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Gravitational effects upon cosmological phase transitions

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The effects of spacetime curvature upon phase transitions in an expanding universe are investigated. We consider a Robertson-Walker model which is a radiation-dominated universe at early times and becomes de Sitter space at later times. In this universe the stability of a field theory containing a pair of interacting scalar fields is studied in first-order perturbation theory. It is noted that the crucial quantity in the stability analysis is $\langle\phi^2\rangle$, where ϕ is a free scalar field. The behavior of $\langle\phi^2\rangle$ as a function of time is investigated, where both thermal and vacuum contributions are taken into account. It is shown that this behavior can be strongly affected by the coupling to the background gravitational field. Such coupling can cause $\langle\phi^2\rangle$ to decrease more slowly or even grow as the universe expands. This behavior can alter the evolution of the system and can result in either stabilization of an otherwise unstable field configuration or destabilization of an otherwise stable configuration.

I. INTRODUCTION

If one combines the standard hot big-bang theory of the origin of the Universe with gauge theories of elementary-particle interactions, one is led to the conclusion that the Universe has undergone a number of phase transitions in its earlier history.[1] Recently, particular attention has been given to grand unified models with Coleman-Weinberg symmetry breaking in which strongly first-order phase transitions occur.[2-8] Such a transition can occur by quantum tunneling through a barrier. Because the barrier penetration probability can be small, it is possible for the Universe to remain in a false vacuum state for some time after the temperature has dropped below the critical temperature T_c, resulting in supercooling.

The false vacuum state has a nonzero energy density ρ_v; when the temperature of the Universe falls below some value θ, ρ_v can become greater than the thermal energy density and begin to dominate the dynamics of the Universe. The Universe then enters a phase of exponential expansion in which the scale factor is approximately that of the de Sitter metric, $a(t)\propto e^{Ht}$, where

$$H=(8\pi G\rho_v/3)^{1/2} .$$

Guth[9] has proposed an "inflationary" universe model in which such exponential expansion is utilized to resolve the long-standing horizon and flatness problems of the standard big-bang cosmology; it requires that the Universe supercool by at least 28 orders of magnitude. The main difficulty with this model is that it is not easy to explain the termination of the de Sitter phase in a way which leads to the observed homogeneity of the present universe. (Some interesting recent suggestions can be found in Refs. 6–8.)

In most work on cosmological phase transitions, the coupling to the background gravitational field is ignored. One deals with quantum field theory in flat spacetime at finite temperature, and the expansion of the Universe serves only to decrease the temperature. However, at sufficiently early times the spacetime curvature can be expected to be important. The effects of spacetime structure upon a field theory can arise both from the coupling to the local curvature and from the global properties of spacetime (such as the presence of horizons). Abbott[3] and Hut and Klinkhamer[5] have argued that such effects may be important in the context of cosmological phase transitions in grand unified models.

The purpose of this paper is to investigate and illustrate gravitational effects in the context of a particular field-theoretic model which consists of a pair of interacting scalar fields. It is shown that the spacetime curvature can drastically change the behavior of the system and produce either stabilization or destabilization of the false vacuum. Some of these results have already been briefly reported in Ref. 10.

In Sec. II the field-theory model is discussed and the analysis of stability to lowest order in perturbation theory is outlined. The essential ingredient of the stability analysis is $\langle\phi^2\rangle$, the expectation value of the square of a free scalar field. In Sec. III, the Einstein equations are solved for a universe containing both radiation and the vacuum energy density ρ_v, and the transition from a radiation-dominated to a de Sitter universe is detailed. In Sec. IV, $\langle \phi^2 \rangle$ in de Sitter spacetime is investigated for various choices of the quantum state. In Sec. V, the results of Sec. IV are applied to the study of phase transitions in our field-theory model. Finally, in Sec. VI the results of the paper are summarized and discussed.

II. THE STABILITY CRITERION

Stability of a field theory in flat spacetime is usually analyzed using an effective potential $V(\Phi)$ calculated for a constant classical field Φ. Here we shall take a different approach which is more suited for essentially nonstatic situations, such as an expanding universe. The stability condition will be deduced from the dynamical field equations for the field operator ϕ. This approach is similar to that of Refs. 11–13.

We consider a model of two interacting real scalar fields ϕ and ψ described by the Lagrangian

$$\mathscr{L}=\mathscr{L}_\phi^{(0)}+\mathscr{L}_\psi^{(0)}-U(\phi,\psi)\ , \tag{2.1}$$

where

$$\begin{aligned}\mathscr{L}_\phi^{(0)}&=\tfrac{1}{2}[(\partial_\mu\phi)^2-m_\phi{}^2\phi^2-\xi_\phi R\phi^2]\ ,\\ \mathscr{L}_\psi^{(0)}&=\tfrac{1}{2}[(\partial_\mu\psi)^2-m_\psi{}^2\psi^2-\xi_\psi R\psi^2]\ ,\end{aligned} \tag{2.2}$$

where $-U(\phi,\psi)$ is the interaction Lagrangian, R is the scalar curvature, and ξ_ϕ,ξ_ψ are the conformal coupling parameters. The Heisenberg field operator ϕ satisfies the equation

$$(\Box+m_\phi{}^2+\xi_\phi R)\phi+U'_\phi(\phi,\psi)=0\ , \tag{2.3}$$

where $U'_\phi=\partial U/\partial\phi$. To investigate the stability of the symmetric phase, $\langle\phi\rangle=\langle\psi\rangle=0$, with respect to the appearance of a nonzero expectation value $\langle\phi(x)\rangle=\Phi(x)$, we study the behavior of small pertubations of $\Phi(x)$ near $\Phi=0$. Here and below the angular brackets mean quantum statistical averaging,

$$\langle\phi\rangle=\mathrm{Tr}(\rho\phi)\ , \tag{2.4}$$

where ρ is the density matrix. Taking the average of Eq. (2.3) and linearizing in Φ we obtain

$$(\Box+m_\phi{}^2+\xi_\phi R)\Phi(x)+\langle U'_\phi\rangle_{\Phi=0}+\langle U''_{\phi\phi}\rangle_{\Phi=0}\Phi(x)=0\ . \tag{2.5}$$

If $U(\phi,\psi)$ is invariant with respect to the transformation $\phi\to-\phi$, then $\langle U'_\phi\rangle_{\Phi=0}=0$ and Eq. (2.5) takes the form

$$(\Box+M^2)\Phi(x)=0\ , \tag{2.6}$$

where

$$M^2(x)\equiv m_\phi{}^2+\xi_\phi R+\langle U''_{\phi\phi}\rangle_{\Phi=0}\ . \tag{2.7}$$

In this paper we shall consider interactions of the form

$$U(\phi,\psi)=\tfrac{1}{2}g^2\phi^2\psi^2+\tfrac{1}{12}\lambda_\phi\phi^4+\tfrac{1}{12}\lambda_\psi\psi^4\ . \tag{2.8}$$

Then

$$M^2=m_\phi{}^2+\xi_\phi R+g^2\langle\psi^2\rangle+\lambda_\phi\langle\phi^2\rangle\ , \tag{2.9}$$

where the averages are taken at $\Phi=0$. The symmetric phase is unstable if Eq. (2.6) has solutions which grow in time and is stable otherwise. For a constant M, Eq. (2.6) has growing solutions in de Sitter spacetime if and only if $M^2<0$ (see Sec. IV). For a time-dependent M^2 the solutions are not known in general. However, it seems reasonable to assume that if, in the course of the system's evolution, M^2 becomes sufficiently negative, an instability will arise. Destabilization of field theories by the effects of spacetime curvature and topology has been discussed for static spacetimes in the papers cited in Ref. 14 and for time-dependent situations in Ref. 13.

We will discuss only models for which it is sufficient to calculate $\langle\phi^2\rangle$ and $\langle\psi^2\rangle$ to the lowest order in perturbation theory, that is, treating ϕ and ψ as free fields. For example, set $m_\phi=m_\psi=0$ and take

$$U(\phi,\psi)=\tfrac{1}{2}g^2\phi^2\psi^2-\tfrac{1}{12}\lambda\phi^4 \tag{2.10}$$

with $g^2>0$, $\lambda>0$. A necessary condition that higher-order terms in λ and g^2 be negligible is that $g\gg\lambda\gg g^4$. The interaction potential (2.10) does not have a minimum but one can still investigate the quasistability of the symmetric phase $\Phi=0$. (By quasistability we mean stability against small perturbations, but not necessarily quantum barrier penetration.) To make a connection with more realistic models, one can asume that Eq. (2.10) is an approximation which breaks down for sufficiently large values of Φ. [We note that for small Φ the Coleman-Weinberg effective potential can be approximated[6] as $V(\Phi)=AT^2\Phi^2-B(T)\Phi^4$, where T is the temperature and $A,B>0$. The interaction (2.10) gives an effective potential of the same form.]

In flat spacetime the finite-temperature expectation values of ϕ^2 and ψ^2 are[11]

$$\langle\phi^2\rangle = \langle\psi^2\rangle = \frac{T^2}{12} , \tag{2.11}$$

and thus

$$M^2 = \tfrac{1}{12}(g^2-\lambda)T^2 . \tag{2.12}$$

The symmetric phase is quasistable at all temperatures if $g^2 > \lambda$ and is unstable if $g^2 < \lambda$. Naively, one would expect that the only difference in de Sitter space is the presence of the curvature-dependent term $\xi_\phi R$ and thus the flat-spacetime conclusions are unaltered for $\xi_\phi = 0$.

However, it will be shown in Sec. IV that in de Sitter space the evolution of $\langle\phi^2\rangle$ and $\langle\psi^2\rangle$ can be very different from that described by Eq. (2.11); as a result, the behavior of the system can strikingly differ from that in flat spacetime.

III. SOLUTION OF EINSTEIN'S EQUATIONS

Let us consider a spatially flat Robertson-Walker metric,

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 . \tag{3.1}$$

Suppose that the energy-momentum tensor of the matter in the Universe is of the form

$$T_{\mu\nu} = T_{\mu\nu}^{(\mathrm{rad})} + \Lambda g_{\mu\nu} , \tag{3.2}$$

where $T_{\mu\nu}^{(\mathrm{rad})}$ is the (trace-free) energy-momentum tensor for radiation and Λ is a constant. The Einstein evolution equation is

$$\left[a^{-1}\frac{da}{dt}\right]^2 = 8\frac{\pi}{3}\rho . \tag{3.3}$$

There, ρ has the form

$$\rho = \frac{3}{8\pi}\left[\frac{A}{a^4}+B\right] , \tag{3.4}$$

where A and B are constants. A solution of Eqs. (3.3) and (3.4) is

$$a(t) = (A/B)^{1/4}[\sinh(2B^{1/2}t)]^{1/2} , \tag{3.5}$$

which has the following asymptotic forms:

$$a(t) \sim \sqrt{2}A^{1/4}t^{1/2}, \quad t \ll \tfrac{1}{2}B^{-1/2} \tag{3.6}$$

and

$$a(t) \sim 2^{-1/2}(A/B)^{1/4}e^{B^{1/2}t}, \quad t \gg \tfrac{1}{2}B^{-1/2} . \tag{3.7}$$

The former is the metric for a radiation-dominated Robertson-Walker universe; the latter is that for de Sitter spacetime. Thus Eq. (3.5) represents a universe which makes a smooth transition between the two regimes on a time scale of the order of $H^{-1} \equiv B^{-1/2}$.

Note that Eq. (3.1) with $a \propto e^{Ht}$ is a metric which covers one-half of the full de Sitter spacetime.[15] This is not relevant for our purposes. The spacetime with which we are concerned is only locally de Sitter space and is entirely covered by the metric of Eqs. (3.1) and (3.5).

In the de Sitter region we can take $a(t) = e^{Ht}$ (a constant rescaling of a is unimportant), or

$$a(\eta) = -(H\eta)^{-1} , \tag{3.8}$$

where

$$\eta = \int a^{-1}(t)dt = -H^{-1}e^{-Ht}$$

and

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2) .$$

The conformal time η is negative and approaches zero as $t \to \infty$. The scalar curvature for the metric Eq. (3.8) is a constant:

$$R = 12H^2 . \tag{3.9}$$

Let θ be the temperature of the Universe at the beginning of the de Sitter phase. At this time the thermal energy density and vacuum energy density are approximately equal, so $\theta^4 \approx H^2$. This occurs at a time much larger than the Planck time t_P (of the order of $10^8\, t_P$ in grand unified models). Consequently $\theta \ll 1$ (Planck units) and $H \ll \theta$. For the purposes of this paper, H will be treated as a free parameter which is unrelated to the parameters appearing in the field theory.

IV. THE BEHAVIOR OF $\langle\phi^2\rangle$

A. The vacuum contribution

In this section we wish to investigate the behavior of $\langle\phi^2\rangle$ in de Sitter spacetime for various choices of the quantum state. Quantum field theory in de Sitter spacetime has been discussed by several authors from various viewpoints.[16–18] The approach which we adopt is most similar to that of Bunch and Davies,[18] several of whose results we will utilize. A free scalar field operator which satisfies

$$\Box\phi + m^2\phi + \xi R\phi = 0 \tag{4.1}$$

may be expanded in creation and annihilation operators, $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$, as

$$\phi=(2\pi)^{-3/2}\int d^3k[a_{\vec{k}}\psi_k(\eta)e^{i\vec{k}\cdot\vec{x}}+\text{H.c.}] . \tag{4.2}$$

The functions $\psi_k(\eta)$, for the metric Eq. (3.8), take the form

$$\psi_k(\eta)=(\pi/4)^{1/2}H\eta^{3/2}[c_1H_\nu^{(1)}(k\eta)+c_2H_\nu^{(2)}(k\eta)] , \tag{4.3}$$

where $|c_2|^2-|c_1|^2=1$ and

$$\nu^2=\tfrac{9}{4}-12(m^2/R+\xi) . \tag{4.4}$$

The formal expectation value of ϕ^2 in the vacuum state defined by the mode functions Eq. (4.3) is

$$\langle\phi^2\rangle=(2\pi^2)^{-1}\int_0^\infty dk\, k^2|\psi_k|^2 . \tag{4.5}$$

Different choices of the constants c_1 and c_2 can be interpreted as different choices for the vacuum state of the quantum field theory. The choice $c_1=0$, $c_2=1$ is of particular interest because it leads to a de Sitter-invariant state. It is this state which was treated by Bunch and Davies and will here be referred to as the Bunch-Davies vacuum. The formal expectation value of ϕ^2 in this state may be regularized by point separation.

The regularized expectation value can be expressed as[18]

$$\langle\phi^2\rangle_{\text{reg}}=-(16\pi^2\epsilon^2)+R/576\pi^2+(16\pi^2)^{-1}[m^2+(\xi-\tfrac{1}{6})R][\ln(\tfrac{1}{12}\epsilon^2\mu^2)+\ln(R/\mu^2)+2\gamma-1+\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)] , \tag{4.6}$$

where γ=Euler's constant, μ is an arbitrary mass, $\psi(x)=\Gamma'(x)/\Gamma(x)$, and ϵ is the regularization parameter (the point separation). The regularization is removed in the limit that $\epsilon\rightarrow 0$. The ϵ^{-2} term may be absorbed by mass renormalization and the $\ln\frac{1}{12}\epsilon^2\mu^2$ term by renormalization of ξ. [That is, in Eq. (2.9) the quadratic divergences in $\langle\phi^2\rangle$ and $\langle\psi^2\rangle$ are absorbed by renormalization of m_ϕ^2: the corresponding logarithmic divergences are absorbed by renormalization of ξ_ϕ.] In addition, terms of the form of (constant) or (constant)$\times R$ may be removed by finite mass and ξ renormalizations, respectively. The renormalized expectation value in the Bunch-Davies vacuum can be expressed as

$$\langle\phi^2\rangle_{\text{BD}}=(16\pi^2)^{-1}\{-m^2\ln(12m^2/\mu^2)+[m^2+(\xi-\tfrac{1}{6})R][\ln(R/\mu^2)+\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)]\} , \tag{4.7}$$

where the additive constant has been chosen so that $\langle\phi^2\rangle_{\text{BD}}=0$ at $R=0$. The term proportional to $\ln(R/\mu^2)$ was discussed in detail in Ref. 13.

The renormalization mass μ appears to introduce an additional free parameter into the theory; however, this is not actually the case. A redefinition of the renormalization mass readjusts the other parameters of the theory in accordance with the renormalization-group transformation. For the theory described by the interaction Lagrangian of Eq. (2.8), let μ_ϕ and μ_ψ denote the renormalization masses associated with the fields ϕ and ψ, respectively. Then if $\mu_\phi\rightarrow\mu'_\phi$ and $\mu_\psi\rightarrow\mu'_\psi$, $m_\phi\rightarrow m'_\phi$, and $\xi_\phi\rightarrow\xi'_\phi$ where to first order

$$m'_\phi{}^2=m_\phi{}^2+(8\pi^2)^{-1}[\lambda_\phi m_\phi{}^2\ln(\mu'_\phi/\mu_\phi)+g^2m_\psi{}^2\ln(\mu'_\psi/\mu_\psi)] \tag{4.8a}$$

and

$$\xi'_\phi=\xi_\phi+(8\pi^2)^{-1}[\lambda_\phi(\xi_\phi-\tfrac{1}{6})\ln(\mu'_\phi/\mu_\phi)+g^2(\xi_\psi-\tfrac{1}{6})\ln(\mu'_\psi/\mu_\psi)] . \tag{4.8b}$$

To first order, the coupling constants λ_ϕ, λ_ψ, and g^2 are not renormalized and are left unchanged by this transformation. All measurable quantities, such as M^2 defined in Eq. (2.9), are invariant under this transformation. Equations (4.8a) and (4.8b) may be derived from the invariance of M^2. A similar pair of equations hold for the parameters of the ψ field, m_ψ and ξ_ψ.

One can eliminate this freedom by fixing μ to be any convenient value. For the massive theory one could, for example, set $\mu^2=m^2$. Another possible renormalization condition for $m\neq 0$ is to require that the linear term in the expansion of Eq. (4.7) in powers of R vanish. This yields

$$\mu^2=12m^2\exp\left[\frac{\xi-\frac{1}{9}}{\xi-\frac{1}{6}}\right] \tag{4.9}$$

or

$$\langle\phi^2\rangle_{\rm BD}=\frac{1}{16\pi^2}\{(\tfrac{1}{9}-\xi)R+[m^2+(\xi-\tfrac{1}{6})R][\ln(R/12m^2)+\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)]\}\ . \tag{4.10}$$

This renormalization is not defined if $m=0$.

In the massless theory, μ cannot be eliminated. If one sets $m=0$ in Eq. (4.7), $\psi(\frac{3}{2}\pm\nu)$ is a constant and may be absorbed by a finite ξ renormalization. Then we have

$$\langle\phi^2\rangle_{\rm BD}=\frac{\xi-\frac{1}{6}}{16\pi^2}R\ln(R/\mu^2)\ . \tag{4.11}$$

Note that one can have $\langle\phi^2\rangle_{\rm BD}<0$. This means that radiative corrections can act in such a way as to destabilize the symmetric vacuum state if $\lambda_\phi>0$. This will be discussed in Sec. V.

In the particular case $m=0$, $\xi=\frac{1}{6}$ no infinite ξ renormalization is required. If one chooses not to perform any finite renormalization then one obtains $\langle\phi^2\rangle_{\rm BD}=R/576\pi^2$. This is equal to $T^2/12$ where $T=H/2\pi$ is the temperature which Gibbons and Hawking[17] have argued should be associated with de Sitter spacetime. From some points of view, it is more natural to retain this term in Eqs. (4.7) and (4.11). If $\xi\neq\frac{1}{6}$, such a term can be created or absorbed by altering μ. If one goes beyond the one-loop approximations, this is true even if $\xi=\frac{1}{6}$. Consequently, the decision to retain or not retain this term does not affect the physics.

Let us note two features of the Bunch-Davies vacuum state. The first is that it is the natural choice for the vacuum state for particles of large mass, $m\gg H^{-1}$. In this case the Compton wavelength is small compared to the local radius of curvature of spacetime (here H^{-1}) and the flat-space definition of particles is applicable. If one matches the de Sitter metric to flat space by allowing H to vary slowly in time, then the Bunch-Davies vacuum is the in-vacuum or the out-vacuum, according to whether the matching occurs in the past or future, respectively. As will be shown below, $\langle\phi^2\rangle$ approaches $\langle\phi^2\rangle_{\rm BD}$ at late times.

However, the Bunch-Davies vacuum is not a physically realizable state in the limit $m=\xi=0$. This can be seen from the fact that $\langle\phi^2\rangle_{\rm BD}$ is singular as $\nu\to\frac{3}{2}$ (i.e., as $m^2+\xi R\to 0$):

$$\langle\phi^2\rangle_{\rm BD}\sim-\frac{R}{64\pi^2}[m^2+(\xi-\tfrac{1}{6})R](m^2+\xi R)^{-1}\ . \tag{4.12}$$

If $\xi=0$, $\langle\phi^2\rangle_{\rm BD}\sim(384\pi^2)^{-1}R^2/m^2$ as $m\to 0$. This is an infrared divergence and can be interpreted as arising from the presence of an infinite number of long-wavelength particles in this state and would cause even the point-separated expression Eq. (4.6) to be infinite. A state which exhibits an infrared divergence cannot arise as a result of dynamical evolution from regular initial conditions.[19,20] An example of a state which is realizable for $m=\xi=0$ is the state which is the conformal vacuum state in the initial radiation-dominated regime ($t\ll H^{-1}$). Here the effect of scalar curvature is negligible, so conformal ($\xi=\frac{1}{6}$) and minimal ($\xi=0$) coupling are equivalent. The conformal vacuum state for a massless field is the analog of the vacuum in Minkowski spacetime where the mode functions are conformal transforms of Minkowski-space positive-frequency mode functions. Although this state is defined only for massless fields, we can consider states for massive fields which approach the conformal vacuum in the massless limit.

Let $\langle\phi^2\rangle_0$ be the expectation value of ϕ^2 in any state associated with mode functions of the form of Eq. (4.3). Then for real ν we have

$$\langle\phi^2\rangle_0-\langle\phi^2\rangle_{\rm BD}=(8\pi)^{-1}H^2|\eta|^3\int_0^\infty dk\,k^2[\,|c_1(k)H_\nu^{(1)}(k\eta)+c_2(k)H_\nu^{(2)}(k\eta)|^2-|H_\nu^{(2)}(k\eta)|^2]\ . \tag{4.13}$$

Provided that $c_1(k)$ vanishes sufficiently rapidly as $k\to\infty$, this integral is convergent at the upper range of integration. This will be the case for physically reasonable states.[19] The integral in Eq. (4.5) will converge at the lower limit for such states even if $m=\xi=0$ ($\nu=\frac{3}{2}$). This requires that c_1 and c_2 approach one another as $k\to 0$. If

$$F(k/H)=|c_2(k)-c_1(k)|^2\ , \tag{4.14}$$

then the integral will converge at the upper range of integration (and state-dependent ultraviolet divergences will be avoided if

$$F\sim 1+O(k^{-\gamma}),\ \ \gamma>1,\ \ k\to\infty\ . \tag{4.15}$$

Infrared divergences will be avoided for the case $\nu=\frac{3}{2}$ ($m=\xi=0$) provided that

$$F\sim k^\beta,\ \ \beta>0,\ \ k\to 0 \tag{4.16}$$

in this case. As noted above, states which arise by dynamical evolution are free of infrared divergences and hence satisfy Eq. (4.16). A simple example of such a state can be constructed by considering a metric of the form

$$a=\begin{cases}2+H\eta, & \eta<\eta_0\\ -(H\eta)^{-1}, & \eta>\eta_0, \quad \eta_0=-H^{-1}\end{cases} \tag{4.17}$$

which is a Robertson-Walker spacetime with $a \propto t^{1/2}$ matched to de Sitter spacetime at $\eta=\eta_0=-H^{-1}$. For $\eta<\eta_0$ we choose the mode functions

$$\psi_k^{(0)}(\eta)=(2k)^{-1/2}a^{-1}(\eta)e^{-ik\eta}\,,$$

which are pure positive frequency, so the state is the in-vacuum. Requiring that $\psi_k(\eta)$ and $\psi_k'(\eta)$ be continuous at $\eta=\eta_0$, we can find the coefficients c_1 and c_2 in Eq. (4.3). This gives

$$F(z)=\frac{1}{4z^4}\,|\,e^{-2iz}(1+2iz)-1-2z^2\,|^2\,. \tag{4.18}$$

At small z, $F(z)\propto z^2$, and hence Eq. (4.16) is fulfilled. This particular choice of state does not satisfy Eq. (4.15) because the scalar curvature for the metric Eq. (4.17) has a discontinuity which creates an excessive number of particles in modes of large k. Hence, the $F(k)$ of Eq. (4.18) is not a good approximation to the corresponding function for a metric such as that of Eq. (3.5) for $k \gtrsim H$. However, it is expected to be a good approximation for $k \ll H$.

The late time behavior of $\langle\phi\rangle_0{}^2$ is found by taking the $\eta\to 0$ limit of Eq. (4.13). In this limit, we may write (here and below we assume ν real)

$$\frac{d\langle\phi^2\rangle_0}{d\eta}\sim\pi^{-3}2^{2\nu-3}H^{5-2\nu}\Gamma^2(\nu)K\,|\,\eta\,|^{2-2\nu}\,, \tag{4.19}$$

where

$$K=\int_0^\infty dz\,z^{3-2\nu}F'(z)\,. \tag{4.20}$$

Equation (4.19) can be derived from Eq. (4.13) by changing the integration variable to $x=k\eta$, using the small argument limit of the Hankel functions, and then differentiating with respect to η. If $\nu=\frac{3}{2}$, then we have

$$K=F(\infty)-F(0)=1 \tag{4.21}$$

and

$$\frac{d\langle\phi^2\rangle_0}{d\eta}=(4\pi^2)^{-1}H^2\,|\,\eta\,|^{-1}\,. \tag{4.22}$$

Thus for $m=\xi=0$, $\langle\phi^2\rangle_0$ is of the form

$$\langle\phi^2\rangle_0\sim-(4\pi^2)^{-1}H^2\ln|\,\eta/\eta_0\,|\,,\quad \eta\to 0 \tag{4.23}$$

where η_0 is a constant. If $\nu<\frac{3}{2}$, we have (recall that $\eta=-|\,\eta\,|$)

$$\langle\phi^2\rangle_0\sim\langle\phi^2\rangle_{\rm BD}-\pi^{-3}2^{2\nu-3}H^2\Gamma^2(\nu)\times(3-2\nu)^{-1}K\,|\,\eta\,|^{3-2\nu},\quad \eta\to 0\,. \tag{4.24}$$

Because $K\to 1$ as $\nu\to\frac{3}{2}$, for ν sufficiently close to $\frac{3}{2}$, $K>0$. Consequently, $\langle\phi^2\rangle_0$ is increasing as $\eta\to 0$ and asymptotically approaches $\langle\phi^2\rangle_{\rm BD}$ from below. For $\frac{3}{2}-\nu\ll 1$, $\langle\phi^2\rangle_{\rm BD}$ is large and $\langle\phi^2\rangle_0$ grows by a large amount at late times, as reflected in the factor of $(3-2\nu)^{-1}$ in Eq. (4.12). When $\nu\to\frac{3}{2}$ $(m=\xi=0)$, $\langle\phi^2\rangle_{\rm BD}$ does not exist and $\langle\phi^2\rangle_0$ grows indefinitely.

In this latter case, one can expand both of the terms on the right-hand side of Eq. (4.24) about $\nu=\frac{3}{2}$; the terms proportional to $(3-2\nu)^{-1}$ cancel and one obtains, for $m=\xi=0$,

$$\langle\phi^2\rangle_0=-\frac{R}{96\pi^2}\left[\ln R/\mu^2+2\ln(\tfrac{1}{2}H\eta)+2\int_0^\infty F'(z)\ln z\,dz\right]. \tag{4.25}$$

B. The thermal contribution

In addition to the vacuum contribution to $\langle\phi^2\rangle$ there is the contribution due to the particles which were present in the initial radiation-dominated phase of the Universe. At the beginning of the de Sitter phase, it dominates the vacuum part and is usually the only contribution taken into account. If one assumes that the Universe is approximately in thermal equilibrium for $t\simeq H^{-1}$, then we can write the thermal contribution as

$$\langle\phi^2\rangle_T=\pi^{-2}\int_0^\infty dk\,k^2 n_k\,|\,\psi_k\,|^2\,, \tag{4.26}$$

where

$$n_k=(e^{k/\theta}-1)^{-1}\,. \tag{4.27}$$

Here, θ is the temperature of the Universe at the beginning of the de Sitter phase $(t\simeq H^{-1})$. It will be assumed that $\theta\gg H,m$. Note that the condition that this integral converge at the lower limit when $\nu=\frac{3}{2}$ is that $F\sim k^{1+\beta}$, $\beta>0$, $k\to 0$, which is fulfilled by Eq. (4.18).

The assumption of thermal equilibrium in the very early Universe may or may not be valid. If the Universe is described by a thermal state at the Planck time, this would presumably have to do with the fundamental relationship between quantum mechanics, statistical mechanics, and gravitation which is not yet well understood. It is also possible that interactions could rapidly thermalize an initially nonthermal state. In any case, Eq. (4.26) is the simplest ansatz for describing the effect of the particles upon $\langle\phi^2\rangle$.

At the beginning of the de Sitter phase ($t\simeq H^{-1}$), because $\theta \gg H$ we can approximate the integrand of Eq. (4.26) by assuming that $|k\eta| \gg 1$, $c_1\simeq 0$, and $|c_2|\simeq 1$. Then one has

$$\langle\phi^2\rangle_T \simeq \frac{T^2}{12}\ , \tag{4.28}$$

where $T=\theta/a$. This is just the usual behavior for a thermal distribution which is being red-shifted by the expansion of the Universe. At late times, $t\to\infty$, we have $\eta\to 0$ and

$$\langle\phi^2\rangle_T \sim 2^{2\nu-3}\pi^{-3}H^{5-2\nu}|\eta|^{3-2\nu} \times\int_0^\infty dz\, z^{2-2\nu} n_k\, |F(z)|^2\ , \tag{4.29}$$

where $z=k/H$ and $F(z)$ is defined in Eq. (4.14). In the Appendix it is shown that if $\theta/H \gg 1$ and $\frac{3}{2}-\nu \ll 1$, the integral in Eq. (4.29) is proportional to $\theta H^{2-2\nu}$. Thus

$$\langle\phi^2\rangle_T \sim A\theta H\, |H\eta|^{3-2\nu},\quad t\to\infty\ , \tag{4.30}$$

where

$$A=\pi^{-3}\int_0^\infty dz\, z^{-2}|F(z)|^2 \tag{4.31}$$

is a numerical factor of the order of unity. For $m=\xi=0$, $\langle\phi^2\rangle_T$ approaches a nonzero constant

$$\langle\phi^2\rangle_T \to A\theta H,\quad t\to\infty\ . \tag{4.32}$$

In fact, it can be shown that Eqs. (4.28) and (4.32) together give a good approximation throughout the de Sitter phase,[10] so

$$\langle\phi^2\rangle_T \simeq \frac{T^2}{12}+A\theta H \tag{4.33}$$

and that corrections to this equation do not exceed $TH\ln[\theta/(T+H)]$.

Thus the behavior of $\langle\phi^2\rangle_T$ at late times is strongly influenced by the coupling to the background gravitational field. For the massless conformal ($\xi=\frac{1}{6}$) scalar field, one would have $\langle\phi^2\rangle_T=T^2/12$ at all times, so the behavior exhibited in Eq. (4.30) reflects the nonconformal character of ϕ. In the massless minimally coupled theory, Eq. (4.32), $\langle\phi^2\rangle_T$ approaches a constant which depends upon the initial temperature. This result is due to the fact that $\psi_k\to$constant as $k\to 0$ for $\nu=\frac{3}{2}$. Normally the expansion of the Universe causes the contribution of a given mode to be red-shifted to zero; in this case each mode is red-shifted until its wavelength is of the order of the horizon size (H^{-1}), and afterwards it gives a constant contribution to $\langle\phi^2\rangle_T$.

Note that $f(x)=ax^\alpha=ae^{\alpha\ln x}\simeq a(1+\alpha\ln x)$ if $\alpha|\ln x|\ll 1$. Thus although $f\to 0$ as $x\to 0$ ($\alpha>0$), f is approximately constant over a large interval if $\alpha\ll 1$. The next correction in Eq. (4.29) is $O(|\eta|^{4-3\nu})$. Consequently, if $\frac{3}{2}-\nu\ll 1$, $\langle\phi^2\rangle_T\approx A\theta H$ over a finite interval in which $|(3-2\nu)\ln|H\eta||\lesssim 1$. For the case $\xi=0$ and $m/H\ll 1$, this condition can be expressed as $|H\eta|\gtrsim e^{-3H^2/2m^2}$. Thus, for small m, the scale factor $(H\eta)^{-1}$ can change by many orders of magnitude while $\langle\phi^2\rangle_T$ remains constant.

Let $\langle\phi^2\rangle$ be the complete expectation value including vacuum and thermal contributions:

$$\langle\phi^2\rangle=\langle\phi^2\rangle_0+\langle\phi^2\rangle_T\ .$$

We are now in a position to describe qualitatively the behavior of $\langle\phi^2\rangle$ in various cases. Consider first the case $0<\frac{3}{2}-\nu\ll 1$, for example, $m\ll H$ and $|\xi|\ll 1$. Initially $\langle\phi^2\rangle\simeq\langle\phi^2\rangle_T$ and decreases as a^{-2}; $\langle\phi^2\rangle_T$ then levels off at a plateau value of $A\theta H$. Meanwhile, $\langle\phi^2\rangle_0$ begins to grow and eventually dominates $\langle\phi^2\rangle_T$. Eventually $\langle\phi^2\rangle$ approaches its asymptotic value $\langle\phi^2\rangle_{\rm BD}$. If ν is sufficiently close to $\frac{3}{2}$, $\langle\phi^2\rangle_{\rm BD}>A\theta H$. This behavior is illustrated in Fig. 1, curve A. In the limit $m=\xi=0$, the time dependence of $\langle\phi^2\rangle$ is essentially the same except that $\langle\phi^2\rangle_{\rm BD}\to\infty$ and $\langle\phi^2\rangle$ continues to grow indefinitely.

Another case which we can treat is that of a massive field with $m\gtrsim H$. Here the Bunch-Davies vacuum is a reasonable choice for the vacuum state, so $\langle\phi^2\rangle_0=\langle\phi^2\rangle_{\rm BD}$. Here $\langle\phi^2\rangle$ decreases monotonically as shown in curve B of Fig. 1. [In this case ν may be imaginary and the explicit formula given above for $\langle\phi^2\rangle_T$, Eq. (4.29), does not apply; however $\langle\phi^2\rangle_T$ will still decrease monotonically.]

The expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$ does not exhibit the exotic behavior found in $\langle\phi^2\rangle$ at $\nu=\frac{3}{2}$. Although the Bunch-Davies vacuum is not well defined at $\nu=\frac{3}{2}$, $\langle T_{\mu\nu}\rangle_{\rm BD}$ does exist in the limit $\nu\to\frac{3}{2}$. The expectation value in a general state approaches $\langle T_{\mu\nu}\rangle_{\rm BD}$ at late times even for $\nu\to\frac{3}{2}$. One can understand this difference between $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ as due to the fact that the former quantity is more sensitive to the contribution of very-long-wavelength modes.

V. DESTABILIZATION OF THE SYMMETRIC PHASE

The results of the previous section show that in de Sitter space the evolution of the expectation values $\langle \phi^2 \rangle$ and $\langle \psi^2 \rangle$ appearing in Eq. (2.9) can be substantially different from that in flat spacetime. This difference can give rise to a qualitative change in the behavior of the system. In particular, the symmetric phase can become unstable, even if it is stable at all temperatures in flat spacetime. As an illustration, let us consider a model with the interaction of Eq. (2.10) and with $m_\phi = m_\psi = \xi_\phi = 0$, $\xi_\psi = \frac{1}{6}$. We shall assume that $g^2 > \lambda > 0$ and g^2 and λ are of the same order of magnitude. Comparing Eqs. (4.25) and (4.33) we see that if $\ln R/\mu^2$ is not too large, then the vacuum part of $\langle \phi^2 \rangle$ is small compared to the thermal part provided that

$$|H\eta| \gg \exp(-4\pi^2 A\theta/H) . \tag{5.1}$$

Assuming that this condition is satisfied, we can write

$$\langle \phi^2 \rangle \approx \frac{T^2}{12} + A\theta H . \tag{5.2}$$

For the conformal field ψ we have

$$\langle \psi^2 \rangle = \frac{T^2}{12} + O(H^2) , \tag{5.3}$$

where $O(H^2)$ refers to the possible presence of the term $R/576\pi^2$. From Eqs. (5.2) and (5.3) we have

$$\begin{aligned} M^2 &= g^2\langle \psi^2 \rangle - \lambda\langle \phi^2 \rangle \\ &\approx \tfrac{1}{12}(g^2-\lambda)T^2 - A\lambda\theta H , \end{aligned} \tag{5.4}$$

where we have neglected g^2H^2 compared to $\lambda\theta H$. At high temperatures, $T \gg (\theta H)^{1/2}$, $M^2 > 0$ and the symmetric phase is quasistable.

It is destabilized at $T = T_*$, where

$$T_*^2 \approx \frac{12A\lambda\theta H}{g^2-\lambda} . \tag{5.5}$$

With g^2/λ and A of order unity, we have T_* of order $(\theta H)^{1/2}$.

The above analysis has been done for a strictly massless, minimally coupled field ϕ ($m_\phi = \xi_\phi = 0$). One expects that the conclusions remain qualitatively unchanged for nonzero m_ϕ and ξ_ϕ if $\kappa^2 = m_\phi{}^2 + \xi_\phi R$ is sufficiently small. Increasing κ^2 has a stabilizing effect on the theory, and above some critical value, $\kappa^2 > \kappa_c{}^2$, destabilization will not occur. We can estimate $\kappa_c{}^2$ in the following way. Suppose that $\kappa^2 > \kappa_c{}^2$, so that at late times M^2 becomes a positive constant and the system approaches the Bunch-Davies state. For $\kappa^2 \ll H^2$, we find, using Eq. (4.12),

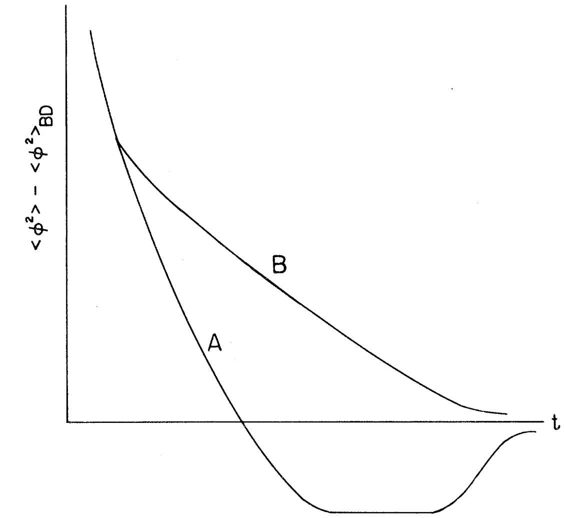

FIG. 1. The behavior of $\langle \phi^2 \rangle$ as a function of time is shown for two cases. If $|m^2+\xi R| \ll H^2$, $\langle \phi^2 \rangle$ behaves as shown in curve A. If $m^2+\xi R \gtrsim H^2$, then the behavior is as shown in curve B.

$$M^2 \approx \kappa^2 - \frac{3\lambda H^4}{8\pi^2\kappa^2} . \tag{5.6}$$

We see that M^2 is positive only if $\kappa^2 > \kappa_c{}^2$, where

$$\kappa_c{}^2 \approx (3\lambda/8\pi^2)^{1/2} H^2 . \tag{5.7}$$

This estimate can be improved if we go beyond the first-order perturbation theory and replace Eq. (5.6) by

$$M^2 \approx \kappa^2 - \frac{3\lambda H^4}{8\pi^2 M^2} . \tag{5.8}$$

Here the expectation value $\langle \phi^2 \rangle$ is calculated using the effective mass M^2 instead of the zeroth-order approximation κ^2. This is analogous to the Hartree approximation in the many-body theory.[21] The solution of Eq. (5.8) is

$$M^2 = \kappa^2/2 + \left[\frac{\kappa^4}{4} - \frac{3\lambda H^4}{8\pi^2} \right]^{1/2} . \tag{5.9}$$

Real solutions exist only for $\kappa^2 > (3\lambda/2\pi^2)^{1/2}H^2$, and thus

$$\kappa_c{}^2 \approx (3\lambda/2\pi^2)^{1/2} H^2 . \tag{5.10}$$

Equation (5.8) goes beyond the first-order approximation and includes some (but by no means all) higher-order contributions. The fact that it yields a value of $\kappa_c{}^2$ which is approximately equal to the first-order value, Eq. (5.7) indicates that this first-order value is reasonably accurate, at least for sufficiently small λ.

As a second example, we take a model with $m_\phi=m_\psi=0$, $0\le\xi_\phi\ll 1$, $\xi_\psi\neq 0,\frac{1}{6}$, and with an interaction of the form

$$U(\phi,\psi)=\tfrac{1}{2}g^2\phi^2\psi^2 . \tag{5.11}$$

Then

$$M^2=\xi_\phi R+g^2\langle\psi^2\rangle . \tag{5.12}$$

At high temperatures $\langle\psi^2\rangle\approx T^2/12$, $M^2>0$ and the symmetric phase is stable. As $T\to 0$, $\langle\psi^2\rangle$ approaches the Bunch-Davies value, Eq. (4.11),

$$\langle\psi^2\rangle_{\rm BD}=\frac{R}{16\pi^2}(\xi_\psi-\tfrac{1}{6})\ln(R/\mu_\psi^2) . \tag{5.13}$$

We have noted earlier that $\langle\psi^2\rangle_{\rm BD}$ can be negative. In this case, M^2 can change sign at some temperature $T=T_*$, and the symmetric phase can become unstable. If the phase transition occurs at a sufficiently high temperature, we may estimate T_* by writing

$$\langle\psi^2\rangle\approx\tfrac{1}{12}T^2+\langle\psi^2\rangle_{\rm BD} . \tag{5.14}$$

Using this expression in Eq. (5.12), we find

$$T_*^2\approx-\frac{3}{4\pi^2}(\xi_\psi-\tfrac{1}{6})R\ln(R/\mu_\psi{}^2) - 12\xi_\phi g^{-2}R . \tag{5.15}$$

The high-temperature form of $\langle\psi^2\rangle$ is a good approximation if $\theta|\eta|=T/H\gg 1$. Thus Eq. (5.15) is valid provided that $T_*\gg H$. Note that T_* is invariant to first order in g^2 under the transformation of Eq. (4.8b). If $\langle\psi^2\rangle_{\rm BD}>0$, then the symmetric phase remains stable down to zero temperature. (Note that even in flat spacetime radiative corrections can cause $\Phi=0$ to be only a local rather than a global minimum of the effective potential[22]; in this case the symmetric phase is only quasistable.)

The condition $\xi_\phi\ll 1$ is imposed in order that first-order perturbation theory be applicable at the phase transition. One expects Eq. (5.13) to contain higher-order corrections such as terms of the form $g^2R\ln^2(R/\mu_\psi{}^2)$.[13] These higher-order contributions will, however, be small compared to the first-order contribution at $T=T_*$ if ξ_ϕ is sufficiently small.

Finally, we shall briefly discuss the time scale for the onset of instabilities. When the temperature drops below T_*, M^2 becomes negative, and Eq. (2.6) has growing solutions. In our model (2.10), for $T\ll T_*$, Eq. (5.4) gives $M^2=-A\lambda\theta H$. The solutions of Eq. (2.6) with M^2=const are $\Phi(x)=\Phi_k(\eta)e^{i\vec{k}\cdot\vec{x}}$, where

$$\Phi_k(\eta)=\eta^{3/2}[C_1H_\nu^{(1)}(k\eta)+C_2H_\nu^{(2)}(k\eta)] \tag{5.16}$$

and

$$\nu=\left[\frac{9}{4}-\frac{M^2}{H^2}\right]^{1/2}\approx\tfrac{3}{2}+\frac{A\lambda\theta}{3H} . \tag{5.17}$$

In the last equation, we have assumed that $\lambda\theta/H\ll 1$. For small η, $\Phi_k(\eta)\propto\eta^{3/2-\nu}$ or

$$\Phi_k(t)\propto\exp(A\lambda\theta t/3) . \tag{5.18}$$

The characteristic time τ can be defined as a typical time of growth of the fluctuations (as measured by a comoving observer):

$$\tau\approx 3/A\lambda\theta . \tag{5.19}$$

During this time the Universe will expand by a factor of $e^{3H/A\lambda\theta}$ which can be large for sufficiently small λ. The actual magnitude of $\Phi_k(t)$ depends on the initial spectrum of fluctuations at $T\sim T_*$. These fluctuations can be of thermal or quantum origins, and we do not attempt to estimate them in this paper.

VI. SUMMARY AND DISCUSSION

In the preceding sections we have seen how the coupling to spacetime curvature can drastically modify the time evolution of $\langle\phi^2\rangle$. This modification arises whenever the scalar field is not conformally invariant (i.e., $m\neq 0$ or $\xi\neq\frac{1}{6}$). In this sense, the massless conformal ($\xi=\frac{1}{6}$) theory is really the case when the coupling to the background gravitational field is minimal, and for the so-called minimal theory ($\xi=0$) gravitational effects are far from minimal. In particular, we have seen that for the massless, minimal ($m=\xi=0$) theory, $\langle\phi^2\rangle$ can grow in an expanding universe rather than decrease as one would normally expect.

This modification in the behavior of $\langle\phi^2\rangle$ can have an important effect upon the stability of the field theory. In the previous section we have seen how, in the context of a particular model field theory, otherwise stable field configurations may be destabilized by the coupling to spacetime curvature. Conversely, this coupling can also have a stabilizing effect upon an otherwise unstable configuration.

In discussions of quantum field theory in de Sitter spacetime, the issues of "Hawking radiation" and of a minimum temperature in de Sitter space often arise. Gibbons and Hawking[17] argued that observers moving along timelike geodesics with a particle detector will measure blackbody radiation at a temperature $T=H/2\pi$. It is important to emphasize that the contribution of this radiation to $\langle\phi^2\rangle$ is included in what we have called the vacuum contribution $\langle\phi^2\rangle_0$ and is not in any way a distinct effect. That is, once one has chosen the quantum state of the system and calculated the expectation value of ϕ^2 in that state (a scalar quantity upon whose value all observers will agree) all possible contributions are included. It is true that the distinction between vacuum and thermal contributions to $\langle\phi^2\rangle$ is observer dependent and hence not unique. A similar situation arises in the case of the radiation emitted by a black hole. Fulling[23] has noted that the decomposition of $\langle T_{\mu\nu}\rangle$ into vacuum and thermal parts is not unique and depends upon the choice of observer.

The growth of $\langle\phi^2\rangle_0$ and the large value of $\langle\phi^2\rangle_{\rm BD}$ when $|\xi+m^2/R|\ll 1$ found in Sec. IV depend upon a large contribution from long-wavelength modes. However, de Sitter space can be represented as a closed Robertson-Walker metric with scale factor $a(t)=H^{-1}\cosh Ht$. One might expect that if one were to use this form of the metric, then the finiteness of the spatial sections would prevent the appearence of infrared divergences. This is not the case. Dowker and Critchley[24] and other authors quoted in Ref. 16 have calculated the de Sitter-invariant Green's function using the closed-space form of de Sitter space. The corresponding expression for the formal expectation value of ϕ^2 with $\xi=0$ is

$$\langle\phi^2\rangle=\frac{H^2}{8\pi^2}\sum_{n=1}^{\infty}\frac{n(n+1)(n+\frac{1}{2})}{(n+2)(n-1)+m^2/H^2}\;. \tag{6.1}$$

This expression can be obtained, for example, from the formulas preceding Eq. (10) of Dowker and Critchley[24]. The $n=1$ term of Eq. (6.1) is $3H^4/8\pi^2m^2$, which is just Eq. (4.12) with $\xi=0$; hence the infrared singularity is still present. Moreover, the Green's function found by Dowker and Critchley turns out to be identical to that of Bunch and Davies, and hence $\langle\phi^2\rangle$ in (6.1) is identical to $\langle\phi^2\rangle_{\rm BD}$.

In this paper we have restricted our attention to first-order perturbation theory. As discussed in Sec. V, this is sufficient for some range of the parameters of the models we consider. However, first-order perturbation theory may not be a good approximation even if the coupling constants are small because the coupling constant can be multiplied by a large dimensionless quantity such as θ/H or $\ln(R/\mu^2)$.

One problem for future work in this area is to understand more fully the higher-order corrections and when they have a significant effect upon the behavior of the system. This is important for the study of phase transitions in grand unified theories in the early Universe with Coleman-Weinberg symmetry breaking, where it appears that higher-order contributions will be important.[10]

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APPENDIX

Here we wish to obtain an approximate expression for the integral appearing in Eq. (4.29). We assume that $\theta/H\gg 1$ and $\frac{3}{2}-\nu\ll 1$. Write the integral as

$$I=I_1+I_2\;, \tag{A1}$$

where

$$I_1=\int_0^{\alpha\theta}dk\,k^{2-2\nu}n_k\,|F(k/H)|^2 \tag{A2}$$

and

$$I_2=\int_{\alpha\theta}^{\infty}dk\,k^{2-2\nu}n_k\,|F(k/H)|^2\;, \tag{A3}$$

where α is chosen so that $0<\alpha\ll 1$ and $\theta\alpha/H\gg 1$. In I_1 we have that $k\ll\theta$, so $n_k\simeq\theta/k$. Thus if $z=k/H$

$$\begin{aligned}I_1&\simeq\theta H^{2-2\nu}\int_0^{\alpha\theta/H}dz\,z^{1-2\nu}|F(z)|^2\\&\simeq\theta H^{2-2\nu}\int_0^{\infty}dz\,z^{1-2\nu}|F(z)|^2\\&=c_1\theta H^{2-2\nu}\;.\end{aligned} \tag{A4}$$

Because $|F|^2\approx 1$ for $k\gtrsim H$, we can write I_2 as

$$I_2\simeq\theta^{3-2\nu}\int_{\alpha}^{\infty}\frac{dx\,x^{2-2\nu}}{e^x-1}\;. \tag{A5}$$

For small α this integral is dominated by the contribution at the lower limit, so

$$I_2 \simeq c_2(\theta\alpha)^{3-2\nu} . \tag{A6}$$

Both c_1 and c_2 are constants of the order of unity, so because $\theta\alpha/H \gg 1$, $I_1 \gg I_2$ and

$$I \simeq c_1 \theta H^{2-2\nu} . \tag{A7}$$

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