# Exact solution of a rotating dyon black hole

Masahiro Kasuya
*Institute for Nuclear Study, University of Tokyo, Midori-cho, Tanashi, Tokyo 188, Japan*
(Received 5 August 1981)

We present an exact solution of a rotating dyon black hole in the Tomimatsu-Sato-Yamazaki space-time and study its special case, i.e., the exact rotating dyon solution for which the space-time metric takes the Kerr-Newman form. This solution is characterized by four physical parameters (mass $M$, angular momentum $S$, electric charge $Q$, and magnetic charge $\Phi$), represents a black hole, and reduces to the rotating monopole in the case $Q=0$.

## I. INTRODUCTION

Dirac pointed out that quantum mechanics does not preclude the existence of magnetic monopoles.[1] Furthermore, Schwinger proposed the dyon, i.e., a pole possessing both electric and magnetic charges.[2] This dyon exists in Abelian theory.

On the other hand, in non-Abelian theory, the 't Hooft magnetic monopole[3] and the Julia-Zee dyon[4] were spherically symmetric classical solutions of SO(3) Yang-Mills theory coupled with a triplet Higgs field.

The solution of the Einstein-Maxwell equation in Kerr space-time was obtained by Newman *et al.*[5] This solution corresponds to a rotating ring of mass and electric charge. Tomimatsu and Sato[6] discovered the series of solutions for the gravitational field of a rotating mass, following Ernst's formulation on axisymmetric stationary fields.[7] Furthermore Yamazaki[8] obtained the charged Kerr-Tomimatsu-Sato family of solutions with arbitrary integer distortion parameter $\delta$ for gravitational fields of rotating masses.

Recently, we found a static spherically symmetric Julia-Zee dyon solution in curved space-time.[9] In this paper we present an exact stationary rotating dyon solution in Tomimatsu-Sato-Yamazaki space-time, and study, in detail, both the Schwinger[10] and the Julia-Zee[11] dyon exact solutions in Kerr-Newman space-time, i.e., in the case $\delta=1$. The solution is characterized by four physical parameters (mass $M$, angular momentum $S$, electric charge $Q$, and magnetic charge $\Phi$), and represents a black hole.[12]

In Sec. II we present an exact dyon solution with the Tomimatsu-Sato-Yamazaki metric. In Secs. III and IV, we take the Schwinger and Julia-Zee dyon, respectively, as the matter fields in Kerr-Newman space-time. In Sec. V we discuss the dyon black hole in Kerr-Newman space-time. We conclude with a short discussion in Sec. VI.

## II. TOMIMATSU-SATO-YAMAZAKI-TYPE SOLUTION

Let us consider the following Lagrangian density, which describes the electromagnetic field induced by a dyon in curved space-time ($\hbar=c=1$):

$$\mathscr{L}=-\frac{1}{16\pi G}\sqrt{-g}R+\mathscr{L}_D . \tag{2.1}$$

The field equation for $g_{\mu\nu}$ is

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=8\pi G T^{\mu\nu} , \tag{2.2}$$

where the energy-momentum tensor of the dyon is given by

$$T^{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_D}{\delta g_{\mu\nu}} . \tag{2.3}$$

Our stationary axisymmetric dyon solution is a straightforward extension of the Tomimatsu-Sato-Yamazaki solutions in Boyer-Lindquist coordinates and is given by (see Refs. 8 and 13)

$$dS^2=\frac{\Sigma}{\Delta}dr^2+\Sigma\, d\theta^2+\frac{B_\delta\Delta\sin^2\theta}{A_\delta}d\varphi^2 -\frac{A_\delta}{B_\delta}\left[dt-\frac{2aC_\delta\sin^2\theta}{\sigma A_\delta}d\varphi\right]^2 , \tag{2.4}$$

where the relations among the Boyer-Lindquist coordinates $(t,r,\theta,\varphi)$, the Weyl coordinates $(t,\rho,z,\varphi)$, and the prolate spheroidal coordinates $(t,x,y,\varphi)$ are

$$\rho=\Delta^{1/2}\sin\theta=\frac{GMp\sigma}{\delta}(x^2-1)^{1/2}(1-y^2)^{1/2} ,$$
$$z=(r-GM)\cos\theta=\frac{GMp\sigma}{\delta}xy ,$$
$$r=\frac{GMp\sigma}{\delta}x+GM, \quad \cos\theta=y ,$$
$$\Delta=(r-GM)^2-\frac{1}{\delta^2}\left[(GM)^2-a^2-\frac{G}{4\pi}(Q^2+\Phi^2)\right]=\left[\frac{GMp\sigma}{\delta}\right]^2(x^2-1) ,$$
$$\Sigma=\frac{(GM\sigma)^{2\delta}B_\delta}{\delta^2[(GM)^2-a^2-(G/4\pi)(Q^2+\Phi^2)]^{\delta-1}}\left[\frac{\delta^2\Delta}{(GM)^2-a^2-(G/4\pi)(Q^2+\Phi^2)}+\sin^2\theta\right]^{1-\delta^2} ,$$
$$(GMp\sigma)^2=(GM)^2-a^2-\frac{G}{4\pi}(Q^2+\Phi^2), \quad p^2+q^2=1 . \tag{2.5}$$

The parameter $a$ is related to the angular momentum $S$ and the mass $M$ of the dyon, $a=S/(GM)=GMq\sigma$, and $\delta$ is an arbitrary integer distortion parameter.

It is well known that the Tomimatsu-Sato-Yamazaki metrics ($\delta\geq 2$) have ring singularities on the equatorial plane outside the nonsingular event horizons, and that concerning the ultimate fate of gravitational collapse, the space-time around a black-hole may always be represented by the Kerr metric. Following the Israel-Carter theorem, all singularities in space-time are hidden behind the nonsingular event horizon in the course of collapse, i.e., the space-time must be represented by the Kerr metric. Therefore we consider the dyon in Kerr-Newman space-time in (2.4) ($\delta=1$).

## III. SCHWINGER DYON

We take the following Lagrangian density, which describes the electromagnetic field induced by a Schwinger dyon[10]:

$$\mathscr{L}_D=-\tfrac{1}{4}\sqrt{-g}\,g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} , \tag{3.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+{}^*G_{\mu\nu} ,$$

and ${}^*G_{\mu\nu}$ is the Dirac string term.[1] The field equation for $A_\mu$ is

$$\partial_\nu(\sqrt{-g}\,g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma})=0 . \tag{3.2}$$

The energy-momentum tensor of the electromagnetic field is given by

$$T^\mu{}_\nu=-(g^{\mu\alpha}g^{\rho\beta}F_{\alpha\beta}F_{\nu\rho}-\tfrac{1}{4}\delta^\mu{}_\nu g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta}) . \tag{3.3}$$

An appropriate vector potential is expressed in Boyer-Lindquist coordinates

$$A_t=\frac{1}{4\pi\Sigma}(-Qr+\Phi a\cos\theta), \quad A_r=A_\theta=0 ,$$
$$A_\varphi=\frac{Q}{4\pi\Sigma}ra\sin^2\theta+\frac{\Phi}{4\pi}\left[1-\frac{r^2+a^2}{\Sigma}\cos\theta\right] , \tag{3.4}$$

where

$$\Sigma=r^2+a^2\cos^2\theta ,$$

in the case $\delta=1$ in (2.5), the space-time indices $\mu,\nu,\rho=t,r,\theta,\varphi$.

The presence of a Dirac string in our solution may be seen from $A_\varphi$ in (3.4). In fact the $\Phi$ term in $A_\varphi$ does not vanish on the semi-infinite line $\theta=\pi$ of the symmetry axis:

$$A_\varphi(\theta=\pi)=\Phi/2\pi .$$

Correspondingly the Dirac string term ${}^*G_{\mu\nu}$ is chosen to be nonzero at $\theta=\pi$. Using differential forms, one can express ${}^*G_{\mu\nu}$ in a compact form as

$$\begin{aligned}{}^*G&\equiv\tfrac{1}{2}{}^*G_{\mu\nu}dx^\mu\wedge dx^\nu\\ &=-\frac{\Phi}{4\pi}\left[1-\frac{r^2+a^2}{\Sigma}\cos\theta\right]d^2\varphi\\ &=-\frac{\Phi}{2\pi}\Theta(-\cos\theta)d^2\varphi ,\end{aligned} \tag{3.5}$$

where $\Theta$ is the step function, and $d^2\varphi$ does not vanish only on the symmetry axis. Since $^*G$ of (3.5) is independent of $a$, it is identical to the $^*G$ in the spherically symmetric case $a=0$. In this special case $a=0$, the $A_\varphi$ takes the same form as in flat space-time:

$$A_\varphi=\Phi(1-\cos\theta)/4\pi .$$

It will be discussed in the next section that $Q$ and $\Phi$ have the physical meaning of the electric and the magnetic charge of our dyon solution, respectively. The magnetic charge $\Phi$ obeys the Dirac quantization condition $eg=n/2$ ($\Phi\equiv4\pi g$), where $e$ is the electric charge of a test particle around our dyon solution.[1]

## IV. JULIA-ZEE DYON

We consider SO(3) Yang-Mills-Higgs theory in curved space-time described by the Lagrangian density[11]

$$\mathscr{L}_D=\sqrt{-g}\,[-\tfrac{1}{4}g^{\mu\rho}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} -\tfrac{1}{2}g^{\mu\nu}D_\mu\phi^aD_\nu\phi^a-\lambda V(\phi)]\ , \quad (4.1)$$

with

$$\begin{aligned} &F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+e\epsilon_{abc}A^b_\mu A^c_\nu\ ,\\ &D_\mu\phi^a=\partial_\mu\phi^a+e\epsilon_{abc}A^b_\mu\phi^c\ ,\\ &V(\phi)=\tfrac{1}{4}(\phi^a\phi^a)^2-\tfrac{1}{2}v^2(\phi^a\phi^a)\ ,\\ &v^2=\mu^2/\lambda\ , \end{aligned} \quad (4.2)$$

where $v^2=\langle\vec{\phi}^{\,2}\rangle$ determines the vacuum expectation value of the triplet Higgs field $\phi^a$. Since gauge fields act as a source of the gravitational field, it is apparent that nontrivial topologies can be generated by the interaction of the Yang-Mills field, the Higgs field, and the associated gravitational field. The equations of motion for the Yang-Mills field $A^a_\mu$ and the Higgs field $\phi^a$ are

$$\begin{aligned} &D_\nu(\sqrt{-g}\,g^{\mu\rho}g^{\nu\sigma}F^a_{\rho\sigma})+\sqrt{-g}\,g^{\mu\nu}e\epsilon_{abc}\phi^bD_\nu\phi^c=0\ ,\\ &D_\mu(\sqrt{-g}\,g^{\mu\nu}D_\nu\phi^a)-\sqrt{-g}\,\lambda[(\phi^b\phi^b)-v^2]\phi^a=0\ , \end{aligned} \quad (4.3)$$

where the isospace indices $a,b,c=1,2,3$. The energy-momentum tensor of this system is given by

$$\begin{aligned} T^\mu{}_\nu=&-(g^{\mu\alpha}g^{\rho\beta}F^a_{\alpha\beta}F^a_{\nu\rho}-\tfrac{1}{4}\delta^\mu{}_\nu g^{\rho\alpha}g^{\sigma\beta}F^a_{\rho\sigma}F^a_{\alpha\beta})\\ &-(g^{\mu\rho}D_\rho\phi^aD_\nu\phi^a-\tfrac{1}{2}\delta^\mu{}_\nu g^{\rho\sigma}D_\rho\phi^aD_\sigma\phi^a)\\ &+\delta^\mu{}_\nu\lambda V(\phi)\ . \end{aligned} \quad (4.4)$$

For the Yang-Mills field $A^a_\mu$ and the Higgs $\phi^a$, in order to obtain an exact rotating Julia-Zee dyon solution with the Kerr-Newman metric, we assume

$$\begin{aligned} &(A^1_t,A^2_t,A^3_t)=\frac{B}{e\Sigma}(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\ ,\\ &A^a_r=0\ ,\\ &(A^1_\theta,A^2_\theta,A^3_\theta)=\frac{D}{e}(\sin\varphi,-\cos\varphi,0)\ ,\\ &(A^1_\varphi,A^2_\varphi,A^3_\varphi)=\frac{1}{e\Sigma}(N\sin\theta\cos\varphi,N\sin\theta\sin\varphi,P)\ ,\\ &(\phi^1,\phi^2,\phi^3)=\frac{U}{e\Sigma}(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\ , \end{aligned} \quad (4.5)$$

where $B$, $D$, $N$ ,$P$, and $U$ are functions of $r$ and $\theta$. When the angular momentum parameter $a$ is zero, the functions $B/\Sigma$, $D$, $N/\Sigma$, $P/\Sigma$, and $U/\Sigma$ reduce to $J/r$, $1-K$, $(1-K)\cos\theta$, $-(1-K)\sin^2\theta$, and $H/r$, respectively.[4] Hence we call (4.5) an extended Wu-Yang-'t Hooft-Julia-Zee ansatz for the fields $A^a_\mu$ and $\phi^a$.[14]

In order to obtain the Kerr-Newman-type solution of our dyon, we can choose the following particular solutions for $B$, $D$, $N$, $P$, and $U$:

$$\begin{aligned} &B=br-a\cos\theta,\quad D=1\ ,\\ &N=(r^2+a^2)\cos\theta-bra\sin^2\theta,\quad P=-(r^2+bra\cos\theta)\sin^2\theta\ ,\\ &U=\pm C\Sigma\ , \end{aligned} \quad (4.6)$$

where $b$ is a constant, and $C^2=e^2v^2$.[14]

From (4.5) and (4.6), we can compute the electric and magnetic fields as follows:

$$(E_r,E_\theta,E_\varphi)=(F_{rt},F_{\theta t},F_{\varphi t})=\frac{1}{e\Sigma^2}(-\beta,-\gamma a\sin\theta,0)\ ,$$
$$(B_r,B_\theta,B_\varphi)=(F_{\theta\varphi},F_{\varphi r},F_{r\theta})=\frac{1}{e\Sigma^2}(\gamma(r^2+a^2)\sin\theta,-\beta a\sin^2\theta,0)\ , \tag{4.7}$$

where 't Hooft's electromagnetic field[3] is

$$F_{\mu\nu}=\frac{\phi^a}{|\phi|}F^a_{\mu\nu}-\frac{1}{e|\phi|^3}\epsilon_{abc}\phi^a D_\mu\phi^b D_\nu\phi^c\ ,$$

and

$$\beta\equiv-2ra\cos\theta+b(r^2-a^2\cos^2\theta)\ ,$$
$$\gamma\equiv-2bra\cos\theta-(r^2-a^2\cos^2\theta)\ .$$

The electric and magnetic fields of (4.7) in the asymptotic rest frame[15] have the following components[10]:

$$(E_{\hat r},E_{\hat\theta},E_{\hat\varphi})=\left[-\frac{\beta}{e\Sigma^2},0,0\right]\to\left[-\frac{b}{er^2},0,0\right]\ ,$$
$$(B_{\hat r},B_{\hat\theta},B_{\hat\varphi})=\left[\frac{\gamma}{e\Sigma^2},0,0\right]\to\left[-\frac{1}{er^2},0,0\right]\ , \tag{4.8}$$

as $r\to\infty$. In the case $a=0$, the arrow signs should be replaced by equalities. Thus $-b/e$ and $-1/e$ have the physical meaning of the electric and the magnetic charge of our dyon solution, respectively,[9]

$$Q\equiv-4\pi\frac{b}{e},\quad \Phi\equiv-4\pi\frac{1}{e}\ , \tag{4.9}$$

i.e.,

$$\frac{4\pi(b^2+1)}{e^2}=\frac{1}{4\pi}(Q^2+\Phi^2)\ .$$

For the Schwinger dyon, we can similarly derive (4.8) using (3.4). The combination $Q^2+\Phi^2$ is both gauge invariant and duality-rotation invariant, and arises from the factor $\beta^2+\gamma^2$, which appears in the calculation of our $T^\mu{}_\nu$:

$$\beta^2+\gamma^2=(b^2+1)\Sigma^2$$

(see next section).

## V. DYON BLACK HOLE

Our stationary axisymmetric dyon solution with the Kerr-Newman form is given in the case $\delta=1$ in (2.4) by[13]

$$A_1=p^2x^2+q^2y^2-1=\frac{1}{(GM\sigma)^2}\left\{\Sigma-G\left[2Mr-\frac{1}{4\pi}(Q^2+\Phi^2)\right]\right\}\ ,$$
$$B_1=\left[px+\frac{1}{\sigma}\right]^2+q^2y^2=\frac{1}{(GM\sigma)^2}\Sigma\ , \tag{5.1}$$
$$C_1=-px-\frac{1}{2}\left[\sigma+\frac{1}{\sigma}\right]=-\frac{\sigma G}{2(GM\sigma)^2}\left[2Mr-\frac{1}{4\pi}(Q^2+\Phi^2)\right]\ ,$$

where

$$\Delta=(r-GM)^2-\left[(GM)^2-a^2-\frac{G}{4\pi}(Q^2+\Phi^2)\right],$$
$$\Sigma=r^2+a^2\cos^2\theta . \tag{5.2}$$

We have checked that Eqs. (3.4) and (5.1) satisfy the equations of motion (2.2) and (3.2) for the Schwinger dyon[10] and that Eqs. (4.6) and (5.1) satisfy the equations of motion (2.2) and (4.3) for the Julia-Zee dyon.[11]

Since we have the metric (2.4), i.e., (5.1), we can show, from (5.2), in the Boyer-Lindquist coordinates that there exists an event horizon at

$$r=GM\pm\left[(GM)^2-a^2-\frac{G}{4\pi}(Q^2+\Phi^2)\right]^{1/2} .$$

Thus our dyon solution represents a black hole provided that

$$a^2+\frac{G}{4\pi}(Q^2+\Phi^2)\leq (GM)^2 . \tag{5.3}$$

We call it a dyon black hole. Furthermore the infinite red-shift surface under (5.3) is given by

$$r=GM\pm\left[(GM)^2-a^2\cos^2\theta-\frac{G}{4\pi}(Q^2+\Phi^2)\right]^{1/2} .$$

The structure of the space-time is schematically shown in Fig. 1. We can see that our dyon black hole is smaller than the Kerr-Newman black hole. The black hole shrinks as the number of physical conserved quantities increases.

From (3.3), (3.4), and (5.1) for the Schwinger dyon and from (4.4), (4.6), and (5.1) for the Julia-Zee dyon, we obtain the same following nonzero components of the energy-momentum tensor $T^\mu{}_\nu$ in the Boyer-Lindquist coordinates:

$$\begin{aligned} T^t{}_t&=-T^\varphi{}_\varphi=\frac{Q^2+\Phi^2}{32\pi^2}\frac{1}{\Sigma^3}[r^2+a^2(1+\sin^2\theta)] ,\\ T^t{}_\varphi&=-\frac{Q^2+\Phi^2}{16\pi^2}\frac{1}{\Sigma^3}(r^2+a^2)a\sin^2\theta,\quad T^\varphi{}_t=\frac{Q^2+\Phi^2}{16\pi^2}\frac{a}{\Sigma^3} ,\\ T^r{}_r&=-T^\theta{}_\theta=\frac{Q^2+\Phi^2}{32\pi^2}\frac{1}{\Sigma^2} . \end{aligned} \tag{5.4}$$

The energy-momentum tensor is traceless. Note that for the Julia-Zee dyon we have neglected in (5.1) and (5.4) an irrelevant constant cosmological term due to the nonvanishing Higgs potential $V(\phi)=-v^4/4$ at its minimum $U=\pm C\Sigma$. In fact, we can take a new Higgs potential $\widetilde{V}(\phi)$, given by

$$\widetilde{V}(\phi)=\frac{1}{4}[(\phi^a\phi^a)-v^2]^2 ,$$

for which $\widetilde{V}(\phi)=0$ with $U=\pm C\Sigma$.[16] Then the cosmological term does not appear in our theory. Furthermore if we discuss our theory in the Prasad-Sommerfield limit $\lambda\to 0$,[14] the cosmological term does not appear. In this limit we can take $U=b'\Sigma$ with $b'$ an arbitrary constant.

## VI. CONCLUSION

We have presented an exact dyon solution with the Tomimatsu-Sato-Yamazaki metric,[13] and obtained an exact rotating dyon solution with the Kerr-Newman metric ($\delta=1$). Our solution represents the gravitational, electric, and magnetic fields of a ring of mass $M$, electric charge $Q$, and magnetic charge $\Phi$ rotating about its axis of symmetry. It is a black-hole solution provided that Eq. (5.3) is satisfied, and reduces to the rotating monopole in the case $Q=0$ (see Fig. 2 for various limiting cases). Our dyon solution leads to "a black hole with four hairs."

Both the Abelian (i.e., Schwinger) and the non-Abelian (i.e., Julia-Zee) dyon solutions have the same metric (5.1). From this point of view, we note that our solution (4.6) for the Yang-Mills and Higgs fields can be obtained from the Schwinger dyon solution $A'_\mu=A^{(s)}_\mu I_3$, $\phi'=vI_3$ [see (3.4) for the expression for $A^{(s)}_\mu$] as a special case of our extended Arafune-Freund-Goebel singular gauge transformation[17]:

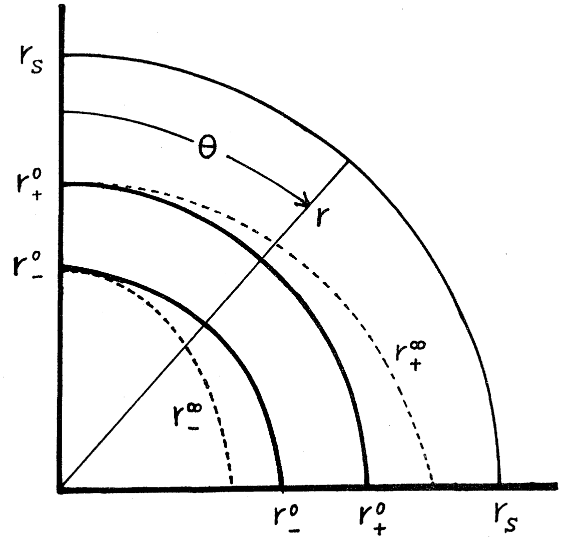


FIG. 1. The structure of the space-time in the Boyer-Lindquist coordinates. The dashed lines and the solid lines denote the infinite red-shift surfaces $r_\pm^\infty$ and the event horizons $r_\pm^0$, respectively,

$$r_\pm^\infty = \mu \pm (\mu^2 - a^2\cos^2\theta - \nu^2)^{1/2} ,$$

$$r_\pm^0 = \mu \pm (\mu^2 - a^2 - \nu^2)^{1/2} ,$$

where $\mu \equiv GM$, $\nu^2 = G(Q^2+\phi^2)/4\pi$, and $r_S$ $(=2\mu)$ is the Schwarzschild radius.

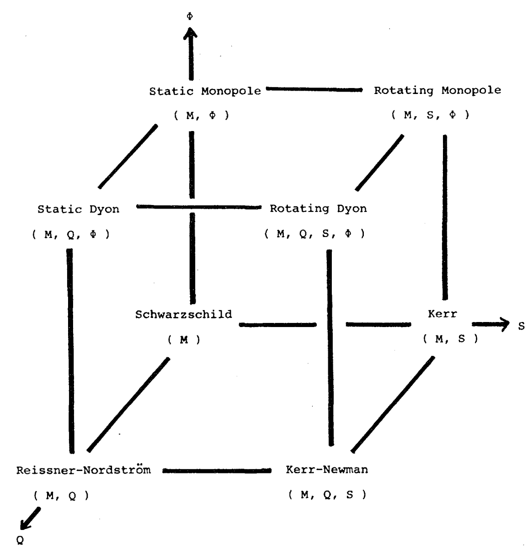


FIG. 2. The relation of our gravitating and rotating dyon solution (characterized by four physical parameters; mass $M$, angular momentum $S$, electric charge $Q$, and magnetic charge $\Phi$) to other solutions is illustrated.

$$w = \exp(-i\varphi I_3)\exp(-i\Xi I_2)\exp(i\varphi I_3) ,$$

$$A_\mu = w A'_\mu w^{-1} - \frac{i}{e}(\partial_\mu w)w^{-1}, \quad \phi = w\phi' w^{-1} , \qquad (6.1)$$

where $A_\mu = I_a A_\mu^a$ and $\phi = I_a \phi^a$ are the Julia-Zee dyon solution, and $\Xi$ is a function of $r$ and $\theta$, by replacing $\Xi = \theta$ in (6.1). If we take the condition $\Xi(r,\theta=0)=0$ and $\Xi(r,\theta=\pi)=\pi$, the equality $\Phi = -4\pi/e$ $(n=-2)$ in (4.9) then guarantees that the resulting $A_\mu$ is free of a Dirac string. Our Julia-Zee dyon has no topological stability and will decay into a monopole of strength $g(=\Phi/4\pi = -1/e)$ possessing such a stability.[17]

Taking $G=0$ in (2.4) and (5.1), the metric reduces to flat space-time.[18] We then have a rotating dyon solution in flat space-time. Furthermore, in the case $a=0$, it reduces to the spherically symmetric dyon solution.[3,4] In particular, the Schwinger dyon solution then reduces to the spherically symmetric solution

$$(A_t, A_r, A_\theta, A_\varphi) = \left[ -\frac{Q}{4\pi r}, 0, 0, \frac{\Phi}{4\pi}(1-\cos\theta) \right] .$$

## ACKNOWLEDGMENTS

The author would like to thank Professor G. W. Gibbons, Professor S. W. Hawking, Professor R. Penrose, Professor G. 't Hooft, Professor A. Tomimatsu, and Professor M. Yamazaki for sending him kind letters about his previous works along this line, and all the members of the theoretical division of INS for the hospitality extended to him.

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