PHYSICAL REVIEW D VOLUME 26, NUMBER 5 1 SEPTEMBER 1982

WKB approach to the Schrödinger equation with relativistic kinematics

P. Cea, P. Colangelo, G. Nardulli, G. Paiano, and G. Preparata
Istituto di Fisica Università di Bari, Bari, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, Bari, Italy
(Received 5 April 1982)

We develop a WKB approximation technique to the Schrödinger equation with relativistic kinematics for central confining potentials. This approach gives simple formulas for the eigenvalues of the equation with a large class of potentials; numerical results are in good agreement with the results obtained with other methods even for small quantum numbers. Approximate expressions for the eigenfunctions are also obtained and the interpolation among different regions is discussed.

I. INTRODUCTION

The theoretical analysis of heavy-quark systems, prompted by the e^+e^- data, has been carried out so far in terms of the nonrelativistic Schrödinger equation supplemented by a variety of more or less phenomenological potentials.[1] It is thus that the Schrödinger equation has crept back, several years ago, in the realm of high-energy physics; and the results are certainly remarkable. It would therefore seem natural to try and extend such treatment to other hadronic states, which do not involve heavy quarks such as the "charmed" and the "bottom" quarks. At this point, however, one encounters the difficulty that one does not know how to deal with quark systems which are far from the nonrelativistic regime and furthermore a phenomenological choice of the potential is seen to imply an intolerably large number of new parameters.

Both difficulties have been recently removed in the theoretical framework of anisotropic chromodynamics (ACD),[2-4] where it has been possible to show that (i) the hadronic states are color singlets made out of confined colored quarks; (ii) the hadronic dynamics can be calculated in terms of a systematic perturbative expansion of the quark-pair-creation term of the Hamiltonian; and (iii) the lowest-order terms in the expansion involve the diagonalization of the unperturbed part of the Hamiltonian, and give rise to relativistic Schrödinger-type equations for hadronic wave functions with a well defined (and calculable) potential. The first results regarding the solution of these equations for quarks of different flavors have been published in a recent paper,[4] where it is shown that a very satisfactory spectrum emerges for mesons (all known states are reproduced within a mass range of $\simeq 100$ MeV) in terms of five parameters only: μ^2, "the string tension," and the four quark masses $m_u = m_d$, m_s, m_c, and m_b.

Owing to the success of this first step and the peculiar smoothness character of the interaction among quarks (for instance its linearity at large distance), we believed it very useful to develop a WKB approach to the solutions of this kind of problem. The reason for this is essentially practical, as our calculational program will necessarily involve a simple and mathematically manageable representation of the spectrum and of the wave functions for hadrons whose excitation is arbitrarily high.

In the following we shall see that our WKB approach has indeed the desired characteristics of simplicity and expedience, and we are confident that it will be of valuable help not only in the accomplishment of the ACD program but also in the general analysis of the spectrum equations for confined-quark systems.

The plan of the paper is as follows. In Sec. II we formulate our problem in its simplest form and for any potential; Sec. III deals with the WKB spectrum for the S wave, while the complete S-wave WKB solution for linear potential is given in Sec. IV. The extension of our method to higher partial waves is presented in Sec. V. The applications to a class of interesting and phenomenologically meaningful potentials comprise Sec. VI. Some brief concluding remarks are given in Sec. VII. Finally some details of our calculations are reported in the Appendix.

II. THE SCHRÖDINGER EQUATION WITH RELATIVISTIC KINEMATICS

Our starting point is the following equation:

$$[(-\hbar^2\vec{\nabla}^2+m^2)^{1/2}+V(\vec{r})]\psi(\vec{r})=E\psi(\vec{r}) , \tag{2.1}$$

where the square root of the operator $(-\hbar^2\vec{\nabla}^2+m^2)$ is defined through the following spectral representation:

$$(-\hbar^2\vec{\nabla}^2+m^2)^{1/2}\psi(\vec{r})=\int \frac{d^3k}{(2\pi\hbar)^3}\int d^3r' e^{i\vec{k}\cdot(\vec{r}-\vec{r}')/\hbar}(\vec{k}^2+m^2)^{1/2}\psi(\vec{r}') . \tag{2.2}$$

As for $V(\vec{r})$ we make the physically relevant assumptions that it is central [i.e., $V(\vec{r})=V(r)$] and that it increases without bound as r increases (i.e., it is a confining potential).

If we now decompose the exponential $e^{i\vec{k}\cdot\vec{r}/\hbar}$ in spherical harmonics [$j_l(kr)$ are the spherical Bessel functions]

$$e^{i\vec{k}\cdot\vec{r}/\hbar}=4\pi\sum_{l,m} i^l j_l\left[\frac{kr}{\hbar}\right]Y_l^{m*}(\hat{k})Y_l^m(\hat{r}) , \tag{2.3}$$

it can be easily checked that $\psi(\vec{r})$ takes on the following form:

$$\psi(\vec{r})=Y_l^m(\hat{r})\Phi_l(r) , \tag{2.4}$$

and that $\Phi_l(r)$ satisfies the integral equation

$$[V(r)-E]\Phi_l(r)+\frac{2}{\pi\hbar}\int_0^\infty dr' r'^2\int_0^\infty dk\frac{k^2}{\hbar^2}(k^2+m^2)^{1/2}j_l\left[\frac{kr}{\hbar}\right]j_l\left[\frac{kr'}{\hbar}\right]\Phi_l(r')=0 . \tag{2.5}$$

By introducing the notations

$$u_l(r)=r\Phi_l(r)$$

and

$$\chi_l(\rho)=\rho j_l(\rho) ,$$

we can rewrite (2.5) as

$$[V(r)-E]u_l(r)+\frac{2}{\pi\hbar}\int_0^\infty dr'\int_0^\infty dk(k^2+m^2)^{1/2}\chi_l\left[\frac{kr}{\hbar}\right]\chi_l\left[\frac{kr'}{\hbar}\right]u_l(r')=0 . \tag{2.6}$$

The problem we must solve is to find solutions of (2.6) subject to the boundary conditions

$$u_l(0)=0 \tag{2.7}$$

and

$$u_l(r)\underset{r\to\infty}{\longrightarrow}0 . \tag{2.8}$$

In order to simplify our discussion of the WKB solutions of Eq. (2.6) we shall need to consider also the region $r<0$. If we set $V(-r)=V(r)$, then from (2.6) we obtain the "parity" relations

$$u_l(-r)=(-1)^{l+1}u_l(r) . \tag{2.9}$$

III. THE WKB S-WAVE SOLUTIONS: SPECTRUM AND EIGENFUNCTIONS

For S wave ($l=0$) Eq. (2.6) becomes

$$[V(r)-E]u_0(r)+\frac{2}{\pi\hbar}\int_0^\infty dr'\int_0^\infty dk\, g(k)\sin\frac{kr}{\hbar}\sin\frac{kr'}{\hbar}u_0(r')=0 , \tag{3.1}$$

where $g(k)=(k^2+m^2)^{1/2}$. We look for solutions of this equation of the form

$$u_0(r)=e^{\sigma_1(r)}e^{(i/\hbar)\sigma(r)} . \tag{3.2}$$

By making use of the "parity" relations (2.9), (3.1) can be rewritten as

$$[V(r)-E]e^{\sigma_1(r)}e^{(i/\hbar)\sigma(r)}+\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}dr'g(k)e^{(i/\hbar)k\cdot(r-r')}e^{\sigma_1(r')}e^{(i/\hbar)\sigma(r')}=0 . \tag{3.3}$$

In the WKB limit ($\hbar\to 0$), the double integral appearing in (3.3) can be evaluated by the saddle-point method. The saddle point is easily seen to be determined by the equations

$$\sigma'(r')+\frac{\hbar}{i}\sigma_1'(r')=k , \quad r'=r . \tag{3.4}$$

At the zeroth order in $\hbar$ we obtain

$$[\sigma'(r)^2+m^2]^{1/2}=E-V(r) . \tag{3.5}$$

Note that the square root appearing in (3.5) is, by definition, non-negative. Thus, unlike the nonrelativistic case, the WKB approximation applies only in the region $r<r_M$, where r_M is defined by

$$V(r_M)=E . \tag{3.6}$$

An equation for σ_1 can be obtained by developing the function $g(k)=(k^2+m^2)^{1/2}$ around the saddle point (3.4). We thus get

$$\left[\left[\sigma'+\frac{\hbar}{i}\sigma_1'\right]^2+m^2\right]^{1/2} +\sigma''\frac{\hbar}{2i}\frac{2m^2}{\{(\sigma'+(\hbar/i)\sigma_1']^2+m^2\}^{3/2}}=0 . \tag{3.7}$$

Equations (3.5) and (3.7) determine the σ and σ_1 functions up to first order in $\hbar$. Note that, as in the nonrelativistic case, the WKB method fails around the "classical" turning point r_0, defined by

$$V(r_0)=E-m . \tag{3.8}$$

The WKB solutions of (3.3) in the regions

(I): $0<r<r_0$,

(II): $r_0<r<r_M$

can be matched in the standard fashion[5,6] and one obtains

$$\begin{aligned} u_{\rm I}(r)&=[E-V(r)]^{1/2}\frac{D}{\sqrt{p}}\times\sin\left[\frac{1}{\hbar}\int_r^{r_0}p(x)dx+\frac{\pi}{4}\right] ,\\ u_{\rm II}(r)&=[E-V(r)]^{1/2}\frac{D}{2\sqrt{p}}\times\exp\left[\frac{-1}{\hbar}\int_{r_0}^{r}|p(x)|dx\right] , \end{aligned} \tag{3.9}$$

where

$$p=p(r)=+\{[E-V(r)]^2-m^2\}^{1/2} . \tag{3.10}$$

The spectrum is determined by imposing the boundary condition $u(0)=0$, which, according to (3.9), becomes

$$\int_0^{r_0}p(x)dx=\pi(n+\tfrac{3}{4}) . \tag{3.11}$$

IV. COMPLETE S-WAVE WKB SOLUTION FOR THE LINEAR POTENTIAL

The method just described provides us (approximate) eigenvalues and eigenfunctions for the equation (3.1) in the region $r\in[0,r_M]$ (with the exception of a small neighborhood of the "classical turning point" r_0). In the case of a linear potential $V(r)=\mu^2 r/\hbar$, the WKB method allows us to write a solution also for $r>r_M$.[7] Let us see how. The starting point is again (3.1). By defining the Fourier transform

$$\tilde{u}(p)=\int_0^\infty dr'\sin\left[\frac{p}{\hbar}r'\right]u(r') , \tag{4.1}$$

we find that $\tilde{u}(p)$ satisfies the equation

$$[(p^2+m^2)^{1/2}-E]\tilde{u}(p)+\frac{2}{\pi\hbar}\int_0^\infty dr\,V(r)\int_0^\infty dp'\sin\frac{pr}{\hbar}\sin\frac{p'r}{\hbar}\tilde{u}(p')=0 , \tag{4.2}$$

which can be extended to the interval $]-\infty,+\infty[$:

$$[(p^2+m^2)^{1/2}-E]\tilde{u}(p)+\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dr\,V(r)\int_{-\infty}^{+\infty}dp'e^{(i/\hbar)r(p-p')}\tilde{u}(p')=0\ . \tag{4.3}$$

In order to apply the saddle-point method to the evaluation of the double integral, we set

$$V(r)=\frac{\mu^2}{\hbar}(r^2+\epsilon)^{1/2}\,|_{\epsilon\to 0^+}\ . \tag{4.4}$$

With this definition (4.3) is formally identical with (3.3) if we substitute $m^2\leftrightarrow\epsilon$ and $V(r)\leftrightarrow(p^2+m^2)^{1/2}$. Looking for solutions $\tilde{u}(p)=e^{\Sigma_1(p)}e^{(i/\hbar)\Sigma(p)}$, we obtain the results

$$\tilde{u}(p)=A\sin G(p)\ ,\quad \forall p\in[0,(E^2-m^2)^{1/2}[\ , \tag{4.5}$$

where

$$G(p)=\frac{1}{\mu^2}\int_0^p[E-(p^2+m^2)^{1/2}]dp\ . \tag{4.6}$$

We observe that, in analogy with the findings of Sec. III, the WKB method gives a solution for $\tilde{u}(p)$ only for $p<(E^2-m^2)^{1/2}$. However, this result can be used to derive the large-r behavior of $u(r)$. The function $u(r)$ is given in fact by

$$u(r)=\frac{2}{\pi\hbar}\int_0^{\infty}\sin\frac{pr}{\hbar}\tilde{u}(p)dp \tag{4.7}$$

and in the large-r region the integral is dominated by the small-p region, where $\tilde{u}(p)$ is known and given by (4.5) and (4.6). So that we can conclude that in the region (III) $r\gg E$, $u(r)$ is given by

$$u_{\rm III}(r)\simeq\int_0^{\infty}\sin\frac{pr}{\hbar}\sin G(p)dp\ . \tag{4.8}$$

The asymptotic behavior of $u(r)$ can be evaluated by the Laplace method of asymptotic estimates[6] (after having deformed the contour in such a way that it avoids the cut of the integrand occurring at $p=im$). One thus gets

$$u_{\rm III}(r)\underset{r\to\infty}{\longrightarrow}\frac{e^{-mr}}{r^{5/2}}\ . \tag{4.9}$$

This is the correct behavior of the wave function, as can independently be proved[8] by studying the general properties of Eq. (3.1) for large r.

By collecting all the results obtained, we remark that the WKB method provides eigenfunctions in the regions (I) $0<r<(E-m)$, (II) $(E-m)<r<E$, and (III) $r\gg E$. We would like to show now that the function[9]

$$u(r)=\int_0^{\infty}\cos\left[\frac{pr}{\hbar}-G(p)\right]dp \tag{4.10}$$

correctly extrapolates to the WKB solutions in the three regions defined above, thus realizing a smooth approximation to the large-quantum-number solution of (3.1) over all the r range. By evaluating (4.10) in the regions (I) and (II) by the saddle-point method (see the Appendix), it is easy to see that it coincides with (3.9). In the region (III) the Laplace method of asymptotic expansions can be used, and again (4.10) is seen to behave as (4.9). Thus we conclude that (4.10) provides a WKB approximation to the solutions of (3.1) without restrictions. Incidentally, we observe that the quantization condition

$$u(0)=\int_0^{\infty}\cos[G(p)]dp=0 \tag{4.11}$$

for large n coincides with (3.11).

V. HIGHER PARTIAL WAVES

The approach described in Secs. III and IV can be naturally extended to the case of $l\neq 0$, i.e., to the solution of Eq. (2.6).

We look for WKB solutions of the type

$$u_l(r)=e^{\sigma_1(r)}\chi_l\left[\frac{\sigma(r)}{\hbar}\right], \tag{5.1}$$

where χ_l are the Riccati-Bessel functions, which, for large values of the argument, behave as

$$\chi_l\left[\frac{\sigma}{\hbar}\right]=\frac{\sigma}{\hbar}j_l\left[\frac{\sigma}{\hbar}\right]\to\sin\left[\frac{\sigma}{\hbar}-\frac{l\pi}{2}\right]$$

or (5.2)

$$\chi_l\left[\frac{\sigma}{\hbar}\right]=\frac{\sigma}{\hbar}y_l\left[\frac{\sigma}{\hbar}\right]\to-\cos\left[\frac{\sigma}{\hbar}-\frac{l\pi}{2}\right].$$

Note that, for $\sigma(r)\gg\hbar[l(l+1)]^{1/2}$, (5.1) can be written as

$$u_l(r)=e^{\sigma_1(r)}\exp\left\{i\left[\frac{\sigma(r)}{\hbar}-\frac{l\pi}{2}\right]\right\}, \tag{5.3}$$

whereas for $\sigma(r)\ll\hbar[l(l+1)]^{1/2}$ we are only interested in the regular solution

$$u_l(r)=e^{\sigma_1(r)}\frac{\sigma(r)}{\hbar}j_l\left[\frac{\sigma(r)}{\hbar}\right]. \tag{5.4}$$

In the latter region, Eq. (2.6) has the solution

$$u_l^{\rm I}(r)=C\frac{\tilde{p}r}{\hbar}j_l\left[\frac{\tilde{p}r}{\hbar}\right], \tag{5.5}$$

where $\tilde{p}=(E^2-m^2)^{1/2}$. This can be checked by direct substitution. Solution (5.4) is compatible with (5.5) provided

$$\sigma(r)\underset{r\to 0}{\longrightarrow}\tilde{p}r\,,\qquad \sigma_1(r)\underset{r\to 0}{\longrightarrow}\ln C=\text{const}\,. \tag{5.6}$$

In the region $\sigma(r)\gg\hbar[l(l+1)]^{1/2}$, using the representation (5.3), we encounter the problem already solved in Sec. III. Thus we may write

$$\begin{aligned} r<r_0:\ u_l^{\rm II}(r)&=\frac{D}{\sqrt{p}}[E-V(r)]^{1/2}\sin\left[\frac{1}{\hbar}F(r)+\frac{\pi}{4}\right],\\ r_0<r<r_M:\ u_l^{\rm III}(r)&=\frac{D}{2\sqrt{p}}[E-V(r)]^{1/2}\exp\left[-\frac{1}{\hbar}\int_{r_0}^{r}|p(x)|\,dx\right], \end{aligned} \tag{5.7}$$

where r_0, r_M, and $p(x)$ are defined in Eqs. (3.8), (3.6), and (3.10), respectively, and

$$F(r)=\int_r^{r_0}p(x)dx\,. \tag{5.8}$$

In order to connect regions (I) and (II) we rewrite $u_l^{\rm II}(r)$ in the form

$$u_l^{\rm II}(r)=D(-1)^{n+1}\frac{[E-V(r)]^{1/2}}{[p(r)]^{1/2}}\chi_l[g(r)+A]\,, \tag{5.9}$$

where $\chi_l(\rho)=\rho j_l(\rho)$, and ($n$ is any integer)

$$\begin{aligned} A&=-\int_0^{r_0}p(x)dx+\frac{l\pi}{2}-\frac{\pi}{4}+n\pi\,,\\ g(r)&=\int_0^{r}p(x)dx\,. \end{aligned} \tag{5.10}$$

It is easy to check that Eq. (5.9), supplemented by the relations (5.10), coincides with (5.7) for any integer n. Furthermore, (5.9) coincides with $u_l^{\rm I}(r)$ if

$$C=D(-1)^{n+1}\sqrt{E}\frac{1}{\sqrt{\tilde{p}}} \tag{5.11}$$

and

$$A=0\,. \tag{5.12}$$

The latter equation gives rise to the quantization condition in the form

$$\int_0^{r_0}p(x)dx=\pi\left[n'+\frac{l}{2}+\frac{3}{4}\right], \tag{5.13}$$

where $n'=(n-1)$ is a non-negative integer. For $r\gg E$ the WKB solution for a linear potential can be obtained by a method analogous to the one employed in Sec. IV. One considers the Schrödinger equation for

$$\tilde{u}_l(p)=\int_0^{\infty}dr'\chi_l\left[\frac{pr'}{\hbar}\right]u_l(r') \tag{5.14}$$

and for $p<(E^2-m^2)^{1/2}$, one obtains the solution

$$\tilde{u}_l(p)=A\chi_l[G(p)]\,, \tag{5.15}$$

where $G(p)$ is defined in (4.6). For larger r, the small-p region dominates and $u_l(r)$ is given by

$$u_l(r)\sim\int_0^{\infty}dp\,\chi_l\left[\frac{pr}{\hbar}\right]\chi_l[G(p)]\,. \tag{5.16}$$

VI. APPLICATIONS

In view of the applications to concrete physical problems it turns out that it is important to generalize (2.1) to

$$[(-\hbar^2\vec{\nabla}^2+m_1{}^2)^{1/2}+(-\hbar^2\vec{\nabla}^2+m_2{}^2)^{1/2}-V(r)]\psi(\vec{r})=E\psi(\vec{r})\,. \tag{6.1}$$

This equation describes in ACD the general $q\bar{q}$ system to lowest order.

Instead of (3.8), the equation for the turning point r_0 is

$$V(r_0)=E-m_1-m_2\,. \tag{6.2}$$

The region where the WKB approximation is applicable is $0<r<r_M$, where r_M is given by

$$V(r_M)=E-|m_1-m_2|\,. \tag{6.3}$$

The WKB wave functions for $l=0$ are as follows:

$$\text{(I)}\ 0<r<r_0:\ u_{\rm I}(r)=\left[\frac{[E-V(r)]^4-(m_1{}^2-m_2{}^2)^2}{[E-V(r)]^3}\right]^{1/2}\frac{D}{\sqrt{p}}\sin\left[\frac{1}{\hbar}\int_r^{r_0}p(x)dx+\frac{\pi}{4}\right] \tag{6.4}$$

and

$$\text{(II)}\ r_0<r<r_M:\ u_{\rm II}(r)=\left[\frac{[E-V(r)]^4-(m_1{}^2-m_2{}^2)^2}{[E-V(r)]^3}\right]^{1/2}\frac{D}{2\sqrt{p}}\exp\left[-\frac{1}{\hbar}\int_{r_0}^{r}|p(x)|dx\right], \tag{6.5}$$

where

$$p(r)=\left[\frac{[E-V(r)]^2}{4}-\frac{m_1{}^2+m_2{}^2}{2}+\frac{(m_1{}^2-m_2{}^2)^2}{4[E-V(r)]^2}\right]^{1/2}. \tag{6.6}$$

The spectrum for $l=0$ and $l\neq 0$ is obtained by substituting (6.6) in (3.11) and (5.13), respectively.

A. The linear potential

Choosing $V(r)=\mu^2 r$, the spectrum is given by

$$\int_0^{(E-m_1-m_2)/\mu^2}\left[\frac{(E-\mu^2 r)^2}{4}-\frac{m_1{}^2+m_2{}^2}{2}+\frac{(m_1{}^2-m_2{}^2)^2}{4(E-\mu^2 r)^2}\right]^{1/2}dr=\pi(n+l/2+\tfrac{3}{4}), \tag{6.7}$$

which for $m_1=m_2=m$ reduces to $(n,l=0,1,\dots)$

$$\frac{E}{2}\left[\frac{E^2}{4}-m^2\right]^{1/2}-m^2\ln\left[\frac{E}{2m}+\left[\frac{E^2}{4m^2}-1\right]^{1/2}\right]=\mu^2\pi(n+l/2+\tfrac{3}{4}). \tag{6.8}$$

From these equations one sees that for light quarks ($m_{1,2}\ll\mu$) one gets a set of linear Regge trajectories without odd daughters. In Fig. 1 we report the Chew-Frautschi plot for the $u\bar{u}$ system (choosing $\mu=0.57$ GeV and $m_u=100$ MeV). The following facts ought to be stressed.

(i) The linearity of Regge trajectories for a linear potential is a peculiar property of relativistic kinematics. This property is lost in the nonrelativistic limit.

(ii) The WKB approximation is a very good approximation (within 1%) for the linear potential also for $n=0$ and $l=0$. This has been checked by comparing our results with the numerical calculation performed by the Multhopp method.[4]

(iii) Our knowledge of the meson spectrum clearly indicates that the linear potential describes correctly only the long-range part of the confining potential.

(iv) For high excitations (n,l) we can compare the structure of the ACD wave function with the one predicted in quark geometrodynamics (QGD).[10] Figures 2(a) and 2(b) show a comparison between the two theories for $n=7$ and $l=0$, for both the wave function and its square. We note that *in the average* the QGD distributions resemble very closely the ACD ones. Deviations are, as expected, concentrated at the "bag boundary," i.e., $r\sim r_M$. The latter fact explains the adequacy of the QGD description of high-energy phenomena.

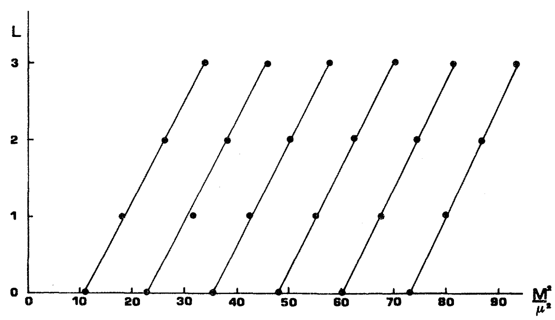

FIG. 1. The Chew-Frautschi plot for the $u\bar{u}$ system ($m_u=100$ MeV, $\mu=0.57$ GeV) showing the linearity of the trajectories and the absence of odd daughters.

B. $V(r)=\mu r^{\alpha}$

This class of potentials has been considered, within the nonrelativistic Schrödinger equation, by many authors.[1] In particular, Martin[11] considers the potential

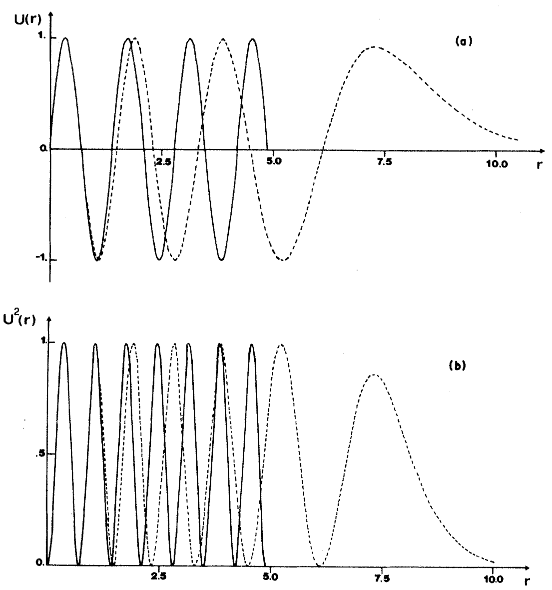

FIG. 2. (a) Comparison between the WKB eigenfunction of the Schrödinger equation with relativistic kinematics and linear potential $V(r)=\mu^2 r$ (dashed line) and the solution calculated within the QGD bag model (solid line). (b) Comparison between the square of the WKB eigenfunction with linear potential (dashed line) and the square of QGD-bag-model solution (solid line). Values of the parameters are radial quantum number $n=7$, angular quantum number $l=0$, $m_q=m_{\bar{q}}=0.18$. All quantities are in units of string tension.

$$V(r)=A+Br^{\alpha} \tag{6.9}$$

with $\alpha=0.1$. By applying our WKB formulas to the potential (6.9) and with the parameters of Ref. 11 we obtain the results which are compared with those of Ref. 11 in Table I. As we can see our WKB results are in good agreement with those of Martin for heavy quarks (good nonrelativistic approximation), whereas for the s quark the discrepancies can be understood by Martin's neglect of relativistic effects. We would like to stress, however, that this kind of potential, with $\alpha\neq 1$, cannot represent a good phenomenological approach for large r, as they fail to give the experimentally observed linearity of Regge trajectories.

We believe in fact that, from the phenomenological point of view, a linear potential plus some deviation at short distances and spin-spin corrections does indeed represent the most likely candidate to accurately account for the most important aspects of the meson spectrum (irrespective of the quark flavor).

C. Linear plus logarithmic potential

It has been pointed out[9] that a linear potential alone cannot explain the (approximately) equal splittings between the low-lying members of the J/ψ and the Υ families. Thus an ACD-inspired potential has been proposed, which has the feature that for $r > 1F$ it is linear, but for short distances it behaves logarithmically.[12] This potential has the following expression:

$$V(r)=\beta^2 r-g^2 K_0(\lambda r)+V_0 , \tag{6.10}$$

where V_0 is a constant shift and $K_0(x)$ is a Bessel function, which for small x diverges logarithmically, while at large x dies off exponentially. In Fig. 3 the potential (6.10) is plotted with realistic values for the parameters β, g^2, and λ (Ref. 9) (see later). In order to apply our WKB formulas we note that $V(r)$ satisfies the requirements of Sec. III, and that the logarithmic potential can be treated as a limit of a r^{α} potential for $\alpha\to 0$. Solving the spectrum equation numerically, for the following parameters

$$\begin{aligned}&\beta=0.4\text{ GeV},\quad g^2=0.7\text{ GeV},\\&\lambda=0.6\text{ GeV},\quad V_0=-0.2\text{ GeV}\\&m_u=m_d=200\text{ MeV},\quad m_s=300\text{ MeV},\\&m_c=1.5\text{ GeV},\quad m_b=4.85\text{ GeV},\end{aligned} \tag{6.11}$$

TABLE I. Comparison between Martin's results (Ref. 11) and WKB approximation for the spectrum of $q\bar{q}$ states with the potential $V(r)=A+B(r/r_0)^{\alpha}$. $A=-8.064$ GeV, $B=6.8698$ GeV, $r_0=1\text{ GeV}^{-1}$, $\alpha=0.1$. Results are in GeV.

$s\bar{s}$ states $m_s=0.518$ GeV	Martin	WKB	$c\bar{c}$ states $m_c=1.8$ GeV	Martin	WKB	$b\bar{b}$ states $m_b=5.174$ GeV	Martin	WKB
ϕ	1.02	0.78	J/ψ	3.095	2.98	Υ	9.46	9.38
$E(1^{++})$	1.42	1.16	$\psi^{\rm I}$	3.687	3.60	$\Upsilon^{\rm I}$	10.025	9.99
ϕ'	1.634	1.42	$\psi^{\rm III}$	4.032	3.96	$\Upsilon^{\rm II}$	10.36	10.33
			Average P state	3.502	3.35	$\Upsilon^{\rm III}$	10.60	10.57
			$\psi^{\rm II}$ (D state)	3.787	3.60	$\Upsilon^{\rm IV}$	10.76	10.75
			$\psi^{\rm IV}$	4.28	4.21	$1P$ state	9.861	9.74
						$2P$ state	10.242	10.17

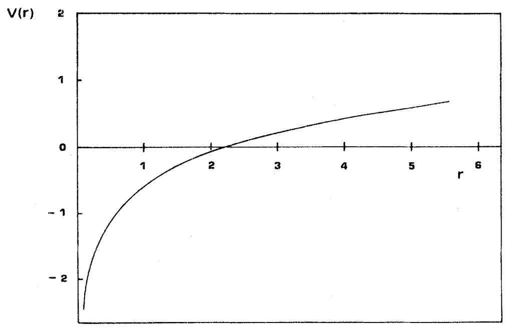

FIG. 3. Plot of the potential $V(r)=\beta^2 r - g^2 K_0(\lambda r) + V_0$. Values of the parameters are $\beta=0.4$ GeV, $g^2=0.7$ GeV, $\lambda=0.6$ GeV, $V_0=-0.2$ GeV. Units of r are in GeV^{-1}.

we derive the results reported in Table II. We have also calculated the $t\bar{t}$ spectrum for two different values of the top-quark mass ($m_t=20$ and 25 GeV).

In view of the absence of spin-spin splitting the results we obtain reproduce quite satisfactorily the experimental data. In particular, the nearly equal splittings of the lowest members of the J/ψ and Υ families, and the linearity of light-quark Regge trajectories are especially noteworthy.

TABLE II. WKB results for the spectrum of $q\bar{q}$ states as a function of quantum numbers n,l. The potential is $V(r)=V_0+\beta^2 r - g^2 K_0(\lambda r)$ [see Eqs. (6.11) and (6.12)]. Results are in GeV. We present also predictions for radial excitations of the $t\bar{t}$ state for two different m_t masses.

n \ l	0	1	2	3	4
			$u\bar{u}$ states		
0	.99	1.37	1.66	1.93	2.15
1	1.66	1.93	2.15	2.36	2.55
2	2.15	2.36	2.55	2.73	2.90
3	2.55	2.73	2.90	3.06	3.21
4	2.90	3.06	3.21	3.36	3.49
			$s\bar{s}$ states		
0	1.11	1.48	1.78	2.03	2.25
1	1.78	2.03	2.25	2.45	2.64
2	2.25	2.45	2.64	2.81	2.98
3	2.64	2.81	2.98	3.14	3.29
4	2.98	3.14	3.29	3.43	3.57
			$c\bar{c}$ states		
0	3.13	3.45	3.68	3.88	4.06
1	3.68	3.88	4.06	4.22	4.38
2	4.06	4.22	4.38	4.52	4.65
3	4.38	4.52	4.65	4.78	4.91
4	4.65	4.78	4.91	5.03	5.14
			$b\bar{b}$ states		
0	9.50	9.82	10.02	10.19	10.33
1	10.02	10.19	10.33	10.46	10.57
2	10.33	10.46	10.57	10.68	10.78
3	10.57	10.68	10.78	10.88	10.97
4	10.78	10.88	10.97	11.07	11.15

m_t \ n	0	1	2	3	4
			$t\bar{t}$ states		
20	39.39	39.90	40.17	40.36	40.52
25	49.32	49.84	50.10	50.29	50.44

D. The linear plus magnetic potential

The first investigations[3] on the structure of the meson spectrum in ACD have shown that, to lowest order in quark-pair creation, the potential consists of a linear part plus a magnetic interaction. Leaving aside its spin-dependent part, which can be treated perturbatively,[4] the S-wave magnetic potential has a highly nonlocal structure, which in momentum space is given by the following expression:

$$V_{\text{mag}}(k) = -\frac{1}{\pi}\frac{\mu^2}{k}\left[\operatorname{arcsinh}\frac{k}{m_1} + \operatorname{arcsinh}\frac{k}{m_2}\right] . \tag{6.12}$$

This potential can easily be treated by adding it to the kinetic term in the Schrödinger equation, and by applying our previous WKB methods to the modified kinetic operator:

$$O(k) = (k^2+m_1{}^2)^{1/2} + (k^2+m_2{}^2)^{1/2} + V_{\text{mag}}(k) . \tag{6.13}$$

We obtain the following wave function:

$$u(r) = \int_0^\infty \cos\left[\frac{pr}{\hbar} - G(p)\right] dp , \tag{6.14}$$

where

$$G(p) = \frac{1}{\mu^2}\int_0^p dt[E - O(t)] . \tag{6.15}$$

In the large-E limit $u(r)$, as given by Eq. (6.14), can be evaluated once more by the saddle-point method; we thus recover the WKB wave function with σ and σ_1 given by Eqs. (A2) and (A5) of the

Appendix. The eigenvalue equation is, as expected,

$$\int_0^\infty dp \cos[G(p)]=0 , \qquad (6.16)$$

which can be also solved by the saddle-point method in the large-quantum-number limit.

Once again we obtain results which agree within a few percent with those obtained by solving the Schrödinger equation numerically by the Multhopp method.[4] This comparison is presented in Table III, where we limit ourselves to the two lowest $q\bar{q}$ states; as a matter of fact, for higher quark masses the contribution of magnetic potential becomes negligible as compared to the linear one.

TABLE III. Comparison between WKB method and Multhopp's method results (Ref. 4) for the spectrum of the two lowest $q\bar{q}$ states ($u\bar{u}$ and $s\bar{s}$), with the potential $V=\mu^2 r+V_{\text{mag}}$, where the magnetic potential V_{mag} is given by Eq. (6.12). All results are in units of "string tension" μ.

n	WKB	Multhopp
	$m_q=m_{\bar{q}}=0.18$	
0	1.032	1.180
1	3.015	3.039
2	4.385	4.395
3	5.487	5.493
4	6.433	6.437
5	7.273	7.276
6	8.036	8.038
7	8.739	8.741
8	9.394	9.396
9	10.010	10.011
	$m_q=m_{\bar{q}}=0.31$	
0	1.796	1.878
1	3.580	3.595
2	4.862	4.869
3	5.912	5.916
4	6.821	6.824
5	7.634	7.635
6	8.375	8.376
7	9.060	9.061
8	9.701	9.702
9	10.304	10.303

VII. CONCLUSIONS

Tables I, II, and III are a relevant summary of the effectiveness of the WKB method, when applied to look for solutions of Schrödinger equations with relativistic kinematics, with a class of confining realistic potentials.

Although we have concentrated our attention in Sec. VI mainly on ACD-inspired potentials, the particular simplicity afforded by the WKB method in dealing with confining potentials of considerable complexity could also be of some aid in those research programs, such as the ones carried out in Ref. 13, where a realistic $q\bar{q}$ potential requires an extremely structured expression.

From work in progress we can already see that with the WKB method high-energy behavior of ACD can be analyzed in a most effective and physically transparent fashion. But to this we shall return in future publications.

APPENDIX

In this appendix we want to describe in some details the application of saddle-point techniques to the study of Eq. (3.1). We restrict ourselves to the case in which $V(r)=\mu^2 r/\hbar$ but $g(k)$ arbitrary (this generalization is useful in the presence of the magnetic potential studied in Sec. VI D).

Moreover we want to prove explicitly that the solution

$$u(r)=\int_0^\infty dp \cos\left[\frac{pr}{\hbar}-G(p)\right] , \qquad (A1)$$

where

$$G(p)=\frac{1}{\mu^2}\int_0^p [E-g(k)]dk \qquad (A2)$$

coincides in the region $0<r<r_M$ with the previous solutions.

Let us start with Eq. (3.3) which reduces to Eq. (3.1) by using the "parity" relations (2.9). The r' integral in (3.3) can be evaluated by the saddle-point method:

$$\frac{1}{2\pi\hbar}\int dr' \exp\left\{\frac{i}{\hbar}\left[\sigma(r')+\frac{\hbar}{i}\sigma_1(r')-r'k\right]\right\}$$
$$=\frac{1}{\sqrt{2\pi\hbar}}\frac{1}{[\,|\sigma''(r_0)+(\hbar/i)\sigma_1''(r_0)|\,]^{1/2}}\exp\left\{\frac{i}{\hbar}\left[\sigma(r_0)+\frac{\hbar}{i}\sigma_1(r_0)-r_0 k\right]\right\}e^{i\theta_1} , \qquad (A3)$$

where $\theta_1=\pi/4$ and r_0 is the saddle point defined by the equation

$$\sigma'(r_0)+\frac{\hbar}{i}\sigma_1'(r_0)=k \ . \tag{A4}$$

The k integral in Eq. (3.3) can be evaluated once again by the saddle-point method. One obtains

$$\begin{aligned}\frac{1}{\sqrt{2\pi\hbar}}\int \frac{dk\, g(k)e^{i\theta_1}}{[\,|\sigma''(r_0)+(\hbar/i)\sigma_1''(r_0)|\,]^{1/2}}\exp\left\{\frac{i}{\hbar}\left[\sigma(r_0)+\frac{\hbar}{i}\sigma_1(r_0)+k(r-r_0)\right]\right\} \\ =\frac{g(k^*)}{\left[\left|\frac{dr_0}{dk}\right|\,|\sigma''(r_0)+(\hbar/i)\sigma_1''(r_0)|\right]^{1/2}_{k=k^*}}e^{i(\theta_1+\theta_2)} \\ \times\exp\left\{\frac{i}{\hbar}\left[\sigma(r_0(k^*))+\frac{\hbar}{i}\sigma_1(r_0(k^*))+k^*[r-r_0(k^*)]\right]\right\},\end{aligned} \tag{A5}$$

where $\theta_2=-\pi/4$ and the saddle point k^* is obtained by

$$0=r-r_0(k^*)+\left[\sigma'(r_0(k^*))+\frac{\hbar}{i}\sigma_1(r_0(k^*))-k^*\right]\frac{dr_0}{dk}\bigg|_{k^*}, \tag{A6}$$

which, using (A4) reduces to

$$r_0(k^*)=r \ . \tag{A7}$$

Now we consider

$$\begin{aligned}\left[\sigma''(r_0)+\frac{\hbar}{i}\sigma_1''(r_0)\right]\frac{dr_0}{dk}\bigg|_{r_0=r}&=\frac{d}{dr_0}\left[\sigma'(r_0)+\frac{\hbar}{i}\sigma_1'(r_0)\right]\frac{dr_0}{dk} \\ &=\frac{d}{dk}\left[\sigma'(r_0)+\frac{\hbar}{i}\sigma_1'(r_0)\right]=1 \ ,\end{aligned} \tag{A8}$$

where the last equality in Eq. (A8) is a consequence of Eq. (A4).

By using result (A8) and the fact that $\theta_1+\theta_2=0$ we obtain, in zeroth order in $\hbar$,

$$g(\sigma')=E-V(r) \ , \tag{A9}$$

which must be compared with Eq. (3.5). Equation (A9) must be solved in order to obtain $\sigma'(r)$. Let us observe, however, that for $V(r)=\mu^2 r/\hbar$ we get, from (A9),

$$\begin{aligned}\sigma(r)&=\int \sigma'(r)dr \\ &=-\frac{\hbar}{\mu^2}\int \sigma' dg \\ &=\sigma'(r)\left[r-\frac{E\hbar}{\mu^2}\right]+\frac{\hbar}{\mu^2}\int g(\sigma')d\sigma' \\ &\quad+\text{const} \ .\end{aligned} \tag{A10}$$

In order to obtain $\sigma_1(r)$ we develop the function $g(k)$ around the saddle point defined by Eqs. (A4) and (A7). Equation (3.7) is generalized as follows:

$$V(r)-E+g\left[\sigma'+\frac{\hbar}{i}\sigma_1'\right]+\frac{\hbar}{2i}\frac{d^2g}{dk^2}\bigg|_{k=\sigma'+(\hbar/i)\sigma_1'}\sigma''=0 \tag{A 11}$$

from which we obtain, by considering $O(\hbar)$ terms and by using (A9),

$$\begin{aligned}\sigma_1'&=-\frac{1}{2}\left[\frac{1}{g'(k)}\frac{dg}{dk}\right]_{k=\sigma'}\sigma'' \\ &=\frac{d}{dr}\left[\ln\left(\left|\frac{dg}{dk}\right|^{-1/2}\right)\right]_{k=\sigma'},\end{aligned} \tag{A12}$$

which gives

$$e^{\sigma_1}=\frac{c}{(\,|dg/dk\,|_{k=\sigma'})^{1/2}}\ . \tag{A13}$$

For the case $g(k)=(k^2+m^2)^{1/2}$, Eqs. (A13) and (A9) solve the problem in the form given in Eq. (3.9).

Finally we evaluate the integral given by Eq. (A1); once again we use the saddle-point method. The saddle point p^* is defined by the equation

$$\mu^2\frac{r}{\hbar}-E+g(p^*)=0\ , \tag{A14}$$

which coincides with Eq. (A9). So it is easy to find that in the large-E limit

$$u(r)=\frac{1}{[g'(p^*)]^{1/2}}\exp\left\{i\left[p^*r-\frac{E}{\mu^2}p^*+\frac{1}{\mu^2}\int_0^{p^*}g(p)dp+\frac{\pi}{4}\right]\right\}. \tag{A15}$$

By taking into account that Eq. (A14) coincides with (A9) for $V(r)=\mu^2r/\hbar$ and that the exponential factor in (A15) is equal to

$$e^{(i/\hbar)\sigma(r)+i\pi/4}$$

[compare with Eq. (A10)], we conclude that, at the same level of approximation, the function $u(r)$ given by Eq. (A1) coincides with the WKB results.

Note added in proof. The angles θ_1 and θ_2 which appear in Eqs. (A3) and (A5) are not equal to $\pi/4$ and $-\pi/4$, respectively, for all the values of k. However it is always true that $\theta_1+\theta_2=0$. Thus the results of the Appendix are unaffected. We thank Professor M. Villani for a discussion on this subject.

[1]C. Quigg and J. L. Rosner, Phys. Rep. 56C, 167 (1979) and references therein.

[2]G. Preparata, Phys. Lett. 102B, 327 (1981); Nuovo Cimento 66A, 205 (1981).

[3]J. L. Basdevant and G. Preparata, Nuovo Cimento 67A, 19 (1982).

[4]P. Colangelo, J. L. Basdevant, and G. Preparata, Nuovo Cimento A (to be published); see also P. Colangelo, thesis, Universitá di Bari (unpublished).

[5]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958).

[6]R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic, London, 1973).

[7]Note that the results of Sec. IV can be extended in a straightforward way to more complicated confining potentials.

[8]G. Paiano, Nuovo Cimento A (to be published).

[9]J. L. Basdevant and G. Preparata, in Proceedings of the European Physical Society International Conference, Lisbon, 1981 (unpublished).

[10]For a review see G. Preparata, in *Understanding the Fundamental Constituents of Matter,* edited by A. Zichichi (Plenum, New York, 1978), p. 115.

[11]A. Martin, Phys. Lett. 100B, 511 (1981).

[12]There exist indications that the short-distance logarithmic behavior in ACD has its origin in the corrections due to quark-pair creation.

[13]D. P. Stanley and D. Robson, Phys. Rev. D 21, 3180 (1980).