Quantum vacuum energy in Taub-NUT (Newman-Unti-Tamburino)-type cosmologies

William A. Hiscock and D. A. Konkowski
Center for Relativity, Physics Department, The University of Texas, Austin, Texas 78712
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The effects of vacuum polarization on the mildest possible sort of cosmological singularity, the Taub-NUT (Newman-Unti-Tamburino)-type singularities, are studied. Unlike stronger sorts of singularities where physical quantities (e.g., curvature, energy density) diverge, in these universes the only barrier is a pathological topology. Quantum effects, known to be important in regions of large spacetime curvature, are found to also be important in these universes, where the curvature may be arbitrarily small or even zero. The vacuum expectation value of the stress-energy tensor for a conformal scalar field is calculated on a flat archetype of the Taub-NUT-type universes, the Misner universe (flat Kasner spacetime with $S^1 \times R^3$ topology). The vacuum stress energy diverges at the singularity and on its associated Cauchy horizons. This divergence, together with the "fixed" nature of the spacetime's topology, suggests that these boundaries will be replaced by curvature singularities in a better approximation to full quantum gravity.

I. INTRODUCTION

There are two outstanding problem areas in our theoretical understanding of gravitation: the nature of singularities[1] and quantum gravity.[2] It is generally accepted that the two subjects are not completely distinct: quantum gravitational effects are expected to be most important in regions of large spacetime curvature, i.e., near curvature singularities.

There are, however, maximally extended spacetimes which possess incomplete geodesics (and hence singularities) yet whose curvature tensors are everywhere well behaved. Such singularities arise due to a pathology in the topological structure of spacetime. In the singularity classification scheme of Ellis and Schmidt,[3] they are known as "quasiregular singularities." Their defining property is that the components of the Riemann tensor in a parallel-propagated frame remain bounded along each curve ending on the singularity. Alternatively, if some component of the Riemann tensor diverges, one has some sort of curvature singularity (e.g., a "big bang" or a "whimper"). It is not clear *a priori* that quantum effects will play a significant role near the topological, i.e., quasiregular, sort of singularity.

In this paper we examine how quantization of matter fields might affect a particularly interesting subset of quasiregular singularities: the "Taub-NUT (Newman-Unti-Tamburino)-type" singularities. These singularities are characterized by incomplete geodesics which spiral an infinite number of times around a topologically closed spatial dimension in a finite proper time, to end on one of a pair of null Cauchy horizons. Even if the spacetime is analytically extended across the Cauchy horizons, the intersection of the Cauchy horizons is a quasiregular singularity on which geodesics end. These spacetimes undergo a sort of "topological collapse" rather than "gravitational collapse" to the singularity. It has long been known that this type of singularity can exist in spatially homogeneous cosmologies; Taub-NUT space is the most thoroughly investigated example. More recently, Moncrief has demonstrated the existence of an entire class of inhomogeneous universes with singularities of this type.[4] Although the global topology of the spacetimes may differ (Taub-NUT spacetime is $R \times S^3$ while Moncrief's spacetimes are $R \times T^3$), the structure of the singularities is similar. A simple archetypal model for all such Taub-NUT-type singularities is the four-dimensional Misner universe[3,5,10] (flat Kasner spacetime with $S^1 \times R^3$ topology).

In this paper we calculate the vacuum expectation value of the stress-energy tensor of a conformal scalar field in the Misner universe. Since the Misner universe is flat, there is no particle creation and the only vacuum polarization present is that induced by the spacetime's topology. The vacuum stress energy diverges at the quasiregular singularity and its associated Cauchy horizons, suggesting that in a self-consistent calculation including quantum effects, these features would be replaced by a curvature singularity.

A number of classical stability analyses[6–9] have shown that Taub-NUT-type singularities are classically unstable towards becoming curvature singularities. In this work, rather than inserting a perturbation into the spacetime, the physics is treated at a deeper level by including quantum effects. We feel this approach is more fundamental than the classical stability analyses, since the "perturbation" is simply the vacuum polarization induced by the topology of the spacetime.

Section II reviews the structure of the Misner universe and its extensions. In Sec. III the vacuum stress energy is calculated for both ordinary and twisted conformal scalar fields. The effects of closing the other spatial dimensions (to yield a three-torus universe) are discussed. The implications of this work for more general Taub-NUT-type cosmologies is considered in Sec. IV.

II. THE MISNER UNIVERSE

The Misner-universe metric is

$$ds^2=-dt^2+t^2(dx^1)^2+(dx^2)^2+(dx^3)^2\,, \qquad (1)$$

with the points (t,x^1,x^1,x^3) and (t,x^1+na,x^2,x^3) identified (n takes on all integer values from $-\infty$ to $+\infty$, and a is any nonzero number). The coordinate t ranges from zero to infinity, x^1 from zero to a (identified as above), and x^2 and x^3 from $-\infty$ to $+\infty$. The metric described by Eq. (1) is the familiar flat Kasner metric with an unusual topology which causes a quasiregular singularity at $t=0$.

For the purposes of this paper, any spacetime with a metric given by Eq. (1) and with the x^1 spatial dimension topologically closed will be defined as a Misner universe. The remaining spatial dimensions (x^2,x^3) may be topologically open or closed; either choice does not affect the quasiregular structure of the singularity. These spacetimes were originally used by Misner[5] in two-dimensional form ($x^2=x^3=$const) as a model for understanding the Taub-NUT universe. In this paper all calculations are performed in the full four-dimensional metric of Eq. (1).

The universal covering space of the Misner universe is a portion of Minkowski space, obtained by the coordinate transformation

$$\begin{aligned} &y^0=t\cosh(x^1),\;\; y^1=t\sinh(x^1)\,,\\ &y^2=x^2,\;\; y^3=x^3, \end{aligned} \qquad (2)$$

yielding the Minkowski metric,

$$ds^2=-(dy^0)^2+(dy^1)^2+(dy^2)^2+(dy^3)^2 \qquad (3)$$

with the points

$$(y^0,y^1,y^2,y^3)\leftrightarrow(y^0\cosh(na)+y^1\sinh(na),y^1\cosh(na)+y^0\sinh(na),y^2,y^3) \qquad (4)$$

identified. The Misner universe occupies the quadrant $y^0>|y^1|$ of Minkowski space.

The Misner spacetime may be analytically extended across the boundaries $y^0=y^1$ $(y^1>0)$ and/or $y^0=-y^1$ $(y^1<0)$; however, if both extensions are performed the resulting spacetime is non-Hausdorff. It is possible to obtain a maximally extended (albeit non-Hausdorff) spacetime by performing both extensions. (For further discussion, see Hawking and Ellis,[10] and Ellis and Schmidt.[3])

Whether either, both, or none of the extensions are performed, the point $y^0=y^1=0$ is not included in the spacetime. Since, even in the maximally extended spacetime, there are timelike and null geodesics which end on the missing point (and hence are incomplete), and since the spacetime is flat, the point $y^0=y^1=0$ is a quasiregular singularity. Further, the physical situation beyond the surfaces $y^0=y^1$ $(y^1>0)$ and $y^0=-y^1$ $(y^1<0)$ is not determinable by evolving Cauchy data given on a spacelike slice within the Misner-universe quadrant $y^0>|y^1|$ (e.g., a $t=$const slice). The boundaries $y^0=|y^1|$ are then Cauchy horizons; one must choose the type of extension (e.g., analytic) done across the boundaries.

A few words concerning time-orientation conventions are perhaps needed at this point to avoid confusion. In this paper, we treat the Minkowski-space quadrant $y^0>|y^1|$ as the Misner universe, as is traditional.[3,4,10] By choosing this quadrant as the original spacetime (i.e., pre-extension), the singularity is an initial singularity, which is the traditional time orientation one uses when discussing cosmological singularities. The Cauchy problem (and Cauchy horizons) are then to be understood in terms of a final-value problem rather than as an initial-value problem; i.e., one chooses Cauchy data on a spacelike slice (e.g., $t=$const>0) within $y^0>|y^1|$, then evolves *backwards* in time towards $t=0$, where the singularity and Cauchy horizons are encountered. Alternatively, one could originally identify the Misner universe with the $y^0<|y^1|$ quadrant of Minkowski space, so that the singularity is a final (rather than initial) singularity and the Cauchy evolution may proceed forward in time, rather than backward. All the physics is of course time-reversal invariant, the conclusions of this paper apply whether $t=0$ is an initial or final singularity.

The global behavior of the extended Misner universe is entirely analogous to the Taub-NUT universe.[6,11] The Misner universe is spatially homogeneous, as is Taub space. It can be extended across the Cauchy horizons (or "Misner bridges"), the surfaces $y^0=|y^1|\neq 0$, to NUT-type static regions ($y^1>|y^0|$; $y^1<|y^0|$) containing closed timelike lines. These static regions are isometric to the Rindler wedge.

III. VACUUM STRESS ENERGY

We have seen that the Misner universe may be regarded as a portion of Minkowski space [Eq. (3)] with an odd topology [Eq. (4)]. Over the past few years, a powerful set of tools has been developed for studying quantum field theory in Minkowski space when boundaries are present (Casimir effect) or a nonstandard topology is chosen.[11–15]

Unfortunately, our study of the Misner universe faces a problem which previous workers did not encounter: the choice of a vacuum state. If one makes a simple identification on Minkowski space, such as $y^1 \leftrightarrow y^1+na$, then the timelike Killing vector $\partial/\partial y^0$ which is used to define positive frequency, and hence the vacuum state, is unaffected. In the Misner universe, while there are local timelike solutions to Killing's equations (e.g., $\partial/\partial y^0$), the nature of the identifications [Eq. (4)] is such that these local solutions cannot be patched together into a global timelike Killing vector field. Without a global timelike Killing vector field, there is no rigorously defined choice of vacuum state. There is, however, a physical argument which shows that the usual Minkowski vacuum state on the covering space is the correct choice for the Misner universe. Since the timelike geodesics of the Misner universe are, apart from identifications, identical with the timelike geodesics of Minkowski space, an "Unruh particle detector"[16] carried by a Misner-space observer will register no particles in the Minkowski vacuum state. If any other state were chosen to represent the vacuum, then geodesic observers in Misner space would detect particles. Clearly the unique quantum state in which all timelike geodesic observers agree there are no particles deserves to be considered the vacuum state. This choice also has a natural expression in terms of a mode decomposition in the coordinate system of Eq. (1).[17,18]

We can now calculate the vacuum expectation value of the stress-energy tensor for a conformally coupled scalar field in the Misner universe. In a Minkowski vaccum, the stress-energy tensor may be written[12–15,19]

$$\langle T_{\mu\nu}\rangle = -i\lim_{y\to\tilde{y}}\left(\tfrac{2}{3}\nabla_\mu\nabla_{\tilde{\nu}}-\tfrac{1}{3}\nabla_\mu\nabla_\nu - \tfrac{1}{6}g_{\mu\nu}\nabla_\alpha\nabla^{\tilde{\alpha}}\right)G_{\text{ren}}(y,\tilde{y}), \qquad (5)$$

where G_{ren} is the renormalized Feynman Green's function. The usual Minkowski-space Feynman Green's function is

$$G_0(y,\tilde{y})=\frac{i}{(2\pi)^2}[-(y^0-\tilde{y}^0)^2+(y^1-\tilde{y}^1)^2 +(y^2-\tilde{y}^2)^2+(y^3-\tilde{y}^3)^2+i\delta]^{-1}\,. \qquad (6)$$

The renormalized Feynman propagator for the Misner universe is obtained from G_0 by adding all the "image charges" at $\tilde{y}_n^\alpha$ according to the prescription of Eq. (4) and subtracting off the singular term G_0:

$$G_{\text{ren}}=\frac{i}{(2\pi)^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{n=+\infty}\{-[y^0-\tilde{y}^0\cosh(na)-\tilde{y}^1\sinh(na)]^2+[y^1-\tilde{y}^1\cosh(na)-\tilde{y}^0\sinh(na)]^2 +(y^2-\tilde{y}^2)^2+(y^3-\tilde{y}^3)^2\}^{-1}\,. \qquad (7)$$

Computation of $\langle T_{\mu\nu}\rangle$ in Minkowskian coordinates is now straightforward. The final result takes a particularly simple form when transformed to the Misner-universe coordinate system, (t,x^1,x^2,x^3):

$$\langle T_\mu{}^\nu\rangle=\frac{K}{12\pi^2}t^{-4}\,\text{diag}(1,-3,1,1)\,, \qquad (8)$$

where

$$K=\sum_{n=1}^{n=\infty}\frac{2+\cosh(na)}{[\cosh(na)-1]^2}\,, \qquad (9)$$

is a finite positive-definite number for all nonzero values of a. The energy density of the vacuum, as given by Eq. (8), is negative.

As expected (and required), the trace of $\langle T_\mu{}^\nu \rangle$ is zero, and $\langle T_\mu{}^\nu \rangle$ is conserved ($\nabla_\nu \langle T_\mu{}^\nu \rangle = 0$) and homogeneous (independent of x^i).

We see from Eq. (8) that every component of $\langle T_\mu{}^\nu \rangle$ diverges as the surface $t=0$ is approached (the surface $t=0$ consists of the Cauchy horizons and the quasiregular singularity). The divergence of the quadratic stress-energy scalar

$$T_{\mu\nu}T^{\mu\nu} = \frac{K^2}{12\pi^4} t^{-8} \tag{10}$$

shows the divergence of $\langle T_\mu{}^\nu \rangle$ is not a coordinate effect.

A generic timelike geodesic (i.e., one which intersects a Cauchy horizon, $y^0 = |y^1| \neq 0$, rather than the singularity at $y^0 = y^1 = 0$) is defined in the covering space by the equation

$$y^i = v^i y^0 + b^i , \tag{11}$$

where v^i and b^i are constants, and $i=1,2,3$. Such a geodesic spirals around the closed x^i dimension an infinite number of times in a finite proper time before reaching the Cauchy horizon at $t=0$. Regardless of whether one has chosen to attempt to extend the spacetime, the vacuum energy density measured along such a geodesic diverges as

$$\rho_{\text{vac}} = \frac{-K\gamma}{24\pi^2 |b^1|} \tau^{-3} + O(\tau^{-2}) , \tag{12}$$

where, as usual, $\gamma = (1 - v^i v_i)^{1/2}$, and τ is the proper time along the geodesic, chosen so that $\tau = 0$ at the Cauchy horizon $t=0$. In the special case when $b^1 = 0$, and the geodesic hits the quasiregular singularity at $y^0 = y^1 = 0$ rather than the Cauchy horizon, the energy density diverges as

$$\rho_{\text{vac}} = \frac{-K}{12\pi^2} \gamma^4 [1 - (v^1)^2]^{-2} \tau^{-4} + O(\tau^{-3}) , \tag{13}$$

where the proper-time coordinate τ has been chosen so that $\tau \to 0$ at the singularity.

Since the vacuum stress energy diverges as seen by all physical observers as the surface $t=0$ is approached, any possible choice of extension is moot; the vacuum energy density diverges on both the quasiregular singularity ($y^0 = y^1 = 0$) and the Cauchy horizons ($y^0 = |y^1| \neq 0$).

It is interesting to consider a twisted scalar field on the Misner universe. The "twist" of the bundle simply changes the boundary condition on the scalar field under the identification given by Eq. (4) from periodic to antiperiodic. The calculation of $\langle T_{\mu\nu} \rangle$ may be carried out precisely as before, and one finds that $\langle T_\mu{}^\nu \rangle$ again has the form given in Eq. (8), except that the constant K is replaced by K_T, where

$$K_T = \sum_{n=1}^{\infty} (-1)^n \frac{2 + \cosh(na)}{[\cosh(na) - 1]^2} . \tag{14}$$

Thus the only effect of the twist is to insert the factor $(-1)^n$ into the summand. The sum in Eq. (12) may be regrouped into the form

$$\begin{aligned} K_T = -\sum_{n=1}^{\infty} & \{\cosh(2na) - \cosh[(2n-1)a]\} \\ & \times \{2\cosh(2na) + 2\cosh[(2n-1)a] + \cosh(2na)\cosh[(2n-1)a] - 5\} \\ & \times \{\cosh(2na) - 1\}^{-2} \{\cosh[(2n-1)a] - 1\}^{-2} . \end{aligned} \tag{15}$$

Since each factor inside the sum in Eq. (15) is now positive definite, K_T is negative definite. The vacuum energy density of a twisted scalar field on the Misner universe is then positive. Further, since the summand defining K [Eq. (9)] is simply the absolute value of the summand in Eq. (14), it follows that $|K_T| < K$ for all values of a. The twisted scalar field then has positive vacuum energy density, but of smaller absolute value than for the untwisted field.

The divergence of $\langle T_\mu{}^\nu \rangle$ as the Cauchy horizons and/or quasiregular singularity is approached is unaffected by the twist; Eqs. (10)–(13) still hold, except K must be replaced by K_T.

We have also calculated $\langle T_\mu{}^\nu \rangle$ for an untwisted scalar field on the closed Misner universe with three-torus topology ($T^3 \times R^1$). In this case, the points to be identified are

$$(t, x^1, x^2, x^3) \leftrightarrow (t, x^1 + na, x^2 + mb, x^3 + lc) , \tag{16}$$

where n, m, and l take on all integer values from $-\infty$ to $+\infty$, b and c are any nonzero lengths, and a is any nonzero number, as before. The only effect the closure of the x^2 and x^3 dimensions has on $\langle T_\mu{}^\nu \rangle$ is to complicate its form; there are additional terms in the stress energy caused by the imposition of periodic boundary conditions in the x^2 and x^3 directions. Since the resulting expression for $\langle T_\mu{}^\nu \rangle$ is rather messy, we will not give its explicit value here. The important point is that all the additional terms are everywhere finite; thus, near the Cauchy horizons and/or quasiregular singularity, the vacuum stress energy will diverge exactly as described by Eqs. (12) and (13), regardless of whether the topology is $S^1 \times R^3$ or $T^3 \times R^1$.

IV. DISCUSSION

Although we have at this point only established explicitly the divergence of the vacuum energy for this one spacetime, we believe the quantum stress energy will probably diverge at any Taub-NUT-type singularity. Curvature and/or other topological complications (which do not affect the singularity structure) will add additional terms to $\langle T_{\mu\nu} \rangle$, representing particle creation and additional vacuum polarization. It seems clear that in any generic spacetime of this sort these additional contributions to $\langle T_{\mu\nu} \rangle$ will not conspire to exactly cancel the divergent terms.

In the absence of any practical scheme for calculating the back reaction to the vacuum stress energy, we can at this point only conclude that the diverging stress energy is grossly incompatible with the nature of a quasiregular singularity. Since the stress-energy scalar diverges strongly, our best guess is that in a self-consistent solution the quasiregular singularity would be replaced by a curvature singularity. While a diverging negative energy density (as in the case of the nontwisted scalar field) can possibly "erase" a curvature singularity (e.g., quantum effects in a Friedman universe[2]), here the pathological global topology which causes the divergence seems fixed, i.e., unalterable by the back reaction via Einstein's equations. Hence, regardless of the sign of the energy density we expect the following "bootstrap" procedure to occur: a pathological topology (quasiregular singularity) gives rise to a divergent vacuum energy which (via Einstein's equations) will cause a curvature singularity.

Of course, it is conceivable that once the radius of curvature (and/or topological identification length) is smaller than the Planck length, the "foamlike" structure of spacetime on such scales[2,20] will alter the topology and allow the universe to become nonsingular. In view of our current limited understanding of the theory of quantum gravity, such a scenario is at best speculative. The calculation presented in this paper certainly seems to show that quantum effects may make a spacetime more singular rather than less, even to the point of changing the type of singularity involved.[3]

Finally, it is interesting to compare this calculation with the work of Deutsch and Candelas,[15] in which they calculated the vacuum stress energy near an arbitrarily curved perfect conductor. They found that the vacuum energy density generically diverges as S^{-3}, where S is the perpendicular distance to the conducting boundary. The τ^{-3} divergence in the energy density in the Misner universe [Eq. (12)] is perhaps the topological (periodic boundary conditions) version of their conducting boundary (Neumann or Dirichlet boundary conditions) result.

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[1]For a recent review, see F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), pp. 97–206.

[2]For several recent reviews, see *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[3]G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Gravit. 8, 915 (1974).

[4]V. Moncrief, Phys. Rev. D 23, 312 (1981).

[5]C. W. Misner, in *Relativity Theory and Astrophysics I: Relativity and Cosmology, Lectures in Applied Mathematics,* edited by J. Ehlers (American Mathematical Society, Providence, R.I., 1967), pp. 160–169.

[6]C. W. Misner and A. H. Taub, Zh. Eksp. Teor. Fiz. 55, 233 (1968) [Sov. Phys. JETP 28, 122 (1969)].

[7]V. Moncrief, Yale University report, 1982 (unpublished).

[8]D. A. Konkowski and L. C. Shepley, Gen. Relativ. Gravit. 14, 61 (1982).

[9]T. M. Helliwell, D. A. Konkowski, and L. C. Shepley (unpublished).

[10]S. W. Hawking and G. F. R Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973), pp. 170—174.

[11]L. H. Ford, Phys. Rev. D 11, 3370 (1975).

[12]B. S. DeWitt, C. F. Hart, and C. J. Isham, Physica (Utrecht) 96A, 197 (1979).

[13]J. S. Dowker and R. Critchley, J. Phys. A 9, 535 (1976).

[14]P. Candelas and D. Deutsch, Proc. R. Soc. London A354, 79 (1977).

[15]D. Deutsch and P. Candelas, Phys. Rev. D 20, 3063 (1979).

[16]W. G. Unruh, Phys. Rev. D 14, 870 (1976).

[17]S. A. Fulling, L. Parker, and B. L. Hu, Phys. Rev. D 10, 3095 (1974).

[18]B. Berger, Phys. Rev. D 12, 368 (1975).

[19]B. S. DeWitt, Phys. Rep. 19C, 295 (1975).

[20]J. A. Wheeler, in *Relativity, Groups and Topology,* edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, New York, 1963).