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Gauge formulation of gravitation theories. I. The Poincaré, de Sitter, and conformal cases

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The gauge formulations of various gravitation theories are discussed. They are based on the approach in which we have the group Diff R^4 acting on x^μ and in which we attach to every x^μ a tangent space with the group of action H. Group H does not act on x^μ and plays the role of an internal (global) symmetry group in the standard Yang-Mills theory. The matter fields in the theory transform according to representations of H and are assumed to be scalars of Diff R^4. The full invariance group of the Lagrangian is then of the form $H^{\text{loc}} \otimes$ Diff R^4. Here H^{loc} is a local gauge group obtained from H exactly as in the Yang-Mills theory. The approach has two characteristic features: (i) The group H^{loc} must be spontaneously broken in order to exclude redundant gauge fields (the Lorentz connections) from the theory in a way covariant with respect to the gauge transformations. (ii) To different H there correspond different gravitational theories, all invariant under Diff R^4 but differing in backgrounds. Thus if H is isomorphic to the Poincaré group the corresponding gauge theory turns out to be equivalent to the usual Einstein or Einstein-Cartan theory of gravity in the Minkowski space as a background. The other choices for H considered in the paper are the de Sitter groups and the conformal group. They yield the Einstein theory with a negative (or positive) cosmological term in the corresponding de Sitter space and the Weyl or Cartan-Weyl theory (depending on realization of the conformal group), respectively.

I. INTRODUCTION

As was pointed out first by Utiyama[1] in 1956 and then by a number of physicists (see, e.g., Refs. 2 and 3 and the references therein), gravitation theory may be looked upon as a gauge theory. However, diverse answers can be found to the following questions:

What is the gauge group?

What are the corresponding gauge potentials?

What is the form of the associated Lagrangian?

What about the metric tensor $g_{\mu\nu}$?

In order to summarize and clarify the situation we recall first a few facts about the group structure of the standard Yang-Mills gauge theories. The minimal Yang-Mills structure of these theories is fully determined by the group G of global internal symmetry transformations. The fields in the Lagrangian transform according to representations of G and the Lagrangian is explicitly invariant with respect to the semidirect product of the groups

$$K_{\text{YM}} = G^{\text{loc}} \otimes P \, . \tag{1.1}$$

Here G^{loc} is an infinite-parameter group of local gauge transformations uniquely determined by G, and P denotes the Poincaré group. Note that K_{YM}, in general, may contain still another internal symmetry group (as a separate factor) and the conformal or other space-time symmetry group instead of P.

Many physicists have tried to treat gravity in an analogous manner. Namely, they put $G^{\text{loc}} = \text{Diff}R^4$, the group of general covariant coordinate transformations. This setting has led to several difficulties which will be discussed later on and which essentially arise from the fact that there is not one but several finite-parameter subgroups in $\text{Diff}R^4$ playing the role of G. As a consequence, there is no unique recipe for how to obtain $\text{Diff}R^4 = G^{\text{loc}}$. (In fact $\text{Diff}R^4$ can be obtained from its many subgroups, e.g., by localizing the Poincaré subgroup or the de Sitter one or even the Galilei one.)

In the spirit of Ref. 4 we, therefore, advocate another gauge approach (see also Ref. 5) in which we have the group $\text{Diff}R^4$ acting on x^μ from the very beginning and in which we attach to every x^μ a tangent space with the group action H. The group H does not act on x^μ and plays the role of G. The matter fields in the theory transform according to representations (even nonlinear) of H and are assumed to be scalars of $\text{Diff}R^4$. The full invariance group of the Lagrangian is then of the form analogous to (1.1)

$$K = H^{\text{loc}} \otimes \text{Diff}R^4 \, , \tag{1.2}$$

where $\text{Diff}R^4$ plays the role of P in (1.1) and H^{loc} is obtained from H exactly as G^{loc} is obtained from G. We show in subsequent sections that this choice of K leads to the following two characteristic features of the approach:

(i) The full group of global symmetries of the theory (i.e., the stability group of the classical vacuum) consists of generators which are linear combinations of the generators from H^{loc} and from $\text{Diff}R^4$. This group coincides with the invariance group of the maximally symmetric solution for metric, i.e., the group of motion (or the isometry group) of the background. It turns out that it is uniquely determined by H. Different gravitation theories correspond to different H's.

All of them are invariant under general coordinate transformations but differ in backgrounds characterized by H. The structure of H itself is very restricted by requiring the theory to be both Lorentz invariant and constructable out of $g_{\mu\nu}$ alone (expressed in terms of vierbeins). From the first requirement we obtain that H must contain the Lorentz group $SO_0(3,1)$. In fact taking the generators of the physical Lorentz group in the usual form $i(x^\mu\partial^\nu - x^\nu\partial^\mu) + L^{\mu\nu}$ we see that the first term represents the Lorentz generators from DiffR^4 and the second one the Lorentz generators from H. The gauge field corresponding to $L_{\mu\nu}$ will be associated with the gravitation connections. From the requirement that the gravitation theory should be constructed from $g_{\mu\nu}$ (or vierbeins) we conclude that still other generators have to be added to those of $SO_0(3,1)$ (in order to obtain vierbeins as another gauge field.) The simplest choice is to add the generators P_a which transform as an $SO_0(3,1)$ four-vector and commute with each other. As a consequence, we obtain H isomorphic to the Poincaré group P which leads to the gauge theory equivalent to the usual Einstein theory of gravity in the Minkowski space as a background. The other choices for H treated in the paper are the de Sitter group $SO_0(3,2)$ [or $SO_0(4,1)$] and the conformal group C. They yield the Einstein theory with a negative (or positive) cosmological term in the corresponding de Sitter space and the Weyl or Cartan-Weyl theory (depending on realization of C), respectively.

(ii) The group $H^{\rm loc}$ obtained from H must be spontaneously broken (in the sense of Ref. 6) in order to exclude redundant gauge fields (namely the Lorentz connections) from the theory in a way covariant with respect to the gauge transformations.

In this and subsequent papers all cases of H will be studied in detail because of the recent revival of interest in gauge-theoretic formulations of gravity in connection with the supergravity theories. Thus Sec. II of the present paper will be devoted to the Poincaré case $H=P$, Sec. III to the de Sitter case $H=SO_0(3,2)$, $SO_0(4,1)$, and finally, Sec. IV to the conformal case $H=C$ (with C realized nonlinearly). [In the following paper we shall consider the most nontrivial case where all transformations of C (special as well as dilatation) are realized on the physical fields of the theory in the purely linear algebraic way.] In this connection let us mention that the Poincaré case has been discussed (in a spirit close to ours) in Ref. 4 and the case $H=SO_0(3,2)$ has been considered in Ref. 7. However, we hope that even in these cases we emphasize here some new (and, it is hoped, crucial) points which are usually overlooked in discussions.

II. THE POINCARÉ CASE

If the group H is isomorphic to P, H is generated by L_{ab} and P_a satisfying the usual commutation relations

$$\begin{aligned}
[L_{ab}, L_{cd}] &= i(\eta_{ad}L_{bc} + \eta_{bc}L_{ad} - \eta_{bd}L_{ac} - \eta_{ac}L_{bd}),\\
[L_{ab}, P_d] &= i(\eta_{bd}P_a - \eta_{ad}P_b),\\
[P_a, P_b] &= 0
\end{aligned} \tag{2.1}$$

with diag$\eta_{ab}=(1,-1,-1,-1)$. [The world (holonomic) indices are denoted by Greek letters, the vierbein (anholonomic) indices by Latin ones, and summation over repeated indices is assumed.] As has already been mentioned the group $H^{\rm loc}$ is obtained from H as $G^{\rm loc}$ from G in the standard Yang-Mills theory. Now assume that the matter fields $\varphi_t(x)$ are world scalars with the transformation properties under $H^{\rm loc}$ and DiffR^4 given by

$$\begin{aligned}
\delta_L\varphi_t(x) &= \frac{i}{2}a^{ab}(x)(L_{ab})_{tm}\varphi^m(x),\\
\delta_P\varphi_t(x) &= 0,
\end{aligned} \tag{2.2}$$

and

$$\delta_R\varphi_t(x) = -\lambda^\mu(x)\partial_\mu\varphi_t(x), \tag{2.3}$$

respectively. Here $\partial_\mu = \partial/\partial x^\mu$ and $a^{ab}(x)$, $\lambda^\mu(x)$ are arbitrary functions.

Notice that the action of the *physical* Poincaré group on $\varphi_t(x)$ is obtained by setting

$$\lambda^\mu(x) = c^\mu - \lambda^\mu_\nu x^\nu, \quad a^{ab}(x) = \delta^a_\mu\delta^b_\nu\lambda^{\mu\nu}, \tag{2.4}$$

where c^μ and λ^μ_ν are constants. Thus the physical Poincaré group does not distinguish the world and vierbein indices.

The gauge fields on H are introduced as usual:

$$\begin{aligned}
L_{ab} &\to \Omega^{ab}_\mu(x),\\
P_a &\to e^a_\mu(x)
\end{aligned}$$

with the transformation properties

$$\begin{aligned}
e'^a_\mu(x)P_a + \tfrac{1}{2}\Omega'^{ab}_\mu(x)L_{ab} \equiv A'_\mu(x) &= h(x)A_\mu(x)h^{-1}(x) + \frac{1}{ig}h(x)\partial_\mu h^{-1}(x),\\
h(x) &\in P^{\rm loc}.
\end{aligned} \tag{2.5}$$

In the infinitesimal form transformations (2.5) are given by

$$\delta_L\Omega^{ab}_\mu(x) = -a^{an}(x)\Omega^b_{\mu\,n}(x) - a^{bn}(x)\Omega^a_{\mu n}(x) - \frac{1}{g}\partial_\mu a^{ab}(x), \tag{2.6}$$

$$\delta_L e^a_\mu(x) = -a^{an}(x)e_{\mu n}(x), \tag{2.7}$$

$$\delta_P \Omega_\mu^{ab}(x) = 0\,, \tag{2.8}$$

$$\delta_P e_\mu^a(x) = \Omega_\mu^{am}(x) c_m(x) - \frac{1}{g}\partial_\mu c^a(x)\,. \tag{2.9}$$

Here $c^a(x)$ characterize the transformations generated by P_a. The gauge fields transform with respect to DiffR^4 as covariant world vectors (i.e., as ∂_μ):

$$\delta_R e_\mu^a(x) = -\lambda^\rho(x)\partial_\rho e_\mu^a(x) - \partial_\mu \lambda^\rho(x) e_\rho^a(x)\,, \tag{2.10}$$

$$\delta_R \Omega_\mu^{ab}(x) = -\lambda^\rho(x)\partial_\rho \Omega_\mu^{ab}(x) - \partial_\mu \lambda^\rho(x)\Omega_\rho^{ab}(x)\,. \tag{2.11}$$

It is easy to check that the covariant derivative of $\varphi_a(x)$,

$$\mathfrak{D}_\rho \varphi_a(x) = \partial_\rho \varphi_a(x) + \frac{i}{2} g \Omega_\rho^{mn}(x)(L_{mn})_a{}^b \varphi_b(x)\,, \tag{2.12}$$

transforms with respect to H^{loc} as the field $\varphi_a(x)$ itself, i.e., according to (2.2).

Now let us construct the covariant curls of the gauge fields. First notice that the quantity

$$A_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]\,, \tag{2.13}$$

where the field $A_\mu(x)$ with the values in the algebra of H is defined in (2.5), transforms homogeneously with respect to transformations (2.5), i.e.,

$$A'_{\mu\nu}(x) = h(x) A_{\mu\nu}(x) h^{-1}(x)\,. \tag{2.14}$$

The covariant curls of $e_\mu^a(x)$ and of $\Omega_\mu^{mn}(x)$ are defined as projections of $A_{\mu\nu}(x)$ on the generator P_a and L_{mn}, respectively:

$$R_{\mu\nu}{}^{ab}(x) = \partial_\mu \Omega_\nu^{ab}(x) - \partial_\nu \Omega_\mu^{ab}(x) - g[\Omega^a_{\mu m}(x)\Omega_\nu^{mb}(x) - \Omega^a_{\nu m}(x)\Omega_\mu^{mb}(x)]\,, \tag{2.15}$$

$$C^a_{\mu\nu}(x) = \partial_\mu e_\nu^a(x) - \partial_\nu e_\mu^a(x) - g\Omega_\mu^{ab}(x) e_{\nu b}(x) + g\Omega_\nu^{ab}(x) e_{\mu b}(x) = \mathfrak{D}_\mu e_\nu^a(x) - \mathfrak{D}_\nu e_\mu^a(x)\,. \tag{2.16}$$

Their transformation properties follow from (2.14) [or from (2.6)–(2.11)]. Here let us only note that with respect to "translations" characterized by $c_m(x)$ they transform as

$$\delta_P R_{\mu\nu}{}^{ab}(x) = 0\,, \tag{2.17}$$

$$\delta_P C^a_{\mu\nu}(x) = R^{am}_{\mu\nu}(x) c_m(x)\,. \tag{2.18}$$

Now physicists usually interpret $e_\mu^a(x)$ as vierbeins and $\Omega_\mu^{ab}(x)$ as the connection and express $\Omega_\mu^{ab}(x)$ in terms of $e_\mu^a(x)$ and its reciprocal by setting $C^a_{\mu\nu}(x) = 0$ (e.g., cf. Ref. 8). [Note that some authors use the terminology $e^a(x)$ = vierbein, $\Omega^{ab}(x)$ = connection, $e_\mu^a(x)$ = translation gauge potential, $\Omega_\mu^{ab}(x)$ = rotation gauge potential.] In this connection let us note two things.

(i) In order to have the reciprocal to any vierbein $e_\mu^a(x)$ physicists have to assume that the 4×4 matrix $e_\mu^a(x)$ contains the constant term $\sim\delta_\mu^a$. However, since $e_\mu^a(x)$ is a field by definition its form is determined by the solution of the corresponding equation of motion. Thus to assume the particular form of $e_\mu^a(x)$ from the very beginning is not justified.

(ii) The constraint $C^a_{\mu\nu}(x) = 0$ is explicitly noncovariant with respect to gauge transformations with parameters $c_m(x)$. Consequently if we perform P transformations, then in order to preserve $C^a_{\mu\nu}(x) = 0$ we also have to put $R^{ab}_{\mu\nu}(x) = 0$ so that $e_\mu^a(x)$ and $\Omega_\mu^{ab}(x)$ reduce to a pure gauge. Otherwise, and it should be especially noticed, $\Omega_\mu^{ab}(x)$ evaluated from $C^a_{\mu\nu}(x) = 0$ does not transform under P transformations according to (2.8) as it should have, provided $e_\mu^a(x)$ is transformed according to (2.9). In order to get rid of this difficulty various authors have modified the transformation law of $e_\mu^a(x)$ (2.9) (see, e.g., Ref. 9).

We think that instead of modifying the transformation law, etc., the correct and more natural procedure consists of the assumption that the group $H = P$ is spontaneously broken down to its Lorentz subgroup $SO_0(3,1)$. This automatically solves both difficulties mentioned above.

A minimal way of breaking the Poincaré group down to $SO_0(3,1)$ is via nonlinear realization of P in the quotient-space $P/SO_0(3,1)$ (see Ref. 6). In our case this leads to the introduction of four Goldstone fields $y^a(x)$ which transform with respect to P transformations as

$$\delta_P y^a(x) = c^a(x)\,. \tag{2.19}$$

Fields $y^a(x)$ play the role of coordinates in the quotient-space $P/SO_0(3,1)$ which is homeomorphic to the Minkowski space. Thus to any x^μ we associate an internal Minkowski space with the coordinates $y^a(x)$ and with the group of motion P.

Now we shall introduce the Cartan form $\omega_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ in the standard way[6]:

$$e^{-iy^a(x)P_a}[\partial_\mu + igA_\mu(x)]e^{iy^a(x)P_a} = i\omega_\mu^a(x)P_a + \frac{i}{2}\omega_\mu^{ab}(x)L_{ab}\,. \tag{2.20}$$

By using (2.14)–(2.18) and (2.19) we can easily find the transformation properties of the forms $\omega_\mu^a(x)$, $\omega_\mu^{ab}(x)$. For instance we can find that they are invariant with respect to P transformations, i.e.,

$$\delta_P \omega_\mu^a(x) = 0\,, \quad \delta_P \omega_\mu^{ab}(x) = 0\,. \tag{2.21}$$

Their explicit forms in terms of $e_\mu^a(x)$ and $\Omega_\mu^{ab}(x)$ are given by

$$\omega_\mu^{ab}(x) = g\Omega_\mu^{ab}(x)\,, \tag{2.22}$$

$$\omega_\mu^a(x) = \partial_\mu y^a(x) + g e_\mu^a(x) - g\Omega_\mu^{ab}(x) y_b(x)\,. \tag{2.23}$$

By using the redefined gauge field $\tilde{e}_\mu^a(x)$ instead of

$e_\mu^a(x)$,

$$\tilde{e}_\mu^a(x) = e_\mu^a(x) - \Omega_\mu^{an}(x) y_n(x) \tag{2.24}$$

with the transformation properties

$$\delta_P \tilde{e}_\mu^a(x) = -\frac{1}{g}\partial_\mu c^a(x), \tag{2.25}$$

$$\delta_L \tilde{e}_\mu^a(x) = -a^{am}(x)\tilde{e}_{\mu n}(x) + \partial_\mu a^{an}(x) y_n(x), \tag{2.26}$$

the expression for $\omega_\mu^a(x)$ becomes simpler:

$$\omega_\mu^a(x) = \partial_\mu y^a(x) + g\tilde{e}_\mu^a(x). \tag{2.27}$$

We may also introduce new covariant curls $\tilde{C}^a_{\mu\nu}(x)$ and $\tilde{R}_{\mu\nu}{}^{ab}(x)$ by using the expression

$$\tilde{C}^a_{\mu\nu}(x)P_a + \tfrac{1}{2}\tilde{R}_{\mu\nu}{}^{ab}(x)L_{ab} = e^{-iy^a(x)P_a}[C^a_{\mu\nu}(x)P_a + \tfrac{1}{2}R_{\mu\nu}{}^{ab}(x)L_{ab}]e^{iy^a(x)P_a} \tag{2.28}$$

so that

$$\tilde{C}^a_{\mu\nu}(x) = C^a_{\mu\nu}(x) - R^{an}_{\mu\nu}(x)y_n(x) = \mathfrak{D}_\mu \omega_\nu^a(x) - \mathfrak{D}_\nu \omega_\mu^a(x),$$

$$\tilde{R}_{\mu\nu}{}^{ab}(x) = R_{\mu\nu}{}^{ab}(x). \tag{2.29}$$

As can be easily checked these new curls are invariant with respect to P transformations.

The form $\omega_\mu^a(x)$ due to its transformation properties [a world four-vector with respect to index μ and transforming with respect to index a only under the transformations from $SO_0^{loc}(3,1)$] can be interpreted as a vierbein. Since $y^a(x)$ represents four independent fields, $\det\partial y^a(x)/\partial x^\mu \neq 0$ and there exists the reciprocal vierbein $\omega^{\mu a}(x)$:

$$\omega_\mu^a(x) = \omega^{\mu b}(x) = \eta^{ab}, \quad \omega_\mu^a(x)\omega_a^\rho(x) = \delta_\mu^\rho. \tag{2.30}$$

Then the metric $g_{\mu\nu}(x)$ can be written as usual,

$$g_{\mu\nu}(x) = \omega_\mu^a(x)\omega_{\nu a}(x), \quad g^{\mu\nu}(x) = \omega^{\mu a}(x)\omega_a^\nu(x). \tag{2.31}$$

It can be easily seen that the condition

$$\tilde{C}^a_{\mu\nu}(x) = 0 \tag{2.32}$$

is invariant with respect to all symmetries of the theory and consequently its solution

$$\Omega_\mu^{mn}(x) = \frac{1}{2g}\{\omega^{\rho n}(x)[\partial_\mu \omega_\rho^m(x) - \partial_\rho \omega_\mu^m(x)] + \omega_\mu^b(x)\omega^{\rho n}(x)\omega^{\nu m}(x)\partial_\nu \omega_{\rho b}(x) - (m \leftrightarrow n)\} \tag{2.33}$$

certainly has the right transformation properties.

Geometrically $\tilde{C}^a_{\mu\nu}(x)$ and $\tilde{R}_{\mu\nu}{}^{ab}(x)$ are a torsion and a curvature, respectively, in the fiber bundle space with the base homeomorphic to the Minkowski space $\{x^\mu\}$ and with the fiber $P/SO_0(3,1)$ homeomorphic to internal Minkowski space. The condition (2.32) defines a connected invariant subspace in the fiber bundle space with torsion equal to zero and gives us the possibility to express connection in terms of vierbeins. Thus we see that we have actually only one gauge field in the theory—the field $\omega_\mu^a(x)$ [or $\tilde{e}_\mu^a(x)$].

The simplest invariant of group $K = P^{loc} \otimes \mathrm{Diff}R^4$ which leads to the second-order equation of motion for $\tilde{e}_\mu^a(x)$ is the scalar curvature

$$R = \omega^{\mu a}(x)\omega^{\rho b}(x)R_{\mu\rho ba}(x). \tag{2.34}$$

The corresponding action is then the familiar Einstein action in the vierbein formalism:

$$S = \frac{1}{16\pi G}\int d^4x \,\det\omega R, \tag{2.35}$$

where G is the Newton constant. The Einstein equations are obtained by variation of (2.35) with respect to $\tilde{e}_\mu^a(x)$:

$$R_{\mu a} - \tfrac{1}{2}\omega_{\mu a}R = 0 \tag{2.36}$$

or

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 0, \tag{2.37}$$

where

$$R_{\mu a} = \omega^{\rho b}(x)R_{\mu\rho ba}, \quad R = \omega^{\mu a}R_{\mu a}, \quad R_{\mu\nu} = \omega_\nu^a R_{\mu a}. \tag{2.38}$$

Summarizing we have seen that the minimal dynamics associated with the group $K = P^{loc} \otimes \mathrm{Diff}R^4$ is the dynamics of the usual Einstein gravitation theory.

Finally let us make several remarks:

(i) The expression (2.33) for the Lorentz connection was obtained by imposing (2.32) but can also be obtained by variation of (2.35) with respect to $\Omega_\mu^{ab}(x)$ as an independent variable. However, if the considered matter fields have spin then these ways of exclusion of $\Omega_\mu^{ab}(x)$ yield nonequivalent theories (since the variation of the action will also contain contributions from the matter field due to Ω dependence of the covariant derivative [see (2.12)]. As a consequence Ω_μ^{ab} will contain beside the pure gravitation part (2.33) also terms quadratic in the matter field, thus $\tilde{C}^a_{\mu\nu}(x) \neq 0$. By inserting the expression for Ω_μ^{ab} in the initial Lagrangian these new terms yield nonminimal interactions and we get Einstein-Cartan gravitation theory.[10] However, this procedure of eliminating Ω_μ^{ab} is not necessary since it is not dictated by the symmetry properties of the theory. It seems natural to put (2.32) forever and to evaluate from it $\Omega_\mu^{ab}(x)$ in terms of vierbeins and use this expression when the interaction with matter is switched on. As we have seen the theory remains torsionless and the nonminimal interactions are not appearing.

(ii) Let us clarify the role of the fields $y^a(x)$. If we consider them as independent variables and vary (2.35) with respect to them we obtain equations which are identically satisfied provided Eq. (2.36) or (2.37) holds, i.e., there are no subsidiary conditions on $\tilde{e}_\mu^a(x)$. The fields $y^a(x)$ are absent in the Lagrangian and in equations of motion as well analogously to Goldstone fields in the spontaneously broken Yang-Mills theories. However, in contrast to the usual Yang-Mills theories the fields $y^a(x)$ did not yield the Higgs effect (the mass of $\tilde{e}_\mu^a$) because of the general covariance of the theory. There are no bilinear combinations of vierbeins different from the constant and invariant with respect to both the gauge transformations and general coordinate transformations. The Goldstone fields y^a in our approach are merely used to redefine the transformation properties of vierbeins with respect to the gauge group which allows us to express covariantly the Lorentz connection $\Omega_\mu^{ab}(x)$ in terms of $\tilde{e}_\mu^a(x)$.

Actually it is not necessary to treat fields $y^a(x)$ as independent. We can put on $y^a(x)$ the condition

$$y^a(x) = \delta_\mu^a x^\mu \qquad (2.39)$$

at every x^μ. This condition identifies the internal Minkowski space with the space-time, breaks the symmetry with respect to the full group $K = P^{\text{loc}} \otimes \text{Diff}R^4$ and can be considered as a choice of gauge. Although both P^{loc} and $\text{Diff}R^4$ do not preserve (2.39), in their semidirect product K there is an infinite-parameter subgroup K_0 which is the invariance group of (2.39) and generators of which are given as a direct sum of certain generators from P^{loc} and $\text{Diff}R^4$. The physical Poincaré subgroup is one of finite-parameter subgroups of K_0. By applying to (2.39) all symmetry transformations we can specify for which constraints on parameters the transformations preserve (2.39). We find

$$\lambda^\mu(x) = \delta_a^\mu c^a(x) - \delta_a^\mu a^{am}(x)\delta_{\rho m} x^\rho . \qquad (2.40)$$

Thus upon imposing (2.40) on parameters of transformations the general translations of coordinates and transformations in the tangent space are no longer independent: translations of x^μ with the parameters $\delta_a^\mu a^{am}(x)\delta_{\rho m}x^\rho$ induce on external indices of fields the SO(3, 1)-gauge rotations with parameters $a^{am}(x)$. The transformation properties of $\varphi_a(x)$ with respect to K_0 are given by

$$\delta\varphi_a(x) = -c^m(x)\partial_m\varphi_a(x) + \frac{i}{2}a^{mn}(x)[i(x_m\partial_n - x_n\partial_m) + L_{mn}]_a{}^b\varphi_b(x) \qquad (2.41)$$

(here $x_m \equiv \eta_{m\mu}x^\mu$). It is clear from here that K_0 is obtained by localizing the transformations of the usual physical Poincaré group. This is the local group considered by Kibble.[2] By taking into account (2.39) the vierbeins become

$$\omega_\mu^a(x) = \delta_\mu^a + g\tilde{e}_\mu^a(x) . \qquad (2.42)$$

The field $\tilde{e}_\mu^a(x)$ transforms with respect to K_0 according to the law:

$$\delta\tilde{e}_\mu^a(x) = -\frac{1}{g}\lambda^\nu(x)\partial_\nu\omega_\mu^a(x) - \frac{1}{g}\partial_\mu\lambda^\nu(x)\omega_\nu^a(x) - \frac{1}{g}a^{an}(x)\omega_{\mu n}(x) \qquad (2.43)$$

with $\lambda^\nu(x)$ given in (2.40). The transformations (2.14) and (2.43) mix the tangent space and x-space-time and therefore, with restriction to them, it has no sense to distinguish the world and vierbein indices. This reflects simply the fact that there are three equivalent realizations of K_0 for tensors. The passage from one to another is the equivalency transformation and it is given by contraction of one vector index with the vierbein or its reciprocal.

The fields $y^a(x)$ can be identified with x^μ from the very beginning. This philosophy is advocated in Ref. 4. In our approach we prefer to separate the gauge and general coordinate transformations in order to follow the parallelism with the usual Yang-Mills theories and to see more clearly the gauge structure of the Einstein theory of gravitation.

(iii) Finally let us remark that we can introduce another natural symmetry group smaller than K_0. This is connected with the following fact. As seen from (2.43) the K_0 variation of the field $\tilde{e}_\mu^a(x)$ contains an antisymmetrical inhomogeneous contribution $\sim a^{an}(x)\delta_{\mu n}$. By using this term we can "gauge away" the antisymmetrical part of $\tilde{e}_\mu^a(x)$. The invariance group of this gauge involves four independent parameter functions $c^a(x)$; however, as shown in Ref. 11 its transformations are nonlinear in $\tilde{e}_{\{\mu a\}}(x)$. This group may be called the minimal group of the Einstein theory. It appears naturally if one treats the Einstein theory as a simultaneous nonlinear realization of the affine and conformal group.[12]

III. THE DE SITTER CASES

The de Sitter groups differ from the Poincaré group in the form of the commutator of generators P_a, namely,

$$[P_a, P_n] = -i\lambda m^2 L_{an} , \qquad (3.1)$$

where $\lambda = +1$ or -1 for $SO_0(3, 2)$ and $SO_0(4, 1)$, respectively. For definiteness let us consider the case $SO_0(3, 2)$, i.e., $\lambda = 1$. The gauge fields $e_\mu^a(x)$, $\Omega_\mu^{ab}(x)$ transform with respect to P transformations as

$$\delta_P \Omega_\mu^{ab}(x) = m^2[c^a(x)e_\mu^b(x) - c^b(x)e_\mu^a(x)], \quad (3.2)$$

$$\delta_P e_\mu^a(x) = \Omega_\mu^{am}(x)c_m(x) - \frac{1}{g}\partial_\mu c^a(x). \quad (3.3)$$

The curls are defined by

$$C_{\mu\nu}^a(x) = \partial_\mu e_\nu^a(x) - \partial_\nu e_\mu^a(x) - g\Omega_\mu^{ab}(x)e_{\nu b}(x) + g\Omega_\nu^{ab}(x)e_{\mu b}(x), \quad (3.4)$$

$$R_{\mu\nu}{}^{ab}(x) = \partial_\mu \Omega_\nu^{ab}(x) - \partial_\nu \Omega_\mu^{ab}(x) - g[\Omega_{\mu m}^a(x)\Omega_\nu^{mb}(x) - \Omega_{\nu m}^a(x)\Omega_\mu^{mb}(x)] + gm^2[e_\mu^a(x)e_\nu^b(x) - e_\nu^a(x)e_\mu^b(x)]. \quad (3.5)$$

We see that in comparison with $H=P$ only the curl of $\Omega_\mu^{ab}(x)$ is modified. The transformation properties of curls with respect to P are given by

$$\delta_P C_{\mu\nu}^a(x) = R_{\mu\nu}^{am}(x)c_m(x), \quad (3.6)$$

$$\delta_P R_{\mu\nu}^{mn}(x) = m^2[c^m(x)C_{\mu\nu}^n(x) - c^n(x)C_{\mu\nu}^m(x)]. \quad (3.7)$$

Thus again the condition $C_{\mu\nu}^a = 0$ is not invariant with respect to (3.6) and (3.7) and as a consequence we cannot eliminate $\Omega_\mu^{mn}(x)$ in an invariant way.

Analogously to the previous case we have to assume that $SO_0(3,2)$ is spontaneously broken down to $SO_0(3,1)$, i.e., we have to consider a nonlinear realization of $SO_0(3,2)$ in the homogeneous space $SO_0(3,2)/SO_0(3,1)$. Now, because of (3.1) the transformation property of $y^a(x)$ with respect to P_a is essentially nonlinear. It is given in accordance with Ref. 6 by

$$e^{iy'^a(y,h)P_a} = h(x)e^{iy^a(x)P_a}e^{(-i/2)u^{mn}(y,h)L_{mn}}, \quad (3.8)$$

where $u^{mn}(y,h)$ is a definite function of group parameters $h(x)$ and fields $y^a(x)$; u^{mn} determines transformations of non-Goldstone fields $\varphi_a(x)$ [a being the SO(3,1) index] to be of the form

$$P_a: \delta\varphi_a(x) = i\tfrac{1}{2}u^{mn}(y(x),\delta h)(L_{mn})_a{}^b\varphi_b(x). \quad (3.9)$$

The explicit form of transformations is given in Refs. 7, 13, and 14. The covariant forms are defined as before, i.e., by

$$e^{-iy^a(x)P_a}\{\partial_\mu + ig[e_\mu^a(x)P_a + \tfrac{1}{2}\Omega_\mu^{mn}(x)L_{mn}]\}e^{iy^a(x)P_a} = i\omega_\mu^a(x)P_a + \frac{i}{2}\omega_\mu^{mn}(x)L_{mn}. \quad (3.10)$$

Under transformations (3.2), (3.3), and (3.8) the form $\omega_\mu^a(x)$ undergoes an SO(3,1) rotation with respect to index a with the parameters $u^{mn}(y(x),\delta h)$; the form $\omega_\mu^{mn}(x)$ transforms according to (2.6) with the same parameters $u^{mn}(y(x),\delta h)$. The covariant derivative is given by

$$\mathfrak{D}_\mu \varphi_a(x) = \partial_\mu \varphi_a(x) + \frac{i}{2}\omega_\mu^{mn}(x)(L_{mn})_a{}^b\varphi_b(x). \quad (3.11)$$

The explicit form of $\omega_\mu^a(x)$ and $\omega_\mu^{ab}(x)$ is rather complicated (see Refs. 7, 13, and 14). For us only that part of the forms which does not depend on gauge fields will be important. It appears that instead of $y^a(x)$ it is better to use the parameters $z^a(x)$ cannonically related to $y^a(x)$, namely,[14]

$$z^a(x) = 2y^a(x)\frac{\tan\frac{1}{2}m\sqrt{y^2}}{m\sqrt{y^2}}. \quad (3.12)$$

We shall write

$$\omega_\mu^a(x) = \omega_\mu^{(0)a}(x) + \tilde{\omega}_\mu^a(x), \qquad \omega_\mu^{ab}(x) = \omega_\mu^{(0)ab}(x) + \tilde{\omega}_\mu^{ab}(x), \quad (3.13)$$

where $\omega_\mu^{(0)a}$, $\omega_\mu^{(0)ab}$ denote the parts of the form which are independent of the gauge fields. They have the form[14]

$$\omega_\mu^{(0)a}(x) = a(x)\delta_\mu^a \quad [\omega^{\mu a}(x) = a^{-1}(x)\eta^{\mu a}], \quad (3.14)$$

$$\omega_\mu^{(0)ab}(x) = -\tfrac{1}{2}m^2 a(x)(z^a\partial_\mu z^b - z^b\partial_\mu z^a), \quad (3.15)$$

where

$$a(x) = \frac{1}{1+\frac{1}{4}m^2z^2(x)}. \quad (3.16)$$

The terms $\tilde{\omega}_\mu^a$, $\tilde{\omega}_\mu^{ab}$ are proportional to the corresponding gauge fields and can be chosen as new independent variables.

By passing from (3.4) and (3.5) to covariant curls according to the general formula (2.28) and by using the definition of forms ω_μ^a, ω_μ^{ab} (3.10) we find

$$e^{-iy^a(x)P_a}[C_{\mu\nu}^a(x)P_a + \tfrac{1}{2}R_{\mu\nu}^{mn}(x)L_{mn}]e^{iy^a(x)P_a} = \tilde{C}_{\mu\nu}^a(x)P_a + \tfrac{1}{2}\tilde{R}_{\mu\nu}^{mn}(x)L_{mn}, \quad (3.17)$$

where

$$\tilde{C}_{\mu\nu}^a(x) = \partial_\mu\omega_\nu^a(x) - \partial_\nu\omega_\mu^a(x) - \omega_\mu^{ab}(x)\omega_{\nu b}(x) + \omega_\nu^{ab}(x)\omega_{\mu b}(x), \quad (3.18)$$

$$\tilde{R}_{\mu\nu}^{am}(x) = \partial_\mu\omega_\nu^{am}(x) - \partial_\nu\omega_\mu^{am}(x) - [\omega_{\mu b}^a(x)\omega_\nu^{bm}(x) - \omega_{\nu n}^a(x)\omega_\mu^{nm}(x)] + m^2[\omega_\mu^a(x)\omega_\nu^m(x) - \omega_\nu^a(x)\omega_\mu^m(x)]. \quad (3.19)$$

The curls $\tilde{C}_{\mu\nu}^a(x)$, $\tilde{R}_{\mu\nu}^{mn}(x)$ transform independently of each other like the covariant derivatives (3.11).

By setting $\tilde{C}_{\mu\nu}^a(x)=0$ we can express $\omega_\mu^{ab}(x)$ in terms of $\omega_\mu^a(x)$ and reciprocal vierbein $\omega^{\mu b}(x)$ in the form (2.33). Notice that the structure of their classical parts $\omega_\mu^{(0)a}$, $\omega_\mu^{(0)ab}$ [(3.14) and (3.15)] agrees with this general form, i.e., torsion $\tilde{C}_{\mu\nu}^a$ is equal to zero on these quantities, which reflects the fact that the de Sitter space is torsionless. Taking into account this remark we see that (2.33) gives the connection between the gauge fields $\tilde{\omega}_\mu^{ab}$ and $\tilde{\omega}_\mu^a$ which has no purely classical piece on its right-hand side (including of course explicit dependence on $\omega_\mu^{(0)a}$).

It turns out to be useful for further discussion to put

$$z^a(x) = \delta_\mu^a x^\mu. \quad (3.20)$$

This condition has as the invariance group a subgroup of the group $SO_0(3,2)^{loc} \otimes \mathrm{Diff}R^4$, namely, the infinite-parameter group K_0 of Sec. II. The physical de Sitter group $SO_0(3,2)$, which acts simultaneously on x^μ and on $SO_0(3,2)$ field indices and which via contraction $m \to 0$ goes to the physical Poincaré group, appears to be one of the constant parameter subgroups of K_0. The group K_0 acts on all fields except the gravitation field $\tilde{\omega}_\mu^a(x)$ according to (2.41). The whole vierbein

$$\omega_\mu^a(x) = \frac{1}{1+\frac{1}{4}m^2x^2}\delta_\mu^a + \tilde{\omega}_\mu^a(x) \qquad (3.21)$$

transforms in it also according to the old law; however, the transformation properties of the gauge fields $\tilde{\omega}_\mu^a(x)$ are in principle different and this appears to be essential for the dynamical content of the theory.

Let us now construct the invariants. First let us remark that the part of the curvature $\tilde{R}_{\mu\nu}^{am}(x)$ which in form coincides with the Einstein curvature tensor [denoted by ${}^E\tilde{R}_{\mu\nu}^{am}(x)$] is covariant in itself since the vierbeins ω_μ^a, $\omega^{\nu b}$ with respect to all transformations from $SO_0(3,2)^{loc}$ transform purely homogeneously [according to the vector representation of $SO(3,1)$]. One might think that the simplest action will be the Einstein action again, i.e.,

$$S = \int d^4x \frac{1}{16\pi G}\det\omega\, {}^E\tilde{R} . \qquad (3.22)$$

However, since the "classical" part of vierbein (3.21) does not reduce to $\delta_\mu^a \cdot$const but is a function of the coordinates x^μ ($\omega_\mu^{(0)a}(x) = 1/(1+\frac{1}{4}m^2x^2)\delta_\mu^a$) in density of action (3.22) there arises a tadpole term proportional to the field $\tilde{\omega}_\mu^a(x)$. This is the reason why the theory connected with (3.22) is poor, i.e., it is not stable due to the possibility of vacuum transitions. [In fact, the action (3.22) is completely equivalent to the Einstein action (2.35). After summing over all vacuum transitions the tadpole term disappears while $\tilde{\omega}_\mu^a(x)$ is shifted by a certain function (analogously to the Higgs field in the standard linear σ models). This function turns out to be such that it cancels the x-dependent part of $\omega_\mu^{(0)a}$ in (3.21) leaving solely δ_μ^a in $\omega_\mu^{(0)a}$. Thus, the Minkowski background is effectively restored.] It turns out that this unpleasant term can be eliminated from the theory in a covariant way by adding to R a cosmological term. The value of this term is uniquely fixed by requiring the absence of a tadpole in the action,

$$S' = \frac{1}{16\pi G}\int d^6x \det\omega({}^E\tilde{R} + 6m^2) . \qquad (3.23)$$

The corresponding equation of motion [obtained by variation with respect to $\tilde{\omega}_\mu^a(x)$] has the form

$${}^E\tilde{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}{}^E\tilde{R} = -3m^2 g_{\mu\nu} , \qquad (3.24)$$

where $g_{\mu\nu} = \omega_\mu^a(x)\omega_{\nu a}(x)$. It is not difficult to check that the classical part of the vierbein (3.21) is a solution of (3.24).

The corresponding metric $g_{\mu\nu}^{(0)}$ has the form

$$g_{\mu\nu}^{(0)}(x) = \omega_\mu^{(0)a}(x)\omega_{\nu a}^{(0)} = \frac{1}{(1+\frac{1}{4}m^2x^2)^2}\eta_{\mu\nu} . \qquad (3.25)$$

The maximal invariance group of (3.25) (the isometry group) appears to be the $SO_0(3,2)$ subgroup of the group K_0.

Let us stress that the Lagrangian in (3.23) does not coincide with the scalar curvature which corresponds to the tensor $\tilde{R}_{\mu\nu}^{nm}$ (3.19) as can easily be seen. But nevertheless this Lagrangian can be expressed in terms of this tensor.[7] Up to the topological invariant

$$\det\omega({}^E\tilde{R} + 6m^2) = -\frac{1}{16m^2}\epsilon^{\mu\nu\rho\lambda}\epsilon_{abcd}\tilde{R}_{\mu\nu}{}^{ab}(x)\tilde{R}_{\rho\lambda}{}^{cd}(x) . \qquad (3.26)$$

Let us note that in Ref. 7 (3.23) and (3.26) were derived differently.

Finally let us make a few remarks concerning the group structure of the obtained theory and a comparison with the previous case. In the gauges (3.20) and (2.39) the full invariance groups of both theories are the same group K_0. The essential difference of both theories consists in the difference of classical parts of the corresponding vierbeins [$\eta_{\mu\nu}$ in the first case and $a(x)\eta_{\mu\nu}$ in the second one]. This is the only trace of the different choices of the initial gauge groups. The classical part of the vierbein defines the background space of the theory ("maximally flat space"), and its isometry group coincides with the group of motion of this space. This group appears to be the maximal group of homogeneous transformations of the corresponding gravitation field (the full vierbein minus the classical part). In the first case this group was the physical Poincaré subgroup of K_0 and, in the second case, the de Sitter subgroup $SO_0(3,2)$. All other transformations from K_0 (except those belonging to the mentioned subgroups) are realized on the corresponding gravitation fields nonhomogeneously and are therefore spontaneously broken. For instance the usual translations act on $\tilde{\omega}_\mu^a(x)$ in the following way:

$$\delta\tilde{\omega}_\mu^a(x) = -c^a\partial_b\tilde{\omega}_\mu^b - \tfrac{1}{2}m^2a^2(x)(xc)\delta_\mu^a . \qquad (3.27)$$

The transformation of the quantity $\partial_\rho\tilde{\omega}_\mu^a(x)\delta_a^\mu$ begins with the pure constant

$$\delta(\partial_\rho\tilde{\omega}_\mu^a\delta_a^\mu) = -\frac{1}{2}m^2c_\rho + \cdots , \qquad (3.28)$$

i.e., this object plays the role of the corresponding Goldstone field. Analogously, in the first case $SO(3,2)$ translations are spontaneously broken with $\partial_\rho\tilde{e}_\mu^a(x)\delta_a^\mu$ as Goldstone fields (the higher derivatives of the gravitation field appear to be Goldstone fields associated with an infinite number of the remaining spontaneously broken generators of the group K_0). Thus although the full invariance groups of both cases coincide the stability groups of their classical vacua are finally different: the first case respects the structure of the quotient space K_0/P and the second that of the quotient-space $K_0/SO_0(3,2)$. Notice that there is a clear analogy with σ models of standard spontaneously broken symmetries where various patterns of spontaneous symmetry breaking are connected with various choices of potential. Notice also a principal difference of the spontaneous symmetry breaking of the gauge group P^{loc}, $SO_0(3,2)^{loc}$, or $SO_0(4,1)^{loc}$ down to $SO_0(3,1)^{loc}$ and the corresponding spontaneous symmetry breaking of K_0 down to the physical Poincaré or de Sitter group. To the first type of symmetry breaking there correspond the Goldstone fields $y^a(x)$ which are finally identified with the coordinates x^μ [via (2.29) and (3.20)] so that the spontaneously broken character of the corresponding generators P_a manifests itself only in inhomogeneity of transformations of x_μ. On all physical fields these generators are realized homogeneously and therefore belong to the vacuum stability group of the theory (the little group).

Finally let us emphasize once more that the proposed gauge approach to gravity has an advantage since the choice of H^{loc} automatically fixes the maximal invariance group of the classical part of the theory and simultaneously also a type of background space in which the corresponding quantum theory has to be constructed. This group is isomorphic to H but, in contrast to it, acts on x^μ as well as on field indices.

IV. THE CONFORMAL CASE

The conformal group $C(\overset{loc}{\approx} SO_0(4,2))$ can be considered as an extension of the Poincaré group P by five generators K_a and D. Its algebra is given by (2.1) and by the following commutation relations (in comparison with the standard notation our generators K_n have the opposite signs):

$$\begin{aligned}
&[L_{ab},K_m]=i(\eta_{bm}K_a-\eta_{am}K_b)\,,\\
&[P_a,K_b]=-2i(\eta_{ab}D+L_{ab})\,,\\
&[K_m,K_n]=0\,,\\
&[L_{ab},D]=0\,,\\
&[P_a,D]=-iP_a\,,\\
&[K_a,D]=iK_a\,.
\end{aligned}\tag{4.1}$$

The algebra of C has various subalgebras besides that of P. For instance the generators $\frac{1}{2}(P_n+m^2K_n)$ and L_{mn} form the Lie algebra of $SO_0(3,2)$, the generators $\frac{1}{2}(P_n-m^2K_n)$ and L_{mn} form the algebra of $SO_0(4,1)$. The operators K_n, L_{mn} generate the second Poincaré algebra. The generators D, P_m, L_{mn} or D, K_m, L_m form two Weyl algebras.

The basic elements of the gauge theory based on the full invariance group

$$K=C^{loc}\otimes \mathrm{Diff}\,R^4 \tag{4.2}$$

are the gauge fields $e^a_\mu(x)$ and $\Omega^{ab}_\mu(x)$ (corresponding to the generators P_a and L_{ab}, respectively) and the gauge fields $f^a_\mu(x)$ and $g_\mu(x)$ associated with the generators K_a and D. The transformation properties of these fields under C^{loc} are characterized as follows: With respect to local L transformations the fields e^a_μ and Ω^{ab}_μ transform according to (2.6) and (2.7) and the fields $f^a_\mu(x)$ and $g_\mu(x)$ as a four-vector and a scalar, respectively. Transformations generated by P_a have the following form:

$$\delta_P\Omega^{ab}_\mu(x)=2[c^a(x)f^b_\mu(x)-c^b(x)f^a_\mu(x)]\,, \tag{4.3}$$

$$\delta_P e^a_\mu(x)=\Omega^{am}_\mu(x)c_m(x)+c^a(x)g_\mu(x)-\frac{1}{g}\partial_\mu c^a(x)\,, \tag{4.4}$$

$$\delta_P f^a_\mu(x)=0\,, \tag{4.5}$$

$$\delta_P g_\mu(x)=2c_a(x)f^a_\mu(x)\,. \tag{4.6}$$

Notice that P transformations of $e^a_\mu(x)$ and $\Omega^{ab}_\mu(x)$ differ from (2.8) and (2.9). The transformation properties under K_a and D are given by

$$\delta_K\Omega^{ab}_\mu(x)=2[b^a(x)e^b_\mu(x)-b^b(x)e^a_\mu(x)]\,, \tag{4.7}$$

$$\delta_K e^a_\mu(x)=0\,, \tag{4.8}$$

$$\delta_K f^a_\mu(x)=\Omega^{am}_\mu(x)b_m(x)-b^a(x)g_\mu(x)-\frac{1}{g}\partial_\mu b^b(x)\,, \tag{4.9}$$

$$\delta_K g_\mu(x)=-2b^a(x)e_{\mu a}(x)\,, \tag{4.10}$$

$$\delta_D\Omega^{ab}_\mu(x)=0\,, \tag{4.11}$$

$$\delta_D e^a_\mu(x)=-t(x)e^a_\mu(x)\,, \tag{4.12}$$

$$\delta_D f^a_\mu(x)=t(x)f^a_\mu(x)\,, \tag{4.13}$$

$$\delta_D g_\mu(x)=-\frac{1}{g}\partial_\mu t(x)\,, \tag{4.14}$$

where $b^a(x)$ and $t(x)$ are the corresponding gauge functions.

The covariant curls are defined in the standard way (2.13). They have the form

$$R^{mn}_{\mu\nu}[L]=R^{mn}_{\mu\nu}+2g(e^m_\mu f^n_\nu-e^n_\mu f^m_\nu)-2g(e^m_\nu f^n_\mu-e^n_\nu f^m_\mu)\,, \tag{4.15}$$

$$R^a_{\mu\nu}[P]=C^a_{\mu\nu}+g(e^a_\mu g_\nu-e^a_\nu g_\mu)\,, \tag{4.16}$$

$$\begin{aligned}R^a_{\mu\nu}[K]=&\,\partial_\mu f^a_\nu-\partial_\nu f^a_\mu-g(\Omega^{ab}_\mu f_{\nu b}-\Omega^{ab}_\nu f_{\mu b})\\ &-g(f^a_\mu g_\nu-f^a_\nu g_\mu)\,,\end{aligned} \tag{4.17}$$

$$R_{\mu\nu}[D]=\partial_\mu g_\nu-\partial_\nu g_\mu+2g(e^a_\mu f_{\nu a}-e^a_\nu f_{\mu a})\,, \quad (4.18)$$

where $R^{mn}_{\mu\nu}(x)$ and $C^a_{\mu\nu}(x)$ are defined by (2.15) and (2.16). By replacing the gauge fields with the corresponding curls, i.e.,

$$\Omega^{ab}_\mu \to R_{\mu\nu}{}^{ab}[L]\,,\quad e^a_\mu \to R^a_{\mu\nu}[P]\,,$$
$$f^a_\mu \to R^a_{\mu\nu}[K]\,,\quad g_\mu \to R_{\mu\nu}[D]\,,$$

in (4.3)–(4.14) and by neglecting all gradient terms we obtain the following transformation properties of the curls under the gauge transformations:

$$\delta_P R^{mn}_{\mu\nu}[L]=2(c^m R^n_{\mu\nu}[K]-c^n R^m_{\mu\nu}[K])\,, \quad (4.19)$$
$$\delta_P R^a_{\mu\nu}[P]=R^{am}_{\mu\nu}[L]c_m+c^a R_{\mu\nu}[D]\,, \quad (4.20)$$
$$\delta_P R^a_{\mu\nu}[K]=0\,, \quad (4.21)$$
$$\delta_P R_{\mu\nu}[D]=2c_a R^a_{\mu\nu}[K]\,, \quad (4.22)$$
$$\delta_K R_{\mu\nu}{}^{ab}[L]=2(b^a R^b_{\mu\nu}[P]-b^b R^a_{\mu\nu}[P])\,, \quad (4.23)$$
$$\delta_K R^a_{\mu\nu}[P]=0\,, \quad (4.24)$$
$$\delta_K R^a_{\mu\nu}[K]=R^{am}_{\mu\nu}[L]b_m-b^a R_{\mu\nu}[D]\,, \quad (4.25)$$
$$\delta_K R_{\mu\nu}[D]=-2b^a R_{\mu\nu a}[P]\,, \quad (4.26)$$
$$\delta_D R_{\mu\nu}{}^{ab}[L]=0\,, \quad (4.27)$$
$$\delta_D R^a_{\mu\nu}[P]=-tR^a_{\mu\nu}[P]\,, \quad (4.28)$$
$$\delta_D R^a_{\mu\nu}[K]=tR^a_{\mu\nu}[K]\,, \quad (4.29)$$
$$\delta_D R_{\mu\nu}[D]=0\,. \quad (4.30)$$

In Ref. 9 field $e^a_\mu(x)$ is identified with the vierbein and the action is taken in the form

$$S=\int d^4x\,\epsilon^{\mu\nu\rho\lambda}\epsilon^{abcd}R_{\mu\nu ab}[L]R_{\rho\lambda cd}[L]\,. \quad (4.31)$$

Expression (4.31) is explicitely invariant with respect to the general coordinate transformations and to the local L and D transformations. On the other hand (4.19) and (4.23) imply that expression (4.31) is also invariant under the local P and K transformations provided the following conditions hold:

$$R^a_{\mu\nu}[P]=0\,, \quad (4.32)$$
$$R^a_{\mu\nu}[K]=0\,. \quad (4.33)$$

However, it follows from (4.20) and (4.21) that conditions (4.32) and (4.33) themselves are not covariant under the local P and K transformations (covariance is possible only in the trivial case $R^{mn}_{\mu\nu}[L]=0, R_{\mu\nu}[D]=0$). Thus, from the group-theoretical point of view, postulation of (4.32) and (4.33) is not justified since these conditions do not agree with the transformation properties of the gauge fields.

In order to avoid this inconsistence the transformation properties of the fields were redefined in Ref. 9 analogously to the Poincaré case (cf. Ref. 15). The redefinition is based on the following observation. Condition (4.32) is invariant under all gauge transformations except those generated by P_a. Hence Ω^{ab}_μ found from this condition has the correct transformation properties with respect to K, L, and D transformations. It follows from (4.23) that K variation of curl $R_{\mu\nu}{}^{ab}[L]$ is equal to zero due to condition (4.32). Thus action (4.31) is invariant under the local K transformations. Then it is claimed in Refs. 9 and 15 that by properly modifying the local P transformations condition (4.32) becomes invariant under P transformations. The redefined P transformations are of standard form (2.10) and (2.11) but with the parameters depending on the fields: $\bar{\lambda}^\rho(x)=(1/g)e^{\rho a}(x)c_a(x)$, where $e^{\rho a}(x)$ is the reciprocal vierbein. For instance

$$\delta'_P e^a_\mu(x)=-\partial_\mu\bar{\lambda}^\rho(x)e^a_\rho(x)-\bar{\lambda}^\rho(x)\partial_\rho e^a_\mu(x)$$
$$=-\frac{1}{g}\partial_\mu c^a(x)+\frac{1}{g}e^{\nu m}(x)c_m(x)[\partial_\mu e^a_\nu(x)-\partial_\nu e^a_\mu(x)]\,. \quad (4.34)$$

Let us stress that with such redefinition of P transformations specificity of the conformal group does not appear (the same procedure is applicable for the Poincaré case[9,15]). It is clear that each curl covariantly transforms through itself under (4.34), and that condition (4.32) and action (4.31) are explicitly invariant so that there is no necessity of the supplementary condition (4.33). Further, field $f^a_\mu(x)$ can be expressed in terms of the other fields by using its equation of motion, i.e., $f^a_\mu(x)$ does not spread and action (4.31) reduces to the well-known Weyl action. On this basis it was concluded in Ref. 9 that equivalence of the Weyl gravitation theory and the above-described gauge theory based on the conformal group was proved.

However, as already mentioned in Sec. II the approach using redefinition of P transformations is rather artificial. Indeed, it is easy to show that the modified local P transformations together with the local L transformation and transformations (4.7)–(4.14) do not possess the original group structure [e.g., the Lie bracket of two infinitesimal variations (4.34) is not equal to zero but is a complicated nonlinear transformation of field $e^a_\rho(x)$]. In other words, transformation law (4.34) by its group structure has nothing to do with the initial law of the local P transformations [although both laws contain the inhomogeneous gradient term $\sim\partial_\mu c_a(x)$]. Thus the trick used in Ref. 9 does not actually solve the problem with invariance under the local P transformations.

Now we show that invariance under all symmetry transformations of the theory can be obtained without breaking the original group structure if, analogously to Secs. II and III, the gauge group $H^{\rm loc}=C^{\rm loc}$ is assumed to be properly spontaneously broken. Furthermore the vierbein will be identified not with $e_\mu^a(x)$ but with a certain form depending on $e_\mu^a(x)$ and the other gauge fields. Finally it will be possible to redefine the gauge fields in such a way that their transformation properties will be in accordance with the condition analogous to (4.32) without any artificial modification of transformation laws.

For this purpose first let us consider nonlinear realizations of the conformal group C. It is clear that in our nonlinear realization of C its subgroup $SO_0(3,1)$ has to be contained in the little group and the generators P_a in the quotient space. However, in contrast to the Poincaré case treated in Sec. II, there exist two inequivalent nonlinear realizations of C satisfying this requirement, namely,

$$C/SO_0(3,1) \tag{4.35}$$

and

$$C/I\otimes SO_0(3,1),\quad I=e^{it(x)D} \tag{4.36}$$

with the spontaneously broken generators $\{P_a, K_b, D\}$ and $\{P_a, K_b\}$, respectively. Note: All transformations generated by K_b, D, L_{mn} form a subgroup in C over which we can make factorization, too. However, the standard method of nonlinear realization[6] (in particular the method of construction of the Cartan forms) is not applicable in this case since one of its very important assumptions is violated (namely, all generators are not orthonormal in the sense of the Cartan inner product with respect to the algebra of the little group). This case will be treated in a subsequent paper.

A. The little group $SO_0(3,1)$

Nonlinear realization (4.35) was considered in Refs. 12 and 16. According to general rules it is necessary in this case to introduce the Goldstone fields $y^a(x)$, $s^b(x)$, and $\sigma(x)$ corresponding to the generators P_a, K_b, and D, respectively, and to define the Cartan form by means of the relation of type (2.20):

$$G^{-1}(y,s,\sigma)(\partial_\mu+igA_\mu)G(y,s,\sigma)=i\omega_\mu^a[P]P_a+i\omega_\mu^a[K]K_a+i\omega_\mu[D]D+\frac{i}{2}\omega_\mu^{ab}[L]L_{ab}\,, \tag{4.37}$$

where now

$$G(y,s,\sigma)=e^{iy^a(x)P_a}e^{is^b(x)K_b}e^{i\sigma(x)D}\,, \tag{4.38}$$

$$A_\mu(x)=e_\mu^a(x)P_a+f_\mu^a(x)K_a+g_\mu(x)D+\tfrac{1}{2}\Omega_\mu^{ab}(x)L_{ab}\,. \tag{4.39}$$

If group $C^{\rm loc}$ acts on cosets $G(y,s,\sigma)$ as a group of left multiplications, forms $\omega_\mu^a[P]$ and $\omega_\mu^a[K]$ undergo an $SO_0(3,1)$ rotation with respect to index a with the parameters depending, in general, on the Goldstone fields. Form $\omega_\mu[D]$ is an invariant of group $C^{\rm loc}$. The form $\omega_\mu^{ab}[L]$ transforms as the Lorentz connection, i.e., according to (2.8), with the same parameters as the forms $\omega_\mu^a[P]$ and $\omega_\rho^a[K]$. All forms are invariant under the local P transformations. Form $\omega_\mu^a[P]$ has the structure

$$\omega_\mu^a[P]=\partial_\mu y^a(x)+ge_\mu^a(x)+\cdots \tag{4.40}$$

and, therefore, can be identified with the vierbein and its part which contains the gauge fields with the gravitation field. The form $\omega_\mu^{ab}[L]$ can be expressed in terms of $\omega_\mu^a[P]$ and its reciprocal $\omega^{\mu a}[P]=(\omega^{-1})_\mu^a[P]$ according to (2.23). The minimal invariant Lagrangian is the Einstein Lagrangian (2.35). The forms $\omega_\mu^a[K]$ and $\omega_\mu[D]$ do not play any dynamical role and can be considered as external matter fields. For instance it is possible to set them equal to zero (which is an invariant operation). As a result, the gauge fields $f_\mu^a(x)$ and $g_\mu(x)$ are expressed in terms of the Goldstone fields and the field $e_\mu^a(x)$ (the inverse Higgs effect[17]).

In this sense, the gauge theory corresponding to the realization of $C^{\rm loc}$ on the quotient space (4.35) is identical with that based on the Poincaré group. We can see it in more details by taking into account the following facts. As already mentioned field $y^a(x)$ does not completely have the Goldstone properties since via (3.39) it can be identified with the coordinate x^μ and, consequently, P_a become the generators of the usual translations which annihilate the vacuum. At the same time fields $s^b(x)$ and $\sigma(x)$ [after imposing (2.39)] remain the inhomogeneously transforming Goldstone fields, i.e., the spontaneously broken character of generators K_a and D is absolute. Thus we have the situation typical for Yang-Mills theories associated with nonlinear realizations of local symmetries—the structure of invariants is completely determined by the transformation properties of the fields under transformations from the stability group of the vacuum, i.e., the spontaneously broken part of the local symmetry has no dynamical effect. Furthermore, the Goldstone fields do not explicitly occur in the Lagrangian and the gauge fields associated with the generators from the quotient space may be supplied with the invariant mass terms (the Higgs effect). In our case we obtain the invariant mass terms

$$(\det\omega)\{\omega^{\mu a}[P]\omega_m^\rho[P](\omega_\mu^b[K]\omega_{\rho b}[K]\delta_a^m+a_1\omega_{\mu a}[K]\omega_\rho^m[K])\}$$

and

$$(\det\omega)\omega^{\mu a}[P]\omega_a^\rho[P]\omega_\mu[D]\omega_\rho[D]$$

for the fields $f^a(x)$ and $g(x)$, respectively. We can construct also the corresponding kinetic terms. In the Einstein theory they can be obtained by the standard method, i.e., by making use of covariant derivatives (2.12) with $\omega_\rho^{mn}[L]$ instead of Ω_ρ^{mn}.

Note that we might associate x with generators $\frac{1}{2}(P-m^2K)$ or $\frac{1}{2}(P+m^2K)$ instead of P and the Goldstone fields $s^b(x)$ with orthogonal combinations of P and K. In these cases we would obtain the gravitation theories with cosmological terms treated in the previous section.

B. The little group $I\otimes SO_0(3,1)$

Now we shall discuss the nonlinear realization (4.36). Its stability group is the group $I\otimes SO_0(3,1)$ where $I=e^{it(x)D}$. As a consequence all matter fields $\varphi_a(x)$ are characterized by both the index a and the number d_φ— the dilatation degree. The former specifies the transformation properties of $\varphi_a(x)$ under $SO_0(3,1)$ and the latter under D transformations:

$$\delta_D\varphi_a(x)=d_\varphi t(x)\varphi_a(x)\,. \qquad (4.41)$$

The Cartan forms are introduced via the decomposition

$$G^{-1}(y,s)(\partial_\mu+iA_\mu)G(y,s)=i\omega_\mu^{\prime a}[P]P_a+i\omega_\mu^{\prime a}[K]K_a+i\omega_\mu^{\prime}[D]D+\frac{i}{2}\omega_\mu^{\prime ab}[L]L_{ab}\,, \qquad (4.42)$$

where

$$G(y,s)=e^{iy^a(x)P_a}e^{is^b(x)K_b}\,.$$

Under the local L transformations the forms transform as usual and under the D transformations according to

$$\begin{aligned}&\delta_D\omega_\mu^{\prime a}[P]=-t(x)\omega_\mu^{\prime a}[P]\,,\\&\delta_D\omega_\mu^{\prime a}[K]=t(x)\omega_\mu^{\prime a}[K]\,,\\&\delta_D\omega_\mu^{\prime}[D]=-\partial_\mu t(x)\,,\\&\delta_D\omega_\mu^{\prime ab}[L]=0\,.\end{aligned} \qquad (4.43)$$

Hence P has the dilatation degree $d=-1$, $\omega_\mu^{\prime a}[K]$ has $d=1$ and $\omega_\mu^{\prime}[D]$ as well as $\omega_\mu^{\prime ab}[L]$ have $d=0$. The infinitesimal K variation of the forms is given by the sum of their local L and D variations with the parameters specifically depending on the Goldstone fields $y^a(x)$. Fields $\varphi_a(x)$ transform under the action of K_a analogously. All forms as well as the fields $\varphi_a(x)$ remain invariant under the local P transformations. The covariant derivative of the field is defined by

$$\mathfrak{D}_\rho^{\prime}\varphi_a(x)=\partial_\rho\varphi_a(x)+d_\varphi\omega_\rho^{\prime}[D]\varphi_a+\frac{i}{2}\omega_\rho^{\prime mb}[L](L_{mb})_{ap}\varphi_p(x)\,. \qquad (4.44)$$

We again interpret form $\omega_\mu^{\prime a}[P]$ as the vierbein. Its covariant curl contains terms which depends on $\omega_\rho[D]$, namely

$$\begin{aligned}\tilde{C}_{\mu\rho}^{\prime a}&=\mathfrak{D}_\mu^{\prime}\omega_\rho^{\prime a}[P]-\mathfrak{D}_\rho^{\prime}\omega_\mu^{\prime a}[P]\\&=\partial_\mu\omega_\rho^{\prime a}[P]-\partial_\rho\omega_\mu^{\prime a}[P]\\&\quad-\omega_\rho^{\prime ab}[L]\omega_{\mu b}^{\prime}[P]-(\omega_\mu^{\prime ab}[L]\omega_{\rho b}^{\prime}[P])\\&\quad+(\omega_\mu^{\prime a}[P]\omega_\rho^{\prime}[D]-\omega_\rho^{\prime a}[P]\omega_\mu^{\prime}[D])\,.\end{aligned} \qquad (4.45)$$

The covariant curl of the form L coincides with the Einstein curvature tensor (2.15) (with $\omega_\mu^{\prime ab}[L]$ instead of Ω_ρ^{ab}). By putting (4.45) equal to zero we can express $\omega_\rho^{\prime ab}[L]$ in terms of $\omega_\mu^{\prime a}[P]$ and $\omega_\mu[D]$. Thus we get $\omega_\rho^{\prime ab}[L]$ in form (2.33) where ∂_ρ is replaced by $\partial_\rho-\omega_\rho[D]$. Though the curvature tensor has the dialation degree equal to zero, the scalar curvature has $d=2$ since the reciprocal vierbein has $d=1$. Determinant det$P$ transforms as a scalar with the $d=-4$ under (4.43). Thus the usual Einstein Lagrangian is not invariant with respect to (4.36). The simplest invariant gravitation Lagrangian is quadratic in curvature

$$\mathcal{L}_{\rm inv}\sim(\det\omega)R^2\,. \qquad (4.46)$$

Instead of R^2 we can take, e.g., $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu}{}^{ab}R_{ab}^{\mu\nu}$ which analogously to R^2 are scalars with $d=4$. Let us stress that tensor $R_{\mu\nu}^{mn}(x)$ depends on $\omega_\rho[D]$ in a very complicated way due to the condition $C_{\mu\nu}^{\prime a}=0$ postulated in order to exclude $\omega_\rho^{ab}[L]$. It is natural to interpret $\omega_\rho[D]$ as a gauge field on the commutative gauge group $I=e^{it(x)D}$.

Thus the gauge theory in which the quotient space is (4.36) leads to the gravitation theory based on the Weyl group $I\otimes P$ studied in detail in Refs. 18 and 19. The local K-transformations are spontaneously broken and are reduced to the local L and D transformations and therefore have no dynamical effect. We may again introduce the invariant mass terms for the field f_μ^a,

$$\sim\det\omega^{\prime}[P]\omega^{\prime\,\mu a}[P]\omega^{\prime\rho b}[P]\omega_{\mu a}^{\prime}[K]\omega_{\rho b}^{\prime}[K]\,,$$

$$\sim\det\omega^{\prime}[P]g^{\mu\rho}[P]\omega_\mu^{\prime a}[K]\omega_{\rho a}^{\prime}[K]\,,$$

so there is the Higgs effect for the field $f_\mu^a(x)$.

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