PHYSICAL REVIEW D VOLUME 25, NUMBER 4 15 FEBRUARY 1982

Gauge formulation of gravitation theories. II. The special conformal case

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(Received 9 March 1981)

The gauge formulation of conformal gravity with the conformally invariant vacuum is discussed.

I. INTRODUCTION

In this paper we discuss the question whether it is possible to generalize the gauge approach to conformally invariant gravitation theory presented in Ref. 1 (further denoted by I) in such a way that the local K transformations are not reduced to L and D transformations as in I. In other words is it possible in the conformal case to avoid spontaneous symmetry breaking with respect to the K transformations, i.e., to avoid the presence of the Goldstone fields $s^a(x)$ but to preserve our interpretation of the local P transformations as translations in the tangent space?

As already mentioned in I we cannot simply include K_a in the stability group of the vacuum since, in that case, the standard method of nonlinear realizations is not applicable. We, therefore, proceed as follows. We leave K_a in the quotient space but we shall not treat the coset parameters associated with K_a as fields but as new independent coordinates.

We have seen in I that in the Poincaré case there is no spontaneous breaking of the symmetry of the vacuum corresponding to the local P transformations due to the genuine coincidence of the Goldstone parameters $y^a(x)$ with the coordinates x^μ. In other words it is always possible to put (2.39) in I, i.e., to glue the tangent space with its base. Since P_a and K_a appear symmetrically in the conformal algebra it looks natural to apply the same mechanism again and to make the spontaneous symmetry breaking of the corresponding K transformations "unobservable," too. This will be discussed in the next section.

II. CONFORMAL GRAVITY WITH THE CONFORMALLY INVARIANT VACUUM

In order to make spontaneous symmetry breaking of K transformations unobservable we extend manifold x^μ into the eight-dimensional space $\{x^\mu, z^\rho\}$. The corresponding general coordinate transformation group acts in this space and thus we should, in principle, start with the very extensive group

$$K' = C^{\text{loc}} \otimes \text{Diff}R^8 \tag{2.1}$$

instead of (4.2) in I; here C^{loc} is localized on the whole space $\{x^\mu, z^\rho\}$. Fortunately, for our purpose, it is sufficient to consider not the whole group $\text{Diff}R^8$ but its special subgroup with the following "flag" structure:

$$\delta x^\mu = \lambda^\mu(x)\,, \quad \delta z^\rho = \gamma^\rho_\lambda(x) z^\lambda + \varphi^\rho(x) \tag{2.2}$$

where $\lambda^\mu(x)$, $\gamma^\rho_\lambda(x)$, and $\varphi^\rho(x)$ are arbitrary functions. This subgroup is the minimal subgroup of $\text{Diff}R^8$ including both the ordinary general transformations of x^μ and the constant parameter conformal transformations of x^μ, z^ρ given in Appendix A [formulas (A2)–(A5)]. Since x^μ form an invariant subspace with respect to (2.2), we may regard gauge transformations of C^{loc} and corresponding gauge fields to depend solely on points of the x space as before. The additional coordinate z^ρ will appear only in the Cartan forms.

Now, we shall proceed as follows. First we consider the realization C^{loc} on the coset $C^{\text{loc}}/(I \otimes SO_0(3,1))^{\text{loc}}$ with the parameter fields $y^a(x,z)$, $s^b(x,z)$. They transform under the global conformal group just as coordinates x^a, z^b in Appendix A. Their local transformations are obtained by replacing constant parameters in the global transformation laws by arbitrary functions of x^μ. Further, the minimal subgroup K_0' mixing the tangent space $C/I \otimes SO(3,1)$ with the $\{x, z\}$ space is found as the group K_0 [the invariance group of condition (2.39)] in I; namely, we identify the coordinates x^μ, z^ρ with the coset parameters $y^a(x,z), s^b(x,z)$:

$$x^\mu = \delta^\mu_a y^a(x,z)\,, \quad z^\mu = \delta^\mu_a s^a(x,z) \tag{2.3}$$

so that there arise certain relations between functions of coordinate transformations (2.2) and gauge functions of C^{loc}. As a result, K_0' is realized on $\{x^\mu, z^\rho\}$ in the following way:

$$K_0'\colon\ \delta x^\mu = c^\mu(x) - a^\mu{}_\nu(x) x^\nu - x^2 b^\mu(x) + 2x^\mu x^\rho b_\rho(x) - t(x) x^\mu\,, \tag{2.4}$$

$$\delta z^\rho = b^\rho(x) - a^\rho{}_\nu(x) z^\nu - 2[b_\lambda(x) x^\lambda \delta^\rho_\kappa - b^\rho(x) x_\kappa + b_\kappa(x) x^\rho] z^\kappa + t(x) z^\rho \tag{2.5}$$

[upper and lower indices are connected by constant tensor $\eta^{\rho\lambda} = \text{diag}(1,-1,-1,-1)$]. On the gauge fields, K_0' acts so that these fields transform with respect to the lower vector index as world vectors with parameters δx^μ (2.4) and with respect to the upper, tangent space indices, according to (4.3)–(4.14) in I, with parameters contained in (2.4) and (2.5).

One may easily check that the Lie bracket of two arbitrary transformations (2.4) and (2.5) can again be written in the same form, i.e., transformations (2.4) and (2.5) indeed constitute a closed group. While the Lie structure of initial gauge conformal transformations satisfies the ordinary commutation relations of the algebra of conformal group (4.1) (see I), the structure of K_0' is more complicated. It is schematically described in Appendix B. For instance, the Lie bracket of two K_0' transformation with parameters $b_\rho^{\mathrm{I}}(x)$, $b_\rho^{\mathrm{II}}(x)$ is the K_0' transformation of the same kind with the bracket parameter

$$b_\rho(x) = (2x^\beta x^\nu - x^2\eta^{\beta\nu})[b_\beta^{\mathrm{II}}(x)\partial_\nu b_\rho^{\mathrm{I}}(x) - b_\beta^{\mathrm{I}}(x)\partial_\nu b_\rho^{\mathrm{II}}(x)]\,.$$

Of course, the constant parameter subgroup of (2.4), (2.5) is the ordinary conformal group. Now we turn to construction of invariants of K_0'.

The covariant Cartan forms are obtained by using the decomposition

$$\begin{aligned} G^{-1}(x,z)[d+igA_\mu(x)dx^\mu]G(x,z) &= i\omega^a[P](x)P_a + i\omega^a[K](x,z)K_a + i\omega[D](x,z)D \\ &\quad + \frac{i}{2}\omega^{ab}[L](x,z)L_{ab} \equiv B(x,z)\,, \end{aligned} \tag{2.6}$$

where $A_\mu(x)$ is given by

$$A_\mu(x) = e_\mu^a(x)P_a + f_\mu^a(x)K_a + g_\mu(x)D + \tfrac{1}{2}\Omega_\mu^{ab}(x)L_{ab} \tag{2.7}$$

and $G(x,z)$ in (A1). Explicitly, they are written as (recall that $z_m = \eta_{m\mu}z^\mu$, $z^m = \eta^{mn}z_n$, etc.)

$$\omega^a[P] = [\delta_\mu^a + g\bar{e}_\mu^a(x)]dx^\mu \equiv \omega_\mu^a[P](x)dx^\mu\,, \tag{2.8}$$

$$\begin{aligned} \omega^a[K] &= \delta_\mu^a dz^\mu + g\Big\{f_\mu^a(x) + z^a A_\mu[D](x) + z_m A_\mu^{ma}[L](x) \\ &\quad + \frac{2}{g}z^a z_b\omega_\mu^b[P](x) - \frac{1}{g}z^2\omega_\mu^a[P](x)\Big\}dx^\mu \\ &\equiv \delta_\mu^a dz^\mu + \omega_\mu^a[K](x,z)dx^\mu\,, \end{aligned} \tag{2.9}$$

$$\omega[D] = g\Big\{A_\mu[D](x) + \frac{2}{g}z^a\omega_{\mu a}[P](x)\Big\}dx^\mu \equiv \omega_\mu[D](x,z)dx^\mu \tag{2.10}$$

$$\begin{aligned} \omega^{ab}[L] &= g\Big\{A_\mu^{ab}[L](x) + \frac{2}{g}(\omega_\mu^a[P](x)z^b - \omega_\mu^b[P](x)z^a)\Big\}dx^\mu \\ &\equiv \omega_\mu^{ab}[L](x,z)dx^\mu\,, \end{aligned} \tag{2.11}$$

with the notation

$$A_\mu[D](x) = g_\mu(x) - 2f_\mu^a(x)x_a\,, \tag{2.12}$$

$$A_\mu^{ab}[L](x) = \Omega_\mu^{ab}[L](x) + 2[f_\mu^a(x)x^b - f_\mu^b(x)x^a]\,, \tag{2.13}$$

$$\begin{aligned} \bar{e}_\mu^a(x) &= e_\mu^a(x) - \Omega_\mu^{ab}(x)x_b - g_\mu(x)x^a \\ &\quad + 2f_\mu^b(x)x_b x^a - x^2 f_\mu^a(x)\,. \end{aligned} \tag{2.14}$$

The transformations (2.4) and (2.5) are induced by the group multiplications of cosets $G(x,z)$ from the left,

$$g(x)G(x,z) = G(x',z')e^{iu(x,z,g)D}e^{(i/2)u^{ab}(x,z,g)L_{ab}}\,, \tag{2.15}$$

where $g(x)$ is an arbitrary element of C^{loc} with the parameters $c^\mu(x)$, $a^{\mu\nu}(x)$, $b^\rho(x)$, or $t(x)$. The functions $u(x,z,g)$ and $u^{ab}(x,z,g)$ reduce to $t(x)$ and $a^{ab}(x)$ when we restrict ourselves to transformations from the little group, to zero for P transformations, and finally to (A6) with $b^m(x)$ instead of b^m for infinitesimal K transformations.

The forms (2.6)–(2.11) transform under (2.15) according to the law

$$\begin{aligned} B'(x',z') &= e^{iu(x,z,g)D}e^{(i/2)u^{ab}(x,z,g)L_{ab}}B(x,z)e^{-iu(x,z,g)D}e^{-(i/2)u^{ab}(x,z,g)L_{ab}} \\ &\quad - idu(x,z,g)D + e^{(i/2)u^{ab}(x,z,g)L_{ab}}de^{-(i/2)u^{ab}(x,z,g)L_{ab}} \end{aligned} \tag{2.16}$$

and the gauge field form $A_\mu(x)dx^\mu$ according to

$$A_\mu'(x')dx'^\mu = g(x)A_\mu(x)dx^\mu g^{-1}(x) + \frac{1}{ig}g(x)dg^{-1}(x)\,. \tag{2.17}$$

For completeness we write the infinitesimal transformation laws corresponding to (2.16),

$$\begin{aligned} &\delta_L^*\omega^a[P] = -a^{am}(x)\omega_m[P](x)\,, \\ &\delta_L^*\omega^a[K] = -a^{am}(x)\omega_m[K](x)\,, \\ &\delta_L^*\omega[D] = 0\,, \\ &\delta_L^*\omega^{mn}[L] = -a^{mt}(x)\omega_t^n[L] - a^{ns}(x)\omega_s^m[L] - da^{mn}(x)\,; \end{aligned} \tag{2.18}$$

$$\delta_P^*\omega^a[P] = \delta_P^*\omega^a[K] = 0\,, \qquad \delta_P^*\omega[D] = \delta_P^*\omega^{ab}[L] = 0\,; \tag{2.19}$$

$$\begin{aligned} &\delta_D^*\omega^a[P] = -t(x)\omega^a[P]\,, \\ &\delta_D^*\omega^a[K] = t(x)\omega^a[K]\,, \\ &\delta_D^*\omega[D] = -dt(x)\,, \\ &\delta_D^*\omega^{mn}[L] = 0\,; \end{aligned} \tag{2.20}$$

$$\delta_K^*\omega^a[P]=2[b^n(x)x^a+b^a(x)x^n]\omega_n[P]+2b^b(x)x_b\omega^a[P]\,,$$
$$\delta_K^*\omega^a[K]=2[b^n(x)x^a-b^a(x)x^n]\omega_n[K]-2b^m(x)x_m\omega^a[K]\,,$$
$$\delta_K^*\omega[D]=2d[b^n(x)\cdot x_n]\,, \tag{2.21}$$
$$\begin{aligned}\delta_K^*\omega^{mn}[L]=&\,2[b^s(x)x^m-b^m(x)x^s]\omega_s^n[L]\\&-2[b^s(x)x^n-b^n(x)x^s]\omega_s^m[L]\\&-2d[b^m(x)x^n-b^n(x)x^m]\,;\end{aligned}$$

where an asterisk denotes the total variation $A'(x',z')-A(x,z)$. Coefficients of dx_μ in all forms except $\omega^a[K]$ transform also as covariant world vectors with respect to indices associated with dx^μ, with parameters δx_μ (2.4). The form $\omega^a[K]$ has xth as well as zth coefficients and therefore the coefficient of dx_μ transforms in a more complicated way. The coefficients of general differential form defined on space $\{x,z\}$,

$$\Omega(x,z,dx,dz)=\Omega_\mu^{(x)}(x,z)dx^\mu+\Omega_\mu^{(z)}(x,z)dz^\mu\,,$$

transform under general coordinate transformations of this space according to

$$\begin{aligned}\delta^*\Omega_\mu^{(x)}(x,z)&=-\frac{\partial\,\delta x^\rho}{\partial x^\mu}\,\Omega_\rho^{(x)}-\frac{\partial\,\delta z^\rho}{\partial x_\mu}\,\Omega_\rho^{(z)}\,,\\ \delta^*\Omega_\mu^{(z)}(x,z)&=-\frac{\partial\,\delta x^\rho}{\partial z^\mu}\,\Omega_\rho^{(x)}-\frac{\partial\,\delta z^\rho}{\partial z^\mu}\,\Omega_\rho^{(z)}\,.\end{aligned} \tag{2.22}$$

In accordance with this general rule,

$$\begin{aligned}\delta^*\omega_\mu^a[K](x,z)=&\,\lambda^{ab}(x)\,\omega_{\mu b}[K](x,z)\\&-\frac{\partial\,\delta x^\rho}{\partial x^\mu}\,\omega_\rho^a[K](x,z)-\frac{\partial\,\delta z^\rho}{\partial x^\mu}\,\delta_\rho^a\,,\end{aligned} \tag{2.23}$$

$$\delta^*\delta_\mu^a=\lambda^{ab}(x)\eta_{\mu b}-\frac{\partial\,\delta z^\rho}{\partial z^\mu}\,\delta_\rho^a\equiv 0\,, \tag{2.24}$$

where the matrix part of the transformations acting on upper indices and defined by (2.18)–(2.21) is denoted by $\lambda^{ab}(x)$. Relation (2.24) demonstrates the self-consistency of our scheme.

Let us discuss the role of coordinates z^ρ in transformations (2.18)–(2.21), (2.23), and (2.24). Variation (2.5) contains the inhomogeneous term $b^\rho(x)$ and therefore the different coefficients in the expansion of the forms with respect to z^ρ are mixed under K transformations. However, the coefficients of dx_μ in all forms undergo with respect to their upper and lower indices transformations which depend only on x^μ. If we take into account that $\partial/\partial z^\rho$ transform purely homogeneously,

$$\delta\frac{\partial}{\partial z^\rho}=-\left(\frac{\partial\,\delta z^\mu}{\partial z^\rho}\right)\frac{\partial}{\partial z^\mu}\,, \tag{2.25}$$

then it might seem at first glance that the z dependence of these forms is not essential. For instance by putting the covariant conditions

$$\frac{\partial}{\partial z^a}\,\omega_\mu[D](x,z)=0\,,\quad \frac{\partial}{\partial z^a}\,\omega_\mu^{ab}[L](x,z)=0 \tag{2.26}$$

we could exclude z dependence in the manner used in the Appendix for arbitrary matter field $\varphi_a(x,z)$ [cf. (A16)]. There is, however, an important difference with an arbitrary matter field—we may treat the coefficients in its expansion with respect to z^μ as we want provided it does not contradict with the transformation properties of the field in contrast to the Cartan form where these coefficients are fixed in terms of the initial gauge fields by (2.6). The conditions (2.26) are, therefore, too rigid. It follows from (2.10) and (2.11) that they are satisfied only in the case $\omega_\mu^a[P](x)=0$. Thus the dependence of the form on z^μ plays a very essential role. Notice that lower coefficients in z expansion of any form transform through higher ones so that the higher coefficients in this expansion always transform among themselves, i.e., appear to be covariant. This fact will be used later when constructing an invariant Lagrangian. Let us also note that there is an analogy with superfields—the superfields are polynomials in Grassmann variables and forms (2.8)–(2.11) are polynomials in z^μ. However, in the superfield case the polynomial character is traced to the nilpotent property of the Grassmann variable, whereas in our case it is traced to special properties of commutators of the conformal algebra.

Let us recall that in Sec. IV of I when we discussed gravity based on the Weyl little group the forms $\omega_\mu'[D]$ and $\omega_\mu'^{ab}[L]$ were considered as gauge fields under D and L transformations since they are connected with the initial gauge fields $g_\mu(x)$ and $\Omega_\mu^{ab}(x)$ via the canonical transformation. In the present case this interpretation is impossible since forms (2.9)–(2.11) explicitly depend on auxiliary coordinate z_ρ. It is natural to identify the dilatation and Lorentz gauge connections with the parts of (2.10) and (2.11) independent of z, i.e., with the fields $A_\mu[D](x)$ and $A_\mu^{ab}[L](x)$ defined by (2.12) and (2.13) and to leave $f_\mu^a(x)$ as the gauge field associated with the special conformal transformation. In contrast to $g_\mu(x),\Omega_\mu^{ab}(x)$ the new gauge fields $A_\mu[D](x),A_\mu^{ab}[L](x)$ transform under P transformations only with respect to μ indices $[\delta x^\mu=c^\mu(x)]$:

$$\begin{aligned}&\delta_P^*A[D]=0\quad (A\equiv A_\mu\cdot dx^\mu)\,,\\&\delta_P^*A^{ab}[L]=0\,,\\&\delta_P^*f^a=0\,,\\&\delta_P^*\omega^a[P](x)=0\,.\end{aligned} \tag{2.27}$$

These fields transform under local D and K transformations as

$$\delta_D^* A[D](x) = \frac{-1}{g} dt(x) ,$$
$$\delta_D^* A^{ab}[L](x) = 0 , \tag{2.28}$$
$$\delta_D^* f^a(x) = t(x) f^a(x) ,$$
$$\delta_D^* \omega^a[P](x) = -t(x)\omega^a[P](x) ,$$

$$\delta_K^* A[D](x) = -\frac{2}{g}\{b^a(x)\omega_a[P](x) - d[b^a(x)x_a]\} , \tag{2.29}$$

$$\begin{aligned}\delta_K^* A^{ab}[L](x) = &-\frac{2}{g} d[b^a(x)x^b - b^b(x)x^a] \\ &+\frac{2}{g}[b^a(x)\omega^b[P](x) - b^b(x)\omega^a[P](x)] \\ &- 2[b^a(x)x_c - b_c(x)x^a]A^{cb}[L](x) \\ &- 2[b^b(x)x_c - b_c(x)x^b]A^{ac}[L](x) ,\end{aligned} \tag{2.30}$$

$$\begin{aligned}\delta_K^* f^a(x) = &-\frac{1}{g} db^a(x) - b^a(x)A[D](x) \\ &- b^\rho(x)A^a_\rho[L](x) \\ &+ 2b^b(x)x_b f^a(x) \\ &- 2[b^a(x)x^m - b^m(x)x^a]f_m(x) ,\end{aligned} \tag{2.31}$$

$$\begin{aligned}\delta_K^* \omega^a[P](x) = &\, 2b^m(x)x_m\omega^a[P](x) \\ &- 2[b^a(x)x_m - b_m(x)x^a]\omega^m[P](x) .\end{aligned} \tag{2.32}$$

It can be checked that these transformation laws have the group structure indicated in Appendix B. For instance

$$\begin{aligned}(\delta_P^*\delta_K^* - \delta_K^*\delta_P^*)A[D](x) = &-\frac{2}{g}c^\mu(x)\partial_\mu b^a(x)\omega_a[P](x) \\ &+\frac{2}{g}d[c^\mu(x)\partial_\mu b^a(x)x_a] \\ &+\frac{2}{g}d[b^a(x)c_a(x)] .\end{aligned} \tag{2.33}$$

Here on the right-hand side we have K transformation with parameters $c^\mu(x)\partial_\mu b^a(x)$ and D transformation with parameter $-2[b^a(x)c_a(x)]$.

It follows from (2.29)–(2.32) that K transformations are realized linearly on gauge fields and do not reduce to D and L transformations although those appear in (2.29)–(2.32). In fact it is impossible to redefine the fields in such a way that K transformations are completely of the form of D and L transformations as in Sec. IV of I since there is no field with the transformation law which would start with parameter $b_\mu(x)$, i.e., the Goldstone field $s^a(x)$. As we see, the nontriviality of K transformations is revealed when constructing the invariant Lagrangian. Since the forms $A[D]$, $A^{ab}[L]$, f^a, and $\omega^a[P]$ are invariant under P transformations it is clear that there is no difficulty with this invariance in contrast to Ref. 2.

Let us now construct the invariants. First we need as usual to construct covariant curls of the Cartan forms (2.8)–(2.11). The curls have more components then in Sec. IV of I since we now have two types of world indices associated with dx^μ and dz^ρ. The indices connected with dx^μ will be denoted by μ_x and those with dz^ρ by μ_z. All possible covariant curls are of the form

$$\begin{aligned}R^a_{\mu_x\rho_x}[P] = &\,\partial_{\mu_x}\omega^a_{\rho_x}[P](x) - \partial_{\rho_x}\omega^a_{\mu_x}[P](x) \\ &+ \omega^{ab}_{\rho_x}[L](x,z)\,\omega_{b\mu_x}[P](x) \\ &- \omega^{ab}_{\mu_x}[L](x,z)\,\omega_{\rho_x b}[P](x) \\ &+ \omega^a_{\mu_x}[P](x)\,\omega_{\rho_x}[D](x,z) \\ &- \omega^a_{\rho_x}[P](x)\,\omega_{\mu_x}[D](x,z) ,\end{aligned} \tag{2.34}$$

$$\begin{aligned}R^a_{\mu_x\rho_x}[K] = &\,\partial_{\mu_x}\omega^a_{\rho_x}[K](x,z) - \partial_{\rho_x}\omega^a_{\mu_x}[K](x,z) \\ &+ \omega^{ab}_{\rho_x}[L](x,z)\,\omega_{b\mu_x}[K](x,z) \\ &- \omega^{ab}_{\mu_x}[L](x,z)\,\omega_{b\rho_x}[K](x,z) \\ &- \omega^a_{\mu_x}[K](x,z)\,\omega_{\rho_x}[D](x,z) \\ &+ \omega^a_{\rho_x}[K](x,z)\,\omega_{\mu_x}[D](x,z) ,\end{aligned} \tag{2.35}$$

$$\begin{aligned}R^a_{\mu_z\rho_x}[K] &= -R^a_{\rho_x\mu_z}[K] \\ &= \partial_{\mu_z}\omega^a_{\rho_x}[K](x,z) - \omega^a_{\rho_x b}[L](x,z)\,\delta^b_{\mu_z} \\ &\quad + \omega_{\rho_x}[D](x,z)\,\delta^a_{\mu_z} ,\end{aligned} \tag{2.36}$$

$$\begin{aligned}R^{mn}_{\mu_x\rho_x}[L] = &\,\partial_{\mu_x}\omega^{mn}_{\rho_x}[L](x,z) - \partial_{\rho_x}\omega^{mn}_{\mu_x}[L](x,z) \\ &- \{\omega^m_{\mu_x s}[L]\omega^{sn}_{\rho_x}[L] - (m \leftrightarrow n)\} ,\end{aligned} \tag{2.37}$$

$$R^{mn}_{\mu_z\rho_x}[L] = -R^{mn}_{\rho_x\mu_z}[L] = \partial_{\mu_z}\omega^{mn}_{\rho_x}(x,z) , \tag{2.38}$$

$$R_{\mu_x\rho_x}[D] = \partial_{\mu_x}\omega_{\rho_x}[D](x,z) - \partial_{\rho_x}\omega_{\mu_x}[D](x,z) , \tag{2.39}$$

$$R_{\mu_z\rho_x}[D] = -R_{\rho_x\mu_z}[D] = \partial_{\mu_z}\omega_{\rho_x}[D](x,z) . \tag{2.40}$$

The other components are equal to zero either because of z independence of $\omega^a_\mu[P](x)$ or the absence of z components in the corresponding forms [only $\omega^a[K]$ in (2.9) contains such a component]. Curls (2.34)–(2.40) transform under gauge transformation with respect to upper indices as the corresponding forms, i.e., according to (2.18)–(2.21), however, without inhomogeneous terms and with respect to lower indices in accordance with the general rules (2.22).

Let us expand these curls in z_a. We see that all z dependence in torsion (2.34) is completely compensated:

$$\begin{aligned}R^a_{\mu_x\rho_x}[P] = &\,\partial_{\mu_x}\omega_{\rho_x}[P](x) - \partial_{\rho_x}\omega_{\mu_x}[P](x) \\ &+ A^{ab}_{\rho_x}[L](x)\,\omega_{b\mu_x}[P](x) \\ &- A^{ab}_{\mu_x}[L](x)\,\omega_{b\rho_x}[P](x) + \omega^a_{\mu_x}[P](x)A_{\rho_x}[D](x) \\ &- \omega^a_{\rho_x}[P](x)A_{\mu_x}[D](x) .\end{aligned} \tag{2.41}$$

Curl $R^a_{\mu_x\nu_x}[K]$ contains powers of z^μ up to the third one—the coefficients at higher powers are identically equal to zero. The nonvanishing coefficients at lower powers of z^μ are

$$R^{(0)a}_{\mu_x\rho_x}[K]=\partial_{\mu_x}f^a_{\rho_x}(x)-\partial_{\rho_x}f^a_{\mu_x}(x)+A^{ab}_{\rho_x}[L](x)f_{b\mu_x}(x)-A^{ab}_{\mu_x}[L](x)f_{b\rho_x}(x)-f^a_{\mu_x}(x)A_{\rho_x}[D](x)+f^a_{\rho_x}(x)A_{\mu_x}[D](x)\,, \tag{2.42}$$

$$\begin{aligned}R^{(1)a}_{\mu_x\rho_x}[K]=z_m\big(&\eta^{ma}\{\partial_{\mu_x}A_{\rho_x}[D](x)-\partial_{\rho_x}A_{\mu_x}[D](x)\}+R^{(0)ma}_{\mu_x\rho_x}\\&+2\{\omega^a_{\rho_x}[P]f^m_{\mu_x}(x)-\omega^a_{\mu_x}[P](x)f^m_{\rho_x}(x)\\&-\omega^m_{\rho_x}[P](x)f^a_{\mu_x}(x)+\omega^m_{\mu_x}[P](x)f^a_{\rho_x}(x)\}\\&-2\eta^{ma}\{\omega^b_{\rho_x}[P](x)f_{\mu_xb}(x)-\omega^b_{\mu_x}[P](x)f_{\rho_xb}(x)\}\big)\,,\end{aligned} \tag{2.43}$$

$$R^{(2)a}_{\mu_x\rho_x}[K]=(z^2\eta^a_m-z^az_m)R^m_{\mu_x\rho_x}[P]\,, \tag{2.44}$$

where $R^{(0)ma}_{\mu_x\nu_x}$ is the ordinary Einstein curvature tensor constructed by means of fields $A^{ma}_\mu[L](x)$. Curl $R^a_{\mu_z\rho_z}[K]$ is identically equal to zero,

$$R^a_{\mu_z\rho_x}[K]\equiv 0\,. \tag{2.45}$$

Before discussing the structure of other curls let us note that it is useful to replace the world indices by the indices from the tangent space by contracting tensors with vierbeins in order to have the same transformation law under gauge transformations for all indices. However we have to be careful because of the presence of world indices of two types. For example, if we take two-component quantity $[V_{\mu_x}(x,z), V_{\rho_z}(x,z)]$ which transforms under (2.4) and (2.5) according to (2.22) then one-component quantity

$$V^a(x,z)=\omega^{a\mu_x}[P](x)\{V_{\mu_x}(x,z)-\omega^m_{\mu_x}[K](x,z)\delta^{\rho_z}_mV_{\rho_z}(x,z)\} \tag{2.46}$$

($\omega^{a\mu}[P]$-the reciprocal vierbein) transforms under D and K transformations according to

$$\delta^*V^a(x,z)=t(x)V^a(x,z)-2b^m(x)x_mV^a(x,z)-2[b^a(x)x^m-b^m(x)x^a]V_m(x,z)\,, \tag{2.47}$$

i.e., as the field with dilation degree $d=1$. In this manner we can replace the usual derivative ∂_{μ_x} which is transformed through ∂_{μ_z} by the covariant derivative

$$\nabla^a_{(x)}=\omega^{a\mu_x}[P](x)\{\partial_{\mu_x}-\omega^m_{\mu_x}[K](x,z)\delta^{\rho_z}_m\partial_{\rho_z}\} \tag{2.48}$$

which transforms through itself according to (2.47). It is possible now to define tensors which transform with respect to world indices as usual, i.e., according to (2.9) of I and not to (2.22). Really the quantity

$$V_\mu(x,z)=V_{\mu_x}(x,z)-\omega^m_{\mu_x}[K](x,z)\delta^{\rho_z}_mV_{\rho_z}(x,z) \tag{2.49}$$

has the following transformation law:

$$\delta^*V_\mu(x,z)=-\frac{\partial}{\partial x^\mu}\delta x^\rho V_\rho(x,z)\,, \tag{2.50}$$

where δx^ρ is taken from (2.4). By using (2.49) we can define curls which transform as ordinary world tensors with respect to the lower indices, i.e., according to (2.50):

$$\tilde R^a_{\mu\nu}[P]=R^a_{\mu_x\nu_x}[P]\,, \tag{2.51}$$

$$\tilde R^a_{\mu\nu}[K]=R^a_{\mu_x\nu_x}[K]\quad\text{[due to (2.45)]}\,, \tag{2.52}$$

$$\tilde R^{mn}_{\mu\nu}[L]=R^{mn}_{\mu_x\nu_x}[L]-\omega^l_{\mu_x}[K]\delta^{\rho_z}_lR^{mn}_{\rho_z\nu_x}[L]+\omega^l_{\nu_x}[K]\delta^{\rho_z}_lR^{mn}_{\rho_z\mu_x}[L]\,, \tag{2.53}$$

$$\tilde R_{\mu\nu}[D]=R_{\mu_x\nu_x}[D]-\omega^l_{\mu_x}[K]\delta^{\rho_z}_lR_{\rho_z\nu_x}[D]+\omega^l_{\nu_x}[K]\delta^{\rho_z}_lR_{\rho_z\mu_x}[D]\,. \tag{2.54}$$

Now it is not difficult to show that

$$\tilde R^{(0)mn}_{\mu\nu}[L]=R^{(0)mn}_{\mu\nu}+2\{f^m_\mu(x)\omega^n_\nu[P](x)-f^n_\mu(x)\omega^m_\nu[P](x)\}-2\{f^m_\nu(x)\omega^n_\mu[P](x)-f^n_\nu(x)\omega^m_\mu[P](x)\}\,, \tag{2.55}$$

$$\tilde R^{(1)mn}_{\mu\nu}[L]\equiv 0\,, \tag{2.56}$$

$$\tilde R^{(2)mn}_{\mu\nu}[L]\equiv 0\,, \tag{2.57}$$

$$\tilde R^{(0)}_{\mu\nu}[D]=R^{(0)}_{\mu\nu}+2\{f^m_\nu(x)\omega_{m\mu}[P](x)-f^m_\mu(x)\omega_{m\nu}[P](x)\}\,, \tag{2.58}$$

$$\tilde R^{(1)}_{\mu\nu}[D]=2z_aR^a_{\mu\nu}[P]\,, \tag{2.59}$$

$$\tilde R^{(2)}_{\mu\nu}[D]\equiv 0\,. \tag{2.60}$$

Thus the nonvanishing curls are $\tilde R^a_{\mu\nu}[P]$, $\tilde R^a_{\mu\nu}[K]$, $\tilde R^{(0)mn}_{\mu\nu}[L]$, $\tilde R^{(0)}_{\mu\nu}[D]$, and $\tilde R^{(1)}_{\mu\nu}[D]$. We see that $\tilde R^a_{\mu\nu}[P]$, $\tilde R^{(0)ab}_{\mu\nu}[L]$, $\tilde R^{(0)a}_{\mu\nu}[K]$, and $\tilde R^{(0)}_{\mu\nu}[D]$ have the same structure in terms of fields $\omega^a_\mu[P](x)$, $f^b_\mu(x)$, $A_\mu[D](x)$, and $A^{ab}_\mu[L]$ as the curls (4.15)–(4.18) of I in terms of initial gauge fields. However, since the corresponding gauge fields transform differently, transformation properties of both the systems of the curls are completely different. While the old curls $R^a_{\mu\nu}[P], R^{ab}_{\mu\nu}[L]$ transform among themselves and the remaining curls according to (4.19)–(4.30) of I, the new curls $\tilde R^a_{\mu\nu}[P]$ and $\tilde R^{(0)mn}_{\mu\nu}$ transform purely homogeneously, i.e., each through itself. Therefore we can set

$$\tilde R^a_{\mu\nu}[P]=0 \tag{2.61}$$

which is a covariant condition. From (2.61) we can express $A^{ab}_\mu[L](x)$ in terms of fields $\omega^a_\mu[P](x)$, $A_\mu[D](x)$:

$$A^{ab}_\mu[L](x)=A^{(0)ab}_\mu[L](x)+\{\omega^{\rho b}[P](x)\omega^a_\mu[P](x)-\omega^{\rho a}[P](x)\omega^b_\mu[P](x)\}A_\rho[D](x)\,, \tag{2.62}$$

where $A^{(0)ab}_\mu[L](x)$ is expressed in terms of

$\omega^a_\mu[P](x)$ and $\omega^{\rho b}[P](x)$ via (2.33) of I. Owing to (2.61) curl $\tilde{R}^{(0)}_{\mu\nu}[D]$ in (2.59) vanishes and curl $\tilde{R}^{(0)}_{\mu\nu}[D]$ becomes a covariant (remember that $\tilde{R}^{(0)}_{\mu\nu}[D]$ transforms through $\tilde{R}^{(1)}_{\mu\nu}[D]$). The component of torsion $\tilde{R}^{(2)a}_{\mu\nu}[K]$ (2.44) vanishes, too, and as a consequence curl $\tilde{R}^{(1)a}_{\mu\nu}[K]$ defined in (2.43) is covariant. It can be written in the form

$$\tilde{R}^{(1)a}_{\mu\nu}[K] = -z_m\{\eta^{ma}\tilde{R}^{(0)}_{\mu\nu}[D] + \tilde{R}^{(0)ma}_{\mu\nu}[L]\}. \qquad (2.63)$$

The gauge field $f^a_\mu(x)$ can also be eliminated before using the equation of motion by putting the manifestly covariant conditions

$$R^{(0)}_{\mu a}[L] \overset{\text{def}}{=} R^{(0)}_{\mu\nu ma}[L]\omega^{\nu m}[P](x) = 0. \qquad (2.64)$$

By solving this equation we find the same expression for $f^a_\mu(x)$ as in Ref. 2, namely,

$$f_{\mu b}(x) = \tfrac{1}{4}\{R^{(0)}_{\mu b} - \tfrac{1}{6}R^{(0)}\omega_{\mu b}[P](x)\}. \qquad (2.65)$$

Then the curl $\tilde{R}^{ab}_{\mu\nu}[L]$ (2.55) can be written as

$$\begin{aligned}\tilde{R}^{ab}_{\mu\nu}[L] = R^{(0)ab}_{\mu\nu} - \tfrac{1}{2}(R^{(0)b}_{\mu}\omega^a_\nu[P] - R^{(0)a}_{\mu}\omega^b_\nu[P] \\ - R^{(0)b}_{\nu}\omega^a_\mu[P] + R^{(0)a}_{\nu}\omega^b_\mu[P]) \\ - \tfrac{1}{6}R^{(0)}(\omega^a_\mu[P]\omega^b_\nu[P] - \omega^b_\mu[P]\omega^a_\nu[P]).\end{aligned} \qquad (2.66)$$

The simplest invariant action can be taken in the form

$$S = \gamma\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd}\tilde{R}^{(0)}_{\mu\nu ab}[L]\tilde{R}^{(0)}_{\rho\sigma cd}[L]. \qquad (2.67)$$

It is invariant under all symmetries of the theory since in constructing the curls and in eliminating the nonpropagating fields we have used only a covariant recipe. Inserting (2.66) into the action we obtain

$$S = \gamma\int d^4x\, \det\omega[P]\{R^{(0)}_{\mu\nu}R^{(0)\nu\mu} - \tfrac{1}{3}(R^{(0)})^2\}, \qquad (2.68)$$

i.e., the well-known Weyl action but derived quite differently and more adequately than in Ref. 2.

Thus the dynamical role of the K transformations consists in extracting the combination (2.68) of two invariants which are equally allowed from the point of view of D and L transformations.

As already mentioned in Ref. 2 the field $A_\mu[D](x)$ does not actually appear in the action (2.68); $A_\mu[D]$ dependence in its two terms is mutually compensated. However, we can add an independent term $\sim\tilde{R}^{(0)}_{\mu\nu}[D]\tilde{R}^{(0)\mu\nu}[D]$ to Lagrangian (2.68) which is invariant in itself and produces the kinetic term for $A_\mu[D]$.

APPENDIX A

This Appendix is devoted to the nonlinear realization of the global conformal symmetry in the coset space $C/I\otimes SO_0(3,1)$. The parameters of this coset space will be considered as independent coordinates. The parameters connected with P will be denoted by x^a and those connected with K_b by z^b.

The group C is realized by means of left multiplication of cosets $C/I\otimes SO_0(3,1)$,

$$\begin{aligned}G(x,z) = e^{ix^aP_a}e^{iz^bK_b} \to gG(x,z) \\ = G(x',z')e^{iu(x,z,g)D}e^{(i/2)u^{ab}(x,z,g)L_{ab}},\end{aligned} \qquad (A1)$$

where g is an arbitrary element of C. If $g\in I\otimes SO_0(3,1)$ the coordinates $\{x,z\}$ transform linearly and homogeneously,

$$\begin{aligned}\delta_L x^a = -a^{am}x_m, \quad \delta_L z^n = -a^{nm}z_m, \\ \delta_D x^a = -tx^a, \quad \delta_D z^n = tz^n\end{aligned} \qquad (A2)$$

where a^{nm}, t are the corresponding constant group parameters. Under the usual translations

$$\delta_P x^a = c^a, \quad \delta_P z^a = 0, \qquad (A3)$$

i.e., coordinate z^a remains invariant. The special conformal transformations are realized nonlinearly

$$\delta_K x^a = 2x^a(x\cdot b) - x^2b^a, \qquad (A4)$$

$$\delta_K z^a = b^a - 2z^a(x\cdot b) + 2x^a(z\cdot b) - 2b^a(x\cdot z). \qquad (A5)$$

Moreover

$$\begin{aligned}u(x,z,\delta g) = -2(x\cdot b), \\ u^{mn}(x,z,\delta g) = 2(b^mx^n - b^nx^m).\end{aligned} \qquad (A6)$$

The Cartan forms are defined as usual by

$$\begin{aligned}G^{-1}(x,z)dG(x,z) = i\omega^{(0)a}[P]P_a + i\omega^{(0)a}[K]K_a \\ + i\omega^{(0)}[D]D + \frac{i}{2}\omega^{(0)mn}[L]L_{mn}\end{aligned} \qquad (A7)$$

and transform according to

$$\begin{aligned}(G^{-1}dG)' = e^{iuD}e^{(i/2)u^{mn}L_{mn}}(G^{-1}dG)e^{-(i/2)u^{mn}L_{mn}}e^{-iuD} \\ + e^{(i/2)u^{mn}L_{mn}}de^{-(i/2)u^{mn}L_{mn}} - idu D.\end{aligned} \qquad (A8)$$

By using commutation relation (4.1) in I it is not difficult to find the explicit expressions of the forms

$$\begin{aligned}\omega^{(0)a}[P] = dx^a, \\ \omega^{(0)a}[K] = dz^a + 2(dx\cdot z)z^a - z^2dx^a, \\ \omega^{(0)}[D] = 2(dx\cdot z), \\ \omega^{(0)mn}[L] = 2(dx^mz^n - dx^nz^m).\end{aligned} \qquad (A9)$$

The fields defined on the coset space $C/I\otimes SO_0(3,1)$ transform under C according to the representations induced by little group $I\otimes SO_0(3,1)$,

$$\varphi'_a(x',z') = e^{d_\varphi\cdot u}(e^{(i/2)u^{mn}L_{mn}})_{ab}\varphi^b(x,z), \qquad (A10)$$

where L_{mn} is a matrix representation of the generators of the group $SO_0(3,1)$ and d_φ is the dilatation degree of the field. Upon restriction to the transformations from the little group, quantities u and u^{mn} coincide with the group parameters; for the infinitesimal K transformations they are defined by (A6).

The covariant differential of field $\varphi_a(x,z)$ is given by

$$\mathfrak{D}\varphi_a(x,z)=d\varphi_a(x,z)+d_\varphi\cdot\omega^{(0)}[D]\varphi_a(x,z)+\frac{i}{2}\,\omega^{(0)mn}[L](L_{mn})_{ab}\varphi^b(x,z)\,. \tag{A11}$$

In the considered nonlinear realization it transforms as the field $\varphi_a(x,z)$ itself. It is natural to regard the coefficients in decomposition (A11) in forms $\omega^{(0)a}[P]$, $\omega^{(0)a}[K]$ (i.e., the covariant differentials of coordinates x^a and z^a) as the covariant derivatives

$$\mathfrak{D}\varphi_a(x,z)=\nabla^b_{(x)}\varphi_a(x,z)\omega^{(0)}_b[P]+\nabla^m_{(z)}\varphi_a(x,z)\omega^{(0)}_m[K]\,, \tag{A12}$$

$$\begin{aligned}\nabla^b_{(x)}\varphi_a(x,z)=&\frac{\partial}{\partial x_b}\varphi_a(x,z)-2z^b\left(z\cdot\frac{\partial}{\partial z}\right)\varphi_a(x,z)\\&+z^2\frac{\partial}{\partial z_b}\varphi_a(x,z)\\&-2iz_m(L^{mb})_{an}\varphi^n(x,z)+2z^b\cdot d_\varphi\cdot\varphi_a(x,z)\,,\end{aligned} \tag{A13}$$

$$\nabla^b_{(z)}\varphi_a(x,z)=\frac{\partial}{\partial z_b}\varphi_a(x,z)\,. \tag{A14}$$

Let us note that the covariant derivatives satisfy the following commutation relations:

$$\begin{aligned}&[\nabla^s_{(x)},\nabla^a_{(x)}]=0\,,\\&[\nabla^s_{(z)},\nabla^a_{(z)}]=0\,,\\&[\nabla^a_{(x)},\nabla^b_{(z)}]=-2i(L^{ab}+\eta^{ab}\overline{D})\,,\end{aligned} \tag{A15}$$

where $\overline{D}$ is the "matrix" part of generator D ($\overline{D}=-id_\varphi I$).

It follows from the form of the covariant derivative (A14) that the condition

$$\nabla^b_{(z)}\varphi_a(x,z)=\frac{\partial}{\partial z_b}\varphi_a(x,z)=0 \tag{A16}$$

is covariant under the action of the whole group. Moreover, from the transformation properties (A2)–(A4) it is seen that coordinates x_μ form the invariant subspace in the coset space $C/I\otimes SO_0(3,1)$. The fields φ_a can therefore be considered as independent on z. Thus the transformation law (A10) reduces to the standard conformal transformations of fields in the Minkowski space.

APPENDIX B

Here we shall write down the results of the bracket operation to the various transformations (2.4) and (2.5). Denoting the variation with parameters $c^\mu(x)$, $a^\mu{}_\nu(x)$, $b^\mu(x)$, and $t(x)$ by P, L, K, and D, respectively it appears that all bracket operations have the general structure

$$\begin{aligned}&[K,K]\subset K\,,\quad[P,P]\subset P\,,\quad[K,L]\subset K+L\,,\\&[P,L]\subset P+L\,,\quad[D,D]\subset D\,,\quad[D,L]\subset D+L\,,\\&[P,D]\subset P+D\,,\quad[K,D]\subset K+D\,,\quad[P,K]\subset K+P+D+L\,.\end{aligned} \tag{B1}$$

This algebra differs from the initial gauge algebra of group C which is of course connected with the fact that the transformations now mix the tangent space with the x space. Its restriction to x-independent parameters coincides with the ordinary conformal algebra. Notice that the bracket operations of generators of the minimal group K_0 in Sec. II of I have a similar structure. This is in accordance with the fact that K_0 is a subgroup of the above discussed group.

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