interesting to note that this equation can be integrated easily if the following method is adopted. By the substitution

$$\frac{\nu'}{2r}=u,\quad e^{-\lambda}=v \tag{2.6}$$

the above equation assumes the form

$$\frac{dv}{dr}+2r^2\frac{(du/dr+u^2r-1/r^3)v}{1+ur^2}=-\frac{2}{r(1+ur^2)} \tag{2.7}$$

which is a linear equation of the first order. On integration (2.7) yields

$$v\exp\left[\int\frac{2r^2(du/dr+u^2r-1/r^3)}{1+ur^2}dr\right]=\int-\frac{2}{r(1+ur^2)}\exp\left[\int 2r^2\frac{(du/dr+u^2r-1/r^3)}{1+ur^2}dr\right]dr+A, \tag{2.8}$$

where A is a constant of integration. By introducing an *ad hoc* relation between u and r in such a form as to make the right-hand side of (2.8) integrable, the above equation may readily admit a solution. In the present case we assume

$$\exp\left[\int 2r^2\frac{(du/dr+u^2r-1/r^3}{1+ur^2}dr\right]=r^a(1+ur^2), \tag{2.9}$$

where a is a constant. On integration this gives

$$u=\frac{1}{r^2}\left[\frac{c-b-(c+b)kr^{4b}}{1-kr^{4b}}\right] \tag{2.10}$$

with

$$b=\frac{(a^2+16a+32)^{1/2}}{4}, \quad c=\frac{a+4}{4}, \tag{2.11}$$

k being another constant of integration. From (2.6), (2.8), (2.9), and (2.10) we find the final form of the solution as follows:

$$e^{\lambda}=\frac{1+c-b-(1+c+b)kr^{4b}}{(1-kr^{4b})(A/r^a-2/a)}, \tag{2.12a}$$

$$e^{\nu}=B^2r^{2(c-b)}(1-kr^{4b}), \tag{2.12b}$$

B being the third constant and $a\neq 0$. Clearly, (2.12a) and (2.12b) represent a one-parameter (a) family of solutions with the requisite number of arbitrary constants, to be determined with the help of the usual boundary conditions. For $a=0$ (2.6) and (2.8)–(2.10) yield the following solution:

$$e^{\lambda}=\frac{2-\sqrt{2}-(2+\sqrt{2})kr^{4\sqrt{2}}}{(A-2\ln r)(1-kr^{4\sqrt{2}})},$$
$$e^{\nu}=B^2r^{2(1-\sqrt{2})}(1-kr^{4\sqrt{2}}). \tag{2.13}$$

III. PROPERTIES OF THE SOLUTION (2.12)

With the help of Eqs. (2.4a), (2.4c), and (2.12) we can write the expressions for p and ρ as follows:

$$8\pi p=\frac{1}{r^2}\left[\frac{A}{r^a}-\frac{2}{a}\right]\left[\frac{1+2c-2b-(1+2c+2b)kr^{4b}}{1+c-b-(1+c+b)kr^{4b}}\right]-\frac{1}{r^2}, \tag{3.1}$$

$$8\pi\rho=\frac{1}{r^2}\left[\frac{-4b(1-kr^{4b})(A/r^a-2/a)(1+c+b)kr^{4b}}{(1+c-b-(1+c+b)kr^{4b})^2}\right.$$
$$\left.+\frac{2/a+(a-1)A/r^a-(2/a)(1+4b)kr^{4b}+Ak(1+4b-a)r^{4b-a}}{1+c-b-(1+c+b)kr^{4b}}+1\right]. \tag{3.2}$$

It is found that the central values p_c and ρ_c of the pressure and density are positively infinite for $-8+4\sqrt{2}\leq a<-2$, if we assume k positive. For $a=-2$ our solution reduces to Tolman's IV solution with positive, finite p_c and ρ_c. For $a=-8+4\sqrt{2}$, the solution becomes a particular case of Tolman's V solution. The newly found solution leads to Tolman's I solution (Einstein's cosmological solution) for $k=0$ and $a=-2$. With $a=-2$ and $A=-2k$, the solution degenerates into the de Sitter solution. In case we put $k=0$ we rediscover Tolman's V solution. Thus our solution is a generalization of the four previously known solutions, viz., the de Sitter solution and Tolman's I, IV, and V solutions.

Evidently the solution does not exist for any value of a given by $-8-4\sqrt{2}<a<-8+4\sqrt{2}$. For values of a without $-8-4\sqrt{2}<a\leq -2$ the solution, though it exists, does not behave physically at the center; either pressure or density becomes negative at the center. However, if one succeeds in finding an internal boundary r_i such that $\rho(r)$ and $p(r)$ are positive functions in the region $r_i\leq r\leq r_0$, the solution may be used in representing portions of stellar distribution. In this respect, a brief analysis of the solution will be presented in Sec. V.

For $-2>a>-8+4\sqrt{2}$, we find from (3.1) and (3.2) the asymptotic form of the equation of state in the central region as follows:

$$\left[\frac{p}{\rho}\right]_{r\to 0}=\frac{2+4(c-b)+a(1+c-b)}{-2-a(1+c-b)}. \tag{3.3}$$

In view of (3.3), it is evident that for all values of a in the range $-2>a>-8+4\sqrt{2}$, the following condition is satisfied at the center:

$$\frac{p}{\rho}<\tfrac{1}{3}. \tag{3.4a}$$

Also

$$\left[\frac{dp}{d\rho}\right]_{r\to 0}=\frac{2+4(c-b)+a(1+c-b)}{-2-a(1+c-b)}. \tag{3.5}$$

Thus we find that for values of a under consideration, at the center

$$\left[\frac{dp}{d\rho}\right] < 1. \quad (3.4b)$$

It is to be noted that generally (3.4b) is considered to signify that the speed of sound does not exceed the speed of light. However, it has been shown by Caporaso and Brecher[15] that $(dp/d\rho)^{1/2}$ represents the adiabatic sound speed and not the propagation speed so that even the violation of (3.4b) may not necessarily imply that the fluid is noncausal.

Every fluid distribution described by the family of solutions (2.12) has a boundary $r=r_0$ which is defined by

$$p(r_0)=0. \quad (3.6)$$

The integration constant A can be evaluated by using the condition (3.6). The remaining two constants will be determined with the help of the following boundary conditions:

$$e^{\lambda(r_0)}=\left[1-\frac{2m_0}{r_0}\right]^{-1}, \quad e^{\nu(r_0)}=1-\frac{2m_0}{r_0}. \quad (3.7)$$

We find from (2.12), (3.1), (3.6), and (3.7)

$$A=r_0^{\,a}\left[\frac{2}{a}+\frac{1+c-b-(1+c+b)kr_0^{4b}}{1+2c-2b-(1+2c+2b)kr_0^{4b}}\right], \quad (3.8a)$$

$$B^2=r_0^{\,2(b-c)}\frac{(1-2\epsilon)}{(1-kr_0^{4b})}, \quad (3.8b)$$

where

$$kr_0^{4b}=\frac{c-b-\epsilon(1+2c-2b)}{c+b-\epsilon(1+2c+2b)} \quad (3.9)$$

with

$$\epsilon=\frac{m_0}{r_0}. \quad (3.10)$$

In order that the expressions for e^{λ} and e^{ν} as given by (2.12) be continuous non-negative functions of r in the region $0\leq r\leq r_0$, so as to maintain the signature of the line element (2.1), one must have

$$0<kr_0^{4b}<\min\left\{1,\frac{1+c-b}{1+c+b},\frac{1+2c-2b}{1+2c+2b}\right\}, \quad (3.11)$$

since $k>0$. Consequently (3.9) requires

$$\epsilon<\min\left\{\frac{c+b}{1+2c+2b},\frac{c-b}{1+2c-2b}\right\} \quad (3.12)$$

or

$$\epsilon>\max\left\{\frac{c+b}{1+2c+2b},\frac{c-b}{1+2c-2b}\right\}. \quad (3.13)$$

It can be seen easily that the sets of conditions (3.11) and (3.12) are consistent. The condition (3.12) delimits the maximum value of the mass-radius ratio. For $-8+4\sqrt{2}<a<-2$, this condition reduces to

$$\epsilon<\frac{c-b}{1+2c-2b}. \quad (3.14)$$

Thus each solution corresponds to a least upper bound for the mass-radius ratio. Its value increases as b decreases and peaks for a minimum positive b, keeping well below the physical limit, that is, $\frac{1}{2}$. Precisely the maximum value for the mass-radius ratio will be less than $(\sqrt{2}-1)/(2\sqrt{2}-1)$. It is to be noted that the above considerations do not hold for $a=-2$ and $a=-8+4\sqrt{2}$ which, as such, correspond to Tolman's IV and V solutions, respectively.

For a number of values of a between -2 and $-8+4\sqrt{2}$, we have computed $dp/d\rho$ and ρ at the boundary. In each case $(dp/d\rho)_{r=r_0}$ is found to be positive and less than one and $\rho\ (r_0)$ positive. For the region $0<r<r_0$, the condition (3.4a) leads to the inequality

$$\left[\frac{A}{r^a}-\frac{2}{a}\right]\{kr^{4b}(1-kr^{4b})[4-a+10b+10c+6(b+c)^2-a(b+c)]$$
$$+(kr^{4b}-1)[4-a+10(c-b)-32b^2+6(c-b)^2-a(c-b)]-32b^2kr^{4b}\}$$
$$+2k^2r^{8b}(1+b+c)(3+2b+2c)-4kr^{4b}(3+5c-2b^2+2c^2)$$
$$+2[3+5(c-b)+2(c-b)^2]>0. \quad (3.15)$$

It is not easy to see the above condition satisfied simply by inspection. However, from the following observations we can work out a method. In view of (3.11), $0 \leq kr^{4b} \leq k\, r_0{}^{4b}$. Further, it is evident from (3.8a) and (3.11) that $A/r_0{}^a - 2/a$ is positive. Since a is negative, $0 \leq |A|/r^a \leq |A|/r_0{}^a$. Consequently, $A/r^a - 2/a$ will always be positive. Now by considering various values of a between -2 and $-8+4\sqrt{2}$, one can check the validity of (3.15) within the distribution. One can also verify that for a permissible value of a, $0 < dp/d\rho < 1$ within the distribution.

From (2.12) it is apparent that, in general, g_{44} goes over to zero at the center. Solutions of this type have been referred to by Tolman[1] as having quasistatic character, for in such cases changes occurring at the center would appear very slow to an external observer.

It is to be noted that if we put $A=0$ in (2.12) another family of new analytic solutions is obtained which, like the general solution, is physically meaningful only for $-8+4\sqrt{2} \leq a \leq -2$. Since the number of arbitrary constants in this case is one less than the requisite number, it will amount to putting a condition on ϵ. However, the solution has some interesting features which we shall discuss in the next section.

IV. A SPECIAL CASE OF (2.12)

Considering $A=0$, the solution (2.12) takes the form

$$e^{\lambda} = -\frac{a}{2}\,\frac{1+c-b-(1+c+b)kr^{4b}}{1-kr^{4b}}, \qquad e^{\nu} = B^2 r^{2(c-b)}(1-kr^{4b}). \tag{4.1}$$

Substituting these in Eqs. (2.4a) and (2.4c) we find the expressions for pressure and density as follows:

$$8\pi p = -\frac{2}{ar^2}\left[\frac{1+2c-2b-(1+2c+2b)kr^{4b}}{1+c-b-(1+c+b)kr^{4b}}\right] - \frac{1}{r^2}, \tag{4.2}$$

$$8\pi\rho = \frac{1}{r^2}\left[1+\frac{8b}{a}\,\frac{kr^{4b}(1+c+b)(1-kr^{4b})}{[1+c-b-(1+c+b)kr^{4b}]^2} + \frac{2}{a}\,\frac{1-(1+4b)kr^{4b}}{1+c-b-(1+c+b)kr^{4b}}\right]. \tag{4.3}$$

It can be easily seen that, in general, these expressions for pressure and density become singular at the center. However, it is to be noted that p_c and ρ_c are positive for values of a which lie between -2 and $-8+4\sqrt{2}$. For these values of a, the ratio p_c to ρ_c is given by (3.3) and satisfies the inequality (3.4a). Also the expression for $(dp/d\rho)_{r\to 0}$ is given by (3.5) so that at the center (3.4b) is satisfied in the present case also.

As in the case of the general solution (2.12), the solution (4.1) does not exist for $-8+4\sqrt{2} > a > -8-4\sqrt{2}$. For $a > -2$ $(a \neq 0)$ and $a < -8-4\sqrt{2}$, either pressure or density becomes negative at the center. In the case $a=-2$, the solution becomes a particular case of Tolman's IV solution, while for $a=-8+4\sqrt{2}$ it assumes a form which is included in Tolman's V and VI solutions.

Each fluid distribution, described by (4.1), corresponding to $-2 > a > -8+4\sqrt{2}$, has a boundary $r=r_0$ defined by (3.6); this condition is used in evaluating the constant k. Thus we find from (4.2) and (3.6)

$$k = \frac{1}{r_0{}^{4b}}\left[\frac{a(1+c-b)+2(1+2c-2b)}{a(1+c+b)+2(1+2c+2b)}\right]. \tag{4.4}$$

The constant B is evaluated by matching the solution with Schwarzschild's empty-field solution over the boundary. We thus have

$$B^2 = \frac{(1-2\epsilon)r_0{}^{2(b-c)}}{1-kr_0{}^{4b}}. \tag{4.5}$$

From (4.4) and (4.5) we obtain an expression for ϵ, the mass-radius ratio,

$$\epsilon = 1+\frac{2}{a}. \tag{4.6}$$

We note that for all values of a lying between -2 and $-8+4\sqrt{2}$, ϵ is less than $\frac{1}{2}$. It approaches its minimum value as a approaches -2 and maximum as a approaches $-8+4\sqrt{2}$. Using (4.3) and (4.4), the expression for the surface density can be written as follows:

$$\rho(r_0)=\frac{1}{8\pi r_0^2}\left[a+6+\frac{8}{a}\right]. \tag{4.7}$$

It is obvious that for $-2>a>-8+4\sqrt{2}$, $\rho(r_0)$ is positive. Also, in view of (4.4), it is evident that the density and pressure are positive everywhere within the distribution.

Eliminating r from (4.2) and (4.3), we find the equation of state in the implicit form

$$p=-\frac{1}{8\pi}\left[\frac{k}{M}\right]^{1/2b}\left[1+\frac{2}{a}\frac{1+2c-2b+(1+2c+2b)M}{1+c-b+(1+c+b)M}\right], \tag{4.8}$$

where

$$M=\frac{(4+\frac{3}{2}a)p-(a^2+\frac{11}{2}a+12)p\pm 2b[\rho^2-6(a+3)\rho p+(a^2+6a+17)p^2]^{1/2}}{(1+c+b)\{[2+a(1+c+b)]p+[2(1+2c+2b)+a(1+c+b)]\rho\}}. \tag{4.9}$$

For very small values of b, that is, for a approaching $-8+4\sqrt{2}$, the equation of state assumes the simple form

$$\rho=C_1 p\left[1+\frac{C_3}{p^{2b}-C_2}\right], \tag{4.10}$$

where C_1, C_2, and C_3 are constants given by

$$C_1=\frac{a^2+\frac{11}{2}a+12}{4+\frac{3}{2}a},$$

$$C_2=k\left[-\frac{1}{8\pi a}\right]^{2b}\frac{(1+c)^{1-2b}}{4+\frac{3}{2}a}[a(1+c)+2(1+2c)]^{1+2b}, \tag{4.11}$$

$$C_3=C_2+\left[-\frac{1}{8\pi a}\right]^{2b}k(1+c)^{1-2b}\left[\frac{2+a(1+c)}{a^2+\frac{11}{2}a+12}\right][a(1+c)+2(1+2c)]^{2b}.$$

V. SOLUTIONS DESCRIBING INCOMPLETE STELLAR STRUCTURE

Here we shall explore those values of a for which the solution (2.12) may be used to describe portions of stellar distribution if not the whole stellar structure. For this purpose, it is essential to investigate the nature of the following four functions of a which occur in the solution and in the expressions for density and pressure:

$$f_1(a)=1+c-b,\quad f_2(a)=1+c+b,$$

$$f_3(a)=1+2c-2b,\quad f_4(a)=1+2c+2b.$$

We note that the nature of these functions and hence the behavior of the solution (2.12) can be studied effectively, if we subdivide the entire real line (excluding zero) into the following six intervals:

I_1: $a\le -8-4\sqrt{2}$.
I_2: $-8-4\sqrt{2}< a<-8+4\sqrt{2}$.
I_3: $-8+4\sqrt{2}\le a\le -2$.
I_4: $-2<a<0$.
I_5: $1>a>0$.
I_6: $a\ge 1$.

Obviously, the above-mentioned four functions of a become complex in the interval I_2. Again these functions take positive values in I_3 and we have already discussed the behavior of the solution in this interval in Sec. III. In the rest of the intervals, we have the following observations to make.

I_1: All functions are negative.
I_4: All functions are positive.
I_5: All functions are positive.
I_6: f_1, f_2, f_4 are positive and f_3 is negative.

In view of the above we deduce from (3.8a) that A/r^a-$2/a$ will be positive in the region $0\le r\le r_0$ for $a<1$. This result is true for any real value of

k. Of course, when $k > 0$, (3.11) is to be satisfied.

The intervalwise behavior of the solution is as follows.

I_1: Since f_1 is negative, e^λ becomes negative.

I_4: p_c is positive, ρ_c is negative. In the case $k > 0$, so that (3.11) is to be satisfied, it is not possible to find r_i such that for $r_i \le r \le r_0$, $\rho(r)$ and $p(r)$ are positive functions. However, for $k<0$, (3.11) does not matter and we can always find r_i such that for $r_i \le r \le r_0$, $p(r)$ and $\rho(r)$ are positive.

I_5: Same as in I_4.

I_6: When A is positive we find that p_c is negative and ρ_c is positive for all finite real values of k. In this case it is not possible to find r_i such that for $r_i \le r \le r_0$ both $\rho(r)$ and $p(r)$ are positive. When A is negative, e^λ becomes negative, whatever the value of k is.

Thus we conclude that for $-2<a<0$ and $0<a<1$ the solution (2.12) may be used to describe stellar regions $r_i \le r \le r_0$ if $k<0$.

For $a=0$, the solution is (2.13). In view of (2.4a) and (2.4c), it has been found that p_c is positively infinite and ρ_c negatively infinite; however, it is easy to find r_i such that $\rho(r)$ will be positive for $r_i \le r \le r_0$.

VI. SOLUTIONS FOR STATIC GASEOUS SPHERES

For a solution to describe a spherical distribution of gas, in addition to the three boundary conditions (3.6) and (3.7), a fourth boundary condition is to be satisfied, viz., the density must also vanish on the boundary:

$$\rho(r_0)=0. \tag{6.1}$$

For the solution (2.12) this condition leads to the following quadratic equation in $k\, r_0^{4b}$:

$$k^2 r_0^{8b}[2+a(1+b+c)+6(b+c)+2(b+c)^2]-2kr_o^{4b}[2+a(1+c)+6c+2(b^2+c^2)] +2+a(1-b+c)+6(c-b)+2(c-b)^2=0. \tag{6.2}$$

It can easily be seen that for each value of a lying between $-8+4\sqrt{2}$ and -2, the above equation has at least one positive root which satisfies the condition (3.11). The value of $k\, r_0^{4b}$ so obtained, in view of the relation (3.9), corresponds to a particular value of ϵ. From (2.12), thus, we derive a one-parameter (a) family of solutions which for $-2>a>-8+4\sqrt{2}$ describes static spheres of perfect gas. Each solution corresponds to a particular value of the mass-radius ratio.

VII. CONCLUSION

We present a one-parameter (a) class of new analytic solutions of Einstein's field equations for spherically symmetric fluid distributions, which has three arbitrary constants. Some well-known solutions, viz., Tolman's I, IV, and V solutions and de Sitter's solution, are particular cases of the solution. For $-2>a>-8+4\sqrt{2}$, the pressure and density are positive functions throughout the distribution. At the center, the pressure and density diverge while their ratio remains finite and in all cases is found to be less than $\frac{1}{3}$. Throughout the distribution it is found that $dp/d\rho<1$. Each solution corresponds to a maximum mass which is less than $(\sqrt{2}-1)/(2\sqrt{2}-1)$ times the radius of the fluid sphere. The solution may be used in describing compact objects under extreme central conditions. For each solution, there exists a particular value of mass-radius ratio under which it describes a gaseous sphere. It has been found that for $-2<a<1$, the solution may be used in describing portions of the stellar structure. As a special case, from the newly found solution, another class of new solutions results which has two arbitrary constants. In this case each solution, for $-2>a>-8+4\sqrt{2}$, corresponds to a particular value for the mass-radius ratio and also for the surface mass. The solution has an equation of state which assumes a very simple form as a approaches $-8+4\sqrt{2}$ from the right.

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