

Acceleration radiation and the generalized second law of thermodynamics

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It is shown that the Bekenstein limit, $S/E \leq 2\pi R$, for the entropy-to-energy ratio of matter confined by a box of size R is not needed for the validity of the generalized second law of thermodynamics. If one attempts to slowly lower a box containing rest energy E and entropy S into a black hole, there will be an effective buoyancy force on the box caused by the acceleration radiation felt by the box when it is suspended near the black hole. As a result there is a finite lower bound on the energy delivered to the black hole in this process and thus a minimal area increase which turns out to be just sufficient to ensure that the generalized second law of thermodynamics is satisfied. By reversing this process, we can "mine" energy from a black hole. The nature of these processes is also analyzed from an inertial point of view, and the mechanism by which energy is transported into and out of the black hole is explained. Analogous effects for accelerating boxes in flat spacetime are also analyzed.

I. INTRODUCTION

One of the most intriguing aspects of both the classical and quantum theory of black holes is the relationship between the laws of black-hole physics and thermodynamics. Classically, black holes are found to obey laws which are analogous to the ordinary laws of thermodynamics.¹ When quantum particle creation processes are taken into account, this relationship becomes more than just an analogy: A Schwarzschild black hole of mass M is found to emit like a perfect blackbody at temperature²

$$T_{\text{bh}} = \kappa/2\pi = 1/8\pi M, \quad (1.1)$$

where $\kappa = 1/4M$ is the surface gravity of the black hole. We use natural units such that $\hbar = G = c = k = 1$ here and throughout the paper.

Furthermore, it has been conjectured that a system consisting of ordinary matter interacting with a black hole will obey the *generalized second law of thermodynamics*,³ which states that generalized entropy S' never decreases. Here S' is defined by

$$S' = S + \frac{1}{4}A, \quad (1.2)$$

where S is the entropy of ordinary matter outside

the black hole and A is the area of the black hole. If the generalized second law is valid, then it is very plausible that the laws of black-hole physics are nothing more than the ordinary laws of thermodynamics applied to a self-gravitating quantum system. On the other hand, if the generalized second law can be systematically violated, the analogy between black holes and thermodynamics would break down. Thus, it is of considerable interest to determine if the generalized second law holds.

The generalized second law was first proposed by Bekenstein³ in the context of classical black-hole physics. In a classical context, two basic processes are known by which a violation of the generalized second law can be achieved.

(1) One could immerse the black hole in a radiation bath at temperature $T_r < T_{\text{bh}}$. Classically, the black hole is a perfect absorber but does not emit, so one would get a flow of energy from a cold body (the radiation) to a hotter body (the black hole) and hence a violation of the generalized second law.

(2) One could put matter with energy E and entropy S into a box and lower it towards the black hole. The energy ϵ delivered to the black hole

when the box is finally dropped in (or the box door is opened) will be decreased from E by the red-shift factor $\chi = (1 - 2M/r)^{1/2}$,

$$\epsilon = \chi E, \quad (1.3)$$

and can be made arbitrarily small by letting the dropping radius approach $2M$. As shown by Bekenstein, if one lowers the box to a proper distance R from the black hole such that

$$R < S/2\pi E, \quad (1.4)$$

then the area increase δA gives a black-hole entropy increase of

$$\begin{aligned} \delta S_{\text{bh}} &= \frac{1}{4} \delta A = \frac{1}{4} \delta(16\pi M^2) \\ &= 8\pi M \epsilon = T_{\text{bh}}^{-1} \epsilon \end{aligned} \quad (1.5)$$

which will not be large enough to compensate for the decrease of S in ordinary entropy. Thus the generalized second law will be violated.

Classically, these violations will be of no consequence as they can never be used to run a *perpetuum mobile* of the second kind since one cannot ever cause the classical black hole to return to its original state. However, the above two possible ways of violating the generalized second law become important once quantum processes near the black hole are taken into account. Then the black hole can both increase in size by absorption of energy and decrease by the emission of radiation by quantum processes as has been described by Hawking.² If one can still violate the generalized second law once these quantum effects are taken into account, one could imagine running a *perpetuum mobile* by decreasing the total generalized entropy by one of the above processes and then allowing the black hole to return to its initial state by means of its quantum evaporation.

As is well known, the first proposed violation of the generalized second law is easily resolved when this quantum particle creation process is taken into account.² If $T_r < T_{\text{bh}}$, the black hole will emit more energy than it will absorb, and no violation of the second law will occur. Indeed, since a black hole responds to incoming radiation exactly as an ordinary blackbody,⁴ any method for producing a violation of the generalized second law with black holes by sending in matter or radiation from infinity should also produce a violation of the ordinary second law using ordinary blackbodies. Thus, we can be confident that no violation of the generalized second law can be obtained by using the black hole as an "input-output machine" for any form of

matter and radiation which is allowed to fall freely into the black hole from infinity.

However, the situation with regard to the second proposed violation of the generalized second law is far less clear. A black hole of large mass and thus low temperature should behave essentially classically, so it would seem that the second of the above classical arguments leading to a violation should remain valid; the effects of Hawking radiation on the process of lowering a massive box into a massive black hole should be completely negligible. Bekenstein^{3,5} has proposed a resolution of this difficulty involving not the quantum properties of the black hole but the quantum properties of the matter being lowered into the black hole. He conjectured that any box of size R , energy E , and entropy S must satisfy

$$S/E \leq 2\pi R. \quad (1.6)$$

If this Bekenstein upper limit on S/E were satisfied, then no violation of the second law should be produced by lowering the box towards the black hole because the physical size of the box would be too large to permit one to lower it as close to the black hole as would be necessary. Recently, Bekenstein⁵ has given arguments for the validity of Eq. (1.6) for a variety of physical systems.

However, there are several disturbing features of this proposed resolution. First, Eq. (1.6) cannot be a fundamental law applicable to all conceivable physically reasonable systems. In particular if we imagine increasing the number of species n , of massless particles in nature, then we could make the S/E ratio arbitrarily large for a given R . Indeed, the arguments of Bekenstein⁵ in favor of Eq. (1.6) break down if $n \gtrsim 100$. [Note added in proof. Bekenstein (private communication) has pointed out that this is incorrect if each species has a minimum energy E_0 . If so, one motivation for this work is removed, but our conclusions remain unchanged.] Thus, Bekenstein's proposed resolution suggests that the validity of the generalized second law depends in an essential way on the fact that a reasonably small number of species of massless particles exist in nature. A related point is that the simple resolution of the first of the classical ways of violating the second law is obtained using only the quantum properties of black hole emission, with no assumption necessary about the entropy of physical matter going into the black hole. Even if Eq. (1.6) holds for all matter that is found in nature, it is somewhat disturbing that we would need to use this fact to rule out the second propo-

sal when we do not need to use any corresponding assumption to take care of the first. Finally, as pointed out to one of us by Thorne,⁶ if one uses a rectangular but noncubic box, the relevant dimension of the box for the lowering argument is the shortest length. However, the arguments of Bekenstein⁵ have established Eq. (1.6) for several physical systems where R is the largest length. Thus, even if the arguments of Bekenstein for Eq. (1.6) were valid for all physical systems, one still should be able to violate the second law by using a rectangular box with one side much shorter than the other two.

The main purpose of this paper is to show that the second of the proposed methods of violation of the second law will not work because of the acceleration radiation a box lowered slowly toward the black hole would feel. No assumptions such as Eq. (1.6) need be made on the entropy of matter within the box, and thus the nature of our resolution is very similar to the resolution of the first proposal. Our results lend strong support to the validity of the generalized second law of thermodynamics. In addition, our argument shows that the existence of acceleration radiation is vital for the self-consistency of black-hole thermodynamics.

The basic ideas of our argument are as follows: Consider a black hole surrounded by a thermal bath which is in equilibrium with the black hole. We show in Sec. II that, for such a black hole, a static object outside the horizon will respond as though it were in a thermal bath of temperature $T = T_{\text{bh}}/\chi$. Here $\chi = \sqrt{g_{00}} = (\xi_a \xi^a)^{1/2}$, where ξ^a is the static Killing vector field. [Our metric signature is (+ ---).] Far from the black hole this bath would be interpreted as that due to the "real" Hawking radiation coming out of the black hole and the radiation from the surrounding heat bath. Very near the black hole, however, this temperature would be ascribed to a cause similar to that of the acceleration radiation which an accelerated observer in flat spacetime would feel.⁷ This acceleration radiation is a purely quantum phenomenon and is a result of both the quantum state of the field near the horizon of the black hole and the state of acceleration of the object. For example, a freely falling object near the horizon would feel essentially nothing, while an object with the same acceleration but at rest outside of a static star would feel nothing if the field were in its vacuum state. We will call the thermal radiation seen by our static object "acceleration radiation" since we are primarily interested in the region near the ho-

zison. The physical effects of this radiation are crucial to our rescue of the second law.

Now, for "real" radiation, the condition $T \propto 1/\chi$ is precisely the condition needed for hydrostatic equilibrium as was shown by Tolman.⁸ Thus, a box of matter lowered towards the black hole (whose reflecting walls play the role of particle detectors and thus feel this radiation) would react the same way as if lowered into a real star composed of thermal radiation. There would be an effective buoyancy force on the box resulting from the higher temperature and thus higher pressure felt by the face of the box closer to the black hole. Consequently less work is delivered to infinity by the rope supporting the box during the process of lowering the box, and hence more energy is delivered to the black hole in this process than would occur classically. The energy delivered to the black hole is minimized when the box is dropped from its "equilibrium point," i.e., when the tension in the rope is zero. By the Archimedes principle⁹—rederived in Sec. II—this occurs when the energy of the box equals the energy of the displaced acceleration radiation. This minimum energy delivered to the black hole is found in Sec. II to increase the black-hole entropy by an amount equal to the entropy of the displaced acceleration radiation. However, the entropy contained in the box cannot be greater than the entropy of the displaced acceleration radiation since thermal radiation maximizes entropy at fixed energy and volume. Thus, the total generalized entropy cannot decrease.

Although our argument satisfactorily resolves the question of violating the second law by this process, it raises a number of rather puzzling new issues. Instead of simply dropping the entire box into the black hole in the above process, we could instead open the box door at the "floating point" of the energy inside the box and then pull back the open box to infinity. The energy-balance calculations show that, again, energy would be delivered to the black hole. What is puzzling about this is that if the box is filled with thermal radiation and the box door is opened at the equilibrium point, absolutely nothing should happen since the radiation in the box will be in equilibrium with the acceleration radiation. This becomes even more puzzling when it is realized that by reversing this process—i.e., lowering in an open box, closing the door, and pulling the box back to infinity—energy is extracted from the black hole. By what mechanism does the energy transfer into and out of the black hole occur?

A second, related issue concerns the physical nature of the acceleration-radiation pressure. By using it to calculate a force on the box, we have, in effect, ascribed a large stress energy to it. However, the true stress energy of a quantum field near a black hole is negligible small.¹⁵ Does this mean that the box produces an enormous disturbance near the black hole resulting in a large stress energy of the quantum field? Alternatively, must we ascribe an observer-dependent stress energy¹⁰ to the acceleration radiation?

The above issues are resolved in Sec. IV where we analyze this process from an inertial point of view. Using a two-dimensional model calculation, we show that for a *closed* box with perfect reflecting walls, energy actually flows out of the box as it is lowered toward the black hole on account of a “radiation by moving mirrors” effect. This explains how the energy is delivered to the black hole. Similarly, in the reverse process energy is extracted from the black hole by transfer of negative energy into the black hole as the closed box is removed. This also yields an alternative viewpoint on the acceleration-radiation buoyant force which is consistent with the negligibly small true stress energy present near the black hole: Less force is needed to hold up the box not because of an acceleration-radiation buoyant force but primarily because the box has actually become lighter by transferring its energy into the black hole. We regard the existence of these two alternate viewpoints—which differ greatly in interpretation but agree completely in all physical predictions—as one of the most aesthetically pleasing aspects of our analysis.

Finally, analogous effects to those described above occur for an accelerated box in flat space-time. These analogous effects are discussed in Sec. V.

II. ACCELERATION RADIATION RESCUES THE GENERALIZED SECOND LAW

In this section, we show that the effective buoyancy force provided by acceleration radiation is just sufficient to prevent a violation of the generalized second law when one lowers a box of energy E and entropy S toward a black hole. Taking account of this buoyancy force, we shall calculate the work W done at infinity by lowering the box, and hence the energy $\epsilon = E - W$ which is delivered to the black hole. The resulting change in black-hole entropy is given by Eq. (1.5) and we shall show that it is always greater than the matter entropy S

contained in the box. For simplicity we shall carry out our analysis only for a Schwarzschild black hole but we anticipate no difficulty in generalizing our results to an arbitrary stationary black hole.

Our first task is to spell out more precisely what we mean by “thermal radiation” and the properties we assume it possesses. By definition, the distribution of matter and radiation confined to a volume V will be said to be “thermal” if its entropy S is maximum for the given energy E and volume V . At fixed energy density

$$e = E/V \quad (2.1)$$

we assume that the entropy of thermal radiation is proportional to V (for large V , at least),

$$S = sV \quad (2.2)$$

i.e., that the entropy is additive. The thermodynamic properties of thermal radiation are determined by specifying the entropy density s as a function of energy density e ,

$$s = s(e) . \quad (2.3)$$

We shall place no limit on how large s can be. In particular, in our analysis we shall permit thermal or other distributions of matter and radiation to violate the Bekenstein limit, Eq. (1.6).

The temperature of thermal radiation is given by

$$\frac{1}{T} = \frac{ds}{de} . \quad (2.4)$$

From the first law of thermodynamics, we have

$$d(eV) = Td(sV) - PdV . \quad (2.5)$$

This yields

$$P = sT - e . \quad (2.6)$$

Differentiating Eq. (2.6), we obtain a special case of the Gibbs-Duhem relation¹¹

$$dP = s dT \quad (2.7)$$

Thus, for thermal radiation, the pressure and temperature gradients are related by the proportionality factor of entropy density.

Our next task is to determine precisely how a particle detector (such as the reflecting walls of a box) would behave if held in a stationary position near a Schwarzschild black hole. We can classify the free-field radiation modes around a Schwarzschild black hole into two classes: “white-hole modes” and “past-infinity modes.” By definition, the white-hole modes are those which, in the maximally extended vacuum Schwarzschild

solution, would have zero initial data on past null infinity. Thus, white-hole-mode radiation would appear to an observer to emanate from the black-hole region of the spacetime. Similarly, the past-infinity modes have vanishing initial data on the white-hole horizon and would appear to an observer as having originated as ordinary radiation coming in from infinity. The answer to how a stationary particle detector would behave is already known in two limits. If the past-infinity modes are empty (i.e., in their vacuum state), and if the detector is far from the black hole, it will see Hawking radiation, i.e., a thermal distribution of white-hole modes at temperature $T = T_{\text{bh}}$. On the other hand, if the detector is very near the black hole⁷ it would react as it would in flat spacetime in the vacuum if it were undergoing the same acceleration $a \approx \kappa/\chi = 2\pi T_{\text{bh}}/\chi$, where κ is the surface gravity of the black hole and $\chi = (\zeta^a \zeta_a)^{1/2}$ is the red-shift factor. Thus, a static detector very near the black hole would see a thermal spectrum at temperature $T = a/2\pi \approx T_{\text{bh}}/\chi$.

The exact answer to what a static detector would see at any radius can be found as follows. From the previous analysis of the Hawking effect for free fields,¹² it can be seen that, with respect to an “out” Fock space defined using “positive frequency” with respect to the Killing field coordinate time, if one starts with the “in” vacuum state, the complete state of the system at late times is precisely a thermal density matrix with respect to white-hole modes and the vacuum state with respect to past-infinity modes. [This can be seen from Eq. (4.22) of Ref. 12 for the two-particle amplitude. When one “traces out” over the “early time” states τ_i and ϵ_o , one produces precisely a thermal density matrix for the white-hole modes λ_i . Further “tracing out” over the “late time horizon states” σ_i , of course, yields the previous result that the state of the system at future infinity is described by a thermal density matrix of white-hole modes.] The Killing parameter time times the red-shift factor χ is the relevant notion of time for the behavior of a stationary detector, just as the Rindler time is the relevant time for the behavior of an accelerating detector in flat spacetime. Thus, a repetition of the analysis of Ref. 7 shows that a stationary detector near a Schwarzschild black hole will respond as though placed in a thermal bath of radiation at temperature

$$T = T_{\text{bh}}/\chi \tag{2.8}$$

with respect to the white-hole modes, but it will

see no radiation in the past-infinity modes. [This reduces to the previous result in the limit where the detector is close to the horizon because there the white-hole modes dominate over the past-infinity modes—i.e., the black hole subtends almost the entire sky—so Eq. (2.8) becomes valid for essentially all of the modes.] If thermal radiation at temperature T_{bh} is sent in from past infinity, a stationary detector would respond as though in a radiation bath given by Eq. (2.8) for *all* modes. Thus, if the black hole were placed in a box with perfect reflecting walls and allowed to come to equilibrium, Eq. (2.8) would hold exactly for all modes. For simplicity we will consider this case of a black hole in thermal equilibrium in our analysis of the attempt to violate the generalized second law. It is not difficult to see that having the black hole radiate into empty space only makes it more difficult to violate the generalized second law because of the entropy increase due to Hawking radiation into empty space during the time the box is lowered, and because of the extra work done on the box by the Hawking radiation. In any case, if the black hole is very massive, the Hawking radiation will be negligible and there will be essentially no difference in the analysis of a black hole in thermal equilibrium and a black hole radiating into empty space. Finally, we should point out that Eq. (2.8) has been derived only for free fields, but we shall assume that it continues to hold for interacting fields.

As noted previously, for *real* thermal radiation in a static spacetime, the condition

$$T = T_0/\chi \tag{2.9}$$

is precisely the condition that the radiation be in hydrostatic equilibrium.⁸ For a static spacetime, the hydrostatic equilibrium equation, derived from $\nabla^a T_{ab} = 0$ for a perfect-fluid stress-energy tensor, is

$$\begin{aligned} \nabla_a P &= (e + P) \left[\frac{\zeta^b}{\chi} \right] \nabla_b \left[\frac{\zeta_a}{\chi} \right] \\ &= -(e + P) \frac{1}{\chi} \nabla_a \chi . \end{aligned} \tag{2.10}$$

Substituting the thermodynamic relations (2.6) and (2.7) into Eq. (2.10), we obtain

$$s \nabla_a T = -(sT) \frac{1}{\chi} \nabla_a \chi , \tag{2.11}$$

from which Eq. (2.9) follows immediately.

Let us now put entropy S and energy E into a

box in our laboratory at infinity and slowly lower the box toward the black hole. We will *not* assume that the matter in the box satisfies the Bekenstein limit, Eq. (1.6). The only fact we shall use is that, at fixed volume and total energy, thermal radiation maximizes entropy. Thus the total entropy S contained in the box must satisfy

$$S \leq V_s (E/V), \quad (2.12)$$

where $s(e)$ is the entropy density of thermal radiation. In our analysis, we shall neglect the mass of the box and string. As discussed further in Sec. III, this is *not* a reasonable assumption since large stresses will be required when we lower the box near the black hole, and thus large energy densities of the box and string will be needed if the energy conditions are to be satisfied for the materials that make up the box and string. However, if instead of dropping the box and/or string into the black hole we merely open the door of the box and then pull it back to infinity, the extra work done when lowering the mass of the box and string will be compensated by the work done in bringing them back to infinity. Thus, the energy-balance calculations in this case would be the same as if one neglected the mass of the box and string and dropped both of them into the black hole. Of course, if the box and string are given mass and are dropped into the black hole, the black-hole area will be increased and it will become more difficult to violate the generalized second law.

The final fact we shall need in order to do our energy-balance calculation is that if a locally measured force F_{loc} is exerted by the end of the string near the black hole then, neglecting the mass of the string, the corresponding force F_∞ that must be exerted on the end of the string in our laboratory at infinity to keep the string stationary is

$$F_\infty = \chi F_{\text{loc}}, \quad (2.13)$$

where χ is the red-shift factor. Equation (2.13) follows from the fact that the “energy at infinity” E_∞ is related to the locally measured energy E_{loc} by the same red-shift factor $E_\infty = \chi E_{\text{loc}}$. Thus the work done at infinity in moving the string must be smaller by the factor χ than the locally measured work done on the box by the string. Since the distance moved is the same, the forces must satisfy Eq. (2.13). Equation (2.13) also can be derived by integrating the local conservation of energy-momentum for the string, $\nabla_a T^{ab} = 0$, and setting the mass density of the string to zero.

In the following analysis we shall assume that

the box has a height much less than its distance to the horizon. This implies that the changes in χ and $d\chi/dl$ across the box are much less than their average values. If this condition is not satisfied, we have to consider the detailed distribution of energy within the box. This would complicate the analysis, but we believe it would not affect the validity of our results.

Let us now tally up the forces that need to be exerted at infinity in order to hold the box stationary. Classically, a local force

$$F_{\text{loc}} = Ea = E \frac{1}{\chi} \frac{d\chi}{dl}$$

would need to be exerted to hold the box in position, where a is the proper acceleration of the box, and l is a coordinate measuring proper length in the direction of the string. Thus, the classical force at infinity is

$$F_1 = \chi F_{\text{loc}} = E \frac{d\chi}{dl}. \quad (2.14)$$

However, this classical force is not the only force acting on the box. The walls of the box must be made out of a reflecting material in order to hold the matter and radiation put into it, and these reflecting walls will “feel” the thermal acceleration-radiation pressure discussed above. Assuming, for simplicity, that the box is rectangular in shape and aligned with the string, we find that the front wall closest to the black hole will feel a local force $AP(l_f)$, while the back wall will feel a force $AP(l_b)$, where A denotes the area of these faces. Thus, an “extra” force F_2 (opposite the direction of F_1) will have to be exerted at infinity in order to counteract the acceleration-radiation pressure, where F_2 is given by

$$\begin{aligned} F_2 &= \chi(l_b)AP(l_b) - \chi(l_f)AP(l_f) \\ &\approx AR \frac{d(\chi P)}{dl} \\ &= V \frac{d(\chi P)}{dl}, \end{aligned} \quad (2.15)$$

where R denotes the proper height of the box, and V the volume of the box. Thus, the total force exerted at infinity is

$$F_\infty = F_1 + F_2 = E \frac{d\chi}{dl} + V \frac{d(\chi P)}{dl}. \quad (2.16)$$

Consequently, the work done in lowering the box down to height l (i.e., the energy gained at infinity) is

$$\begin{aligned} W &= \int F_\infty dl = (E\chi + V\chi P) \Big|_l^\infty \\ &= E - (E + PV)\chi \Big|_l, \end{aligned} \quad (2.17)$$

where we have neglected the buoyancy pressure at infinity (or, equivalently, have taken into account the work done in bringing the box into the radiation-filled cavity containing the black hole). This differs from the classical expression by the term $-PV\chi \Big|_l$, which is the work done by the buoyancy force of the acceleration radiation. Note that this term dominates the classical term $E\chi$ as one approaches the horizon since $T \rightarrow \infty$, and hence we have

$$P = \int s dT \rightarrow \infty.$$

Originally, we started with the energy E in the box in our laboratory at infinity. An energy W has now been recovered by lowering the box to height l . Let us now drop the box into the black hole. It is interesting to consider what happens when we release the box. If the dropping point satisfies $F_\infty > 0$ (so that the gravitational attraction dominates the buoyancy force) we would expect the local acceleration of the box to decrease to zero, and the box should fall directly into the black hole since it no longer “feels” the acceleration radiation. On the other hand, if $F_\infty < 0$ (so that buoyancy dominates and the box has been “pushed” in), then the box should actually move away from the black hole initially when it is released. However, our analysis of the buoyancy force applies only in the static or quasistatic limits, so we cannot analyze the dynamical behavior of the box after release. (In addition, radiation reaction forces will act on the box walls when the acceleration changes with time. The nature of these forces as well as the stress properties the box must satisfy in order to withstand the forces acting on it when it is pushed beyond the floating point will be discussed in Sec. V.) For the purposes of the discussion below, we shall assume that the box eventually “turns around” and falls into the black hole. In fact, as mentioned above and also discussed at the beginning of Sec. III, we can avoid the issue of the dynamical behavior of the box entirely in our *Gedankenexperiment* by opening and closing doors in the box rather than releasing the box.

By conservation of energy, the energy ϵ delivered to the black hole in the total process of lowering the box and dropping it into the black hole must be the difference between the original energy E and the work W done at infinity during the process of

lowering. Using Eq. (2.17), we obtain

$$\epsilon = E - W = (E + PV)\chi, \quad (2.18)$$

where the right-hand side of Eq. (2.18) is evaluated at the dropping point. In order to minimize the increase in black-hole entropy, Eq. (1.5), we wish to minimize ϵ . The condition which minimizes ϵ is simply

$$0 = \frac{d\epsilon}{dl} = -F_\infty = -E \frac{d\chi}{dl} - V \frac{d(\chi P)}{dl}, \quad (2.19)$$

i.e., the box is dropped from its floating point. Since the acceleration radiation satisfies the hydrostatic equilibrium Eq. (2.10), we have

$$\frac{d(\chi P)}{dl} = -e \frac{d\chi}{dl}. \quad (2.20)$$

Thus, the floating-point condition is simply

$$E = eV, \quad (2.21)$$

i.e., the box displaces its own weight of acceleration radiation. This principle has been noted previously in the other contexts.⁹

Substituting this result in Eq. (2.18), we find

$$\begin{aligned} \epsilon_{\min} &= (eV + PV)\chi \\ &= VT\chi s (E/V) \\ &= T_{\text{bh}} V s (E/V), \end{aligned} \quad (2.22)$$

where Eq. (2.6) was used in the second line and Eq. (2.8) was used in the third. Hence, the minimum entropy increase of the black hole is

$$\begin{aligned} (\delta S_{\text{bh}})_{\min} &= \frac{1}{T_{\text{bh}}} \epsilon_{\min} \\ &= V s (E/V), \end{aligned} \quad (2.23)$$

i.e., the black-hole entropy increases by at least the entropy of the displaced acceleration radiation at the floating point. Thus, the total change in generalized entropy in the process is

$$\begin{aligned} \delta S' &= -S + \delta S_{\text{bh}} \\ &\geq -S + V s (E/V) \geq 0, \end{aligned} \quad (2.24)$$

where Eq. (2.12) was used to get the final inequality. Thus, the buoyancy force of the acceleration radiation prevents one from achieving a violation of the generalized second law.

III. MINING ENERGY FROM A BLACK HOLE

In the calculation of the previous section, we neglected the mass of the box and string. In fact, large stresses will be placed on the box and string as we lower the box close to the black hole. Thus, if the box and string are composed of matter satisfying, say, the weak energy condition, they must be given a large rest mass. (Indeed, if we require $\mathcal{T} \leq c\mu$ with $c < 1$, where \mathcal{T} is the string tension and μ is its mass per unit length, then we also must “taper” the string at its end near the black hole in order to keep it from breaking under its own weight.) The large rest mass of the box will prevent us from coming close to the maximum entropy limit, Eq. (2.12), and the rest mass of the string will alter our energy-balance calculations.

In order to avoid these difficulties as well as our inability to analyze the dynamics of the box after it is dropped, we can dispose of the contents of the box into the black hole by opening a door on the lower face of the box at the appropriate point and allowing the matter within the box to fall into the black hole, and then withdrawing the opened box to infinity. If we assume the box walls have negligible volume and thus displace a negligible amount of radiation, the box will no longer feel any buoyancy force while it is being withdrawn. In this case one will maximize the entropy increase of the black hole by opening the door at the point where the energy of the radiation displaced by the box is equal to the energy of the material contained within the box, not including the energy of the box itself. The energy gained at infinity from lowering the mass of the box and string will be just equal to the energy lost in pulling the box and string back to infinity.

Now, suppose we fill the box with thermal radiation at infinity, lower it to the radius where its temperature equals the temperature of the ambient acceleration radiation, open its door and return it to infinity. The entropy increase of the black hole will just equal the entropy lost into the black hole by this thermal radiation falling into the black hole. The process is thus isentropic and one can imagine reversing the process. An open box lowered in toward the black hole. The box is then closed at a certain distance from the horizon trapping the thermal radiation present at that radius and then withdrawn from the black hole. In this process one will have extracted a net energy

$$\epsilon = T_{\text{bh}} V_S (T_{\text{bh}} / \chi) \quad (3.1)$$

from the black hole. [See Eq. (2.22).] We will have expended the work W given by Eq. (2.17) but will have recovered the energy, $W + \epsilon$, in the form of thermal radiation in the box when it is far from the black hole. In this process we have literally mined the acceleration radiation that surrounds the black hole.

From the point of view of an accelerated observer, there is a large flux of radiation trapped near the horizon of the black hole by a gravitational and angular momentum potential barrier. The work done in raising the box allows that portion of the radiation trapped within the box to overcome this barrier and be brought to infinity.

How rapidly can this radiation be mined? Our analysis is strictly valid only in the quasistatic limit, but we believe that the mining process can be carried out at a finite rate without affecting the analysis significantly. The only important modification is that the walls of the box will radiate due to the changing acceleration of the walls (see Sec. V). The work done against the radiation reaction forces may be larger than the energy extracted if the rate is too high. However, we can find no reason to believe that this will place any fundamental limitation on the rate at which the energy can be extracted. In particular, there seems to be no barrier to extracting energy from a large black hole at a much greater rate than the ordinary energy loss rate due to the Hawking radiation.

Unfortunately, the technological applications of black-hole mining do not appear to be very promising. The Hawking temperature of a Schwarzschild black hole is $1/8\pi M$. Therefore, the mining of thermal energy at a temperature T requires lowering the front end of the box to a red-shift factor of $\chi = 1/8\pi MT$. Provided that $T_{\text{bh}} \ll T$, the proper distance of the box from the horizon is

$$R = \int_{2M}^r \frac{dr}{(1 - 2M/r)^{1/2}} \approx 4M \left[1 - \frac{2M}{r} \right]^{1/2} = 4M\chi. \quad (3.2)$$

[Note that $R(T)$ is thus independent of the mass of the black hole for $T_{\text{bh}} \ll T$.] Thus, in order to mine thermal energy at, say 100 K, we need to lower the box to within a distance $R \sim 10^{-3}$ cm from the horizon. At $T = 100$ K, the buoyancy pressure on the box (which would help support it) should be negligible. The local acceleration of the box is

$$a = \frac{1}{\chi} \frac{d\chi}{dl} \approx \frac{1}{4M\chi} = \frac{T}{2\pi}. \quad (3.3)$$

Thus, for $T = 100$ K, we need an acceleration of about 10^{22} cm/sec². The spatial gradient of acceleration is

$$\frac{da}{dl} = \frac{d}{dl} \left[\frac{1}{\chi} \frac{d\chi}{dl} \right] \approx a^2 \quad (3.4)$$

so for $T = 100$ K and small Δl the differential acceleration between the front and back faces of the box in cgs units is $\Delta a \sim 10^{23} \Delta l$, where Δl is the height of the box. Thus, the engineering requirements for black-hole mining at, say, $T = 100$ K are truly formidable. We must build a box that can withstand the above differential acceleration, and we must be able to position this box to an accuracy of 10^{-3} cm. We must also build a string that is able to accelerate the box to 10^{22} cm/sec².

IV. THE INERTIAL VIEWPOINT

In the previous two sections, we showed how acceleration radiation around a black hole prevents a violation of the generalized second law and how it can be mined. However, as already mentioned in the Introduction there are several puzzling aspects of our analysis. If we lower a box filled with thermal radiation down to the radius where the acceleration radiation has the same temperature and open the box door absolutely nothing should happen during the opening of the door since the matter inside the box is in thermal equilibrium with the acceleration radiation. Yet, by the time the open box has been returned to infinity an energy ϵ will have been delivered to the black hole. Where, when, and by what mechanism did the energy transfer into the black hole occur? Similarly in our analysis of black-hole mining the mined radiation was the acceleration radiation outside the horizon. However, we know that the energy extracted must ultimately have come from the black hole itself. How was the energy actually extracted from the black hole?

A related puzzle is what the viewpoint of an inertial observer would be. An inertial observer does not see any acceleration radiation. He would deny the existence of the thermal bath near the horizon, and would deny the existence of a buoyancy force. However, he must somehow conclude that the tension in the rope holding up the box is

less than it would be in a classical analysis. Furthermore, when in the mining process the box door is closed, there is for him no acceleration radiation to be trapped in the box. Yet, if our analysis is correct, he will agree that when the box arrives far from the black hole it is full of thermal radiation. How did this energy get into the box?

These puzzles are resolved by a closer examination of the quantum effects of the perfect reflecting walls composing the box. As has been established in studies of "moving mirrors" in two-dimensional spacetimes,^{13,14} if the proper acceleration of a mirror changes with time, energy will be transferred across the mirror via quantum effects. Thus, if a rectangular closed box is lowered toward a black hole energy will be transferred across both faces of the box which are normal to string. However, since there is a larger change in acceleration for the face closer to the black hole, a net amount of energy is transferred out of the box in the lowering process. This provides the required mechanism for transferring energy from the box into the black hole. Similarly, if a closed box is removed from the vicinity of a black hole, negative energy is transferred out of the box and into the black hole. This explains how a black hole can be mined in the inertial viewpoint.

This energy transfer into the black hole by the quantum field also provides a simple explanation in the inertial viewpoint for the reduction in the force necessary to hold up the box. It is not due to a buoyant force (except for a small residual vacuum polarization effect which becomes negligible as the box approaches the horizon) but rather due to the box having radiated its energy and thus its weight into the black hole. At the floating point, all of the box's rest mass has been transferred to the black hole, the box weighs nothing, and no force is required to hold it up. If the box is pushed in beyond its floating point, its mass, from the inertial observers viewpoint, has become negative and the box has to be pushed in to hold at such a radius.

That one should expect an energy transfer out of an accelerated box can be seen from the following considerations. Suppose, for simplicity, we lower an empty box toward a black hole. If we lower the box sufficiently slowly, no particle creation will occur and the interior of the box will remain in the vacuum state with respect to its local time coordinate. However, as we shall show by an explicit model calculation below, this vacuum state will be essentially the *Rindler vacuum* state of flat space-

time. Ignoring the Casimir energy and the true quantum stress energy near the black hole (both of which become negligible for large boxes near large black holes) the stress energy inside the box will be the *negative* stress energy of the Rindler vacuum associated with the local acceleration of the box. Thus, energy must have been transferred out of the box as it was lowered.

This inertial viewpoint may appear to differ radically from the “accelerating viewpoint” given in Secs. II and III. Do these two viewpoints always lead to identical physical predictions? It is not difficult to see that they do. Two analyses which differ by a conserved T_{ab} will make equivalent dynamical predictions as long as self-gravitating effects are ignored (as they were in our analysis above). The “inertial” and “accelerating” analyses are exactly of this character. For the accelerated observer the natural reference state is the vacuum state defined with respect to the Schwarzschild time—i.e., the state in which an observer at constant radius sees no particles. By assigning zero stress energy to this state, the accelerated observer’s stress energy will differ from the true (inertial) stress energy by the true stress energy of this accelerated vacuum. This difference between the two stress energies is a conserved tensor, however, and thus will make identical dynamical predictions. For example, if we lower an empty box close to a black hole, in the accelerating viewpoint of Secs. II and III the box remains empty (no stress energy) but becomes surrounded with thermal radiation (large stress energy). In the inertial viewpoint, the box surroundings remain essentially empty (small stress energy) but the box fills up with a large negative stress energy. However, the forces on the box, the tension on the string, and the energy transmitted to infinity are identical in the two viewpoints. It is the inertial viewpoint that is literally correct, since the stress energy it assigns is presumably the physically correct one, i.e., the one that would also give the correct self-gravitating effects. However, the accelerating viewpoint has considerable intuitive advantages since it avoids dealing with negative stress energies.

It is instructive to give a detailed calculation in the inertial viewpoint of the energy flow into or out of a box as it is moved near a black hole to illustrate the above points. We have not been able to carry out such a calculation in four dimensions because we do not know the true quantum stress energy around a black hole or the energy transfer across a reflecting wall in four-dimensional curved

spacetime. However, we have done a model calculation in two spacetime dimensions which, we believe, contains all the essential features of the problem. We present now this calculation.

The behavior of mirrors on a massless scalar field in two dimensions has been extensively analyzed by Davies and Fulling,¹³ Fulling and Davies,¹⁴ and others. We will use the result derived in Davies, Fulling, and Unruh¹⁵ (DFU) for the energy-momentum tensor for such a massless scalar field. Any two-dimensional metric is conformally flat and may be written as

$$ds^2 = \Omega^{-2} du dv, \quad (4.1)$$

where u, v are null coordinates. If the null coordinates are chosen so that the state of the field in the past is defined to be the vacuum state with respect to the normal modes in u, v coordinates, then the expectation value for the energy-momentum tensor is given by

$$T_{uu} = -\Omega_{,u,u} / (12\pi\Omega), \quad (4.2)$$

$$T_{vv} = -\Omega_{,v,v} / (12\pi\Omega), \quad (4.3)$$

$$T_{uv} = \mathcal{R} / 96\pi\Omega^2, \quad (4.4)$$

where \mathcal{R} is the scalar curvature. The condition of being in the vacuum with respect to the u, v modes means that if the quantum field is expanded in terms of normal modes

$$\Phi = \sum_{\lambda > 0} (a_{\lambda} \phi_{\lambda} + a_{\lambda}^{\dagger} \phi_{\lambda}^*), \quad (4.5)$$

where the ϕ_{λ} have a u dependence of $e^{i\lambda u}$ and/or a v dependence of $e^{i\lambda v}$, then the state of the field $|\psi\rangle$ is defined by

$$a_{\lambda} |\psi\rangle = 0. \quad (4.6)$$

For a moving mirror, we choose the coordinates to the right of the mirror as u, v and to the left as u', v' such that the state of the incoming field is the vacuum state with respect to the u modes to the right of the mirror and the v' modes to the left. The coordinates v and u' are then defined so that the mirror trajectory is given by the equations

$$u - v = 0, \quad v' - u' = 0. \quad (4.7)$$

(u', v can always be chosen in such a way for a two-dimensional spacetime.) Let Ω and Ω' be the conformal factors for the u, v and u', v' coordinates, respectively,

$$\begin{aligned} ds^2 &= \Omega^{-2} du dv \quad \text{to right} \\ &= \Omega'^{-2} du' dv' \quad \text{to left.} \end{aligned} \quad (4.8)$$

We will be interested in the expectation value of the energy flow at the mirror in the frame of the mirror. Let V^μ be the four-velocity of the mirror, and let a^μ be its acceleration. We have

$$V^\mu = (V^u, V^v) = \Omega(1, 1), \tag{4.9}$$

$$a^\mu = \Omega(\Omega_{,v} - \Omega_{,u})(1, -1). \tag{4.10}$$

Thus, we obtain

$$\begin{aligned} T_{\mu\nu} V^\mu a^\nu &= (T_{uu} - T_{vv}) \Omega^2 (\Omega_{,v} - \Omega_{,u}) \\ &= \frac{-\Omega}{12\pi} (\Omega_{,u,u} - \Omega_{,v,v}) (\Omega_{,v} - \Omega_{,u}). \end{aligned} \tag{4.11}$$

However, note that

$$\begin{aligned} \frac{d}{d\tau} (-a^\mu a_\mu) &= \frac{d}{d\tau} (\Omega_{,u} - \Omega_{,v})^2 \\ &= 2(\Omega_{,u,u} - \Omega_{,v,v}) (\Omega_{,u} - \Omega_{,v}) \Omega, \end{aligned} \tag{4.12}$$

where τ denotes the proper time along the world line of the mirror. Therefore, we have

$$T_{\mu\nu} V^\mu \frac{a^\nu}{a} = \frac{1}{12\pi} \frac{da}{d\tau}, \tag{4.13}$$

where $a = (-a^\mu a_\mu)^{1/2}$. The left-hand side is just the quantity of interest: the energy flow in the direction of the acceleration, since the acceleration is orthogonal to the velocity.

Consider now the box which we slowly lower into the black hole. The acceleration increases as the box comes closer to the horizon, and thus $da/d\tau$ is positive for both the top and bottom faces of the box. Since the acceleration vector points away from the black hole, we find that there is a positive energy flow toward the black hole from the bottom mirror, and a negative energy flow into the box. Similarly there is a negative energy flow outward from the top mirror, and a positive energy flow into the box. The net energy flow out of the bottom mirror is given by

$$\int_{\text{bottom}} T_{\mu\nu} V^\mu \frac{a^\nu}{a} d\tau = \frac{\Delta a_b}{12\pi}. \tag{4.14}$$

Similarly the net energy flow into the top of the top mirror is given by

$$\int_{\text{top}} T_{\mu\nu} V^\mu \frac{a^\nu}{a} d\tau = \frac{\Delta a_t}{12\pi}, \tag{4.15}$$

where Δa_b and Δa_t are the changes in acceleration of the bottom and top mirrors, respectively. If we assume that the box was initially at infinity where the initial acceleration was zero for both the top and bottom of the box, we find that the total ener-

gy flow out of the box as it is lowered into the black hole is given by

$$\delta E = (a_b - a_t) / 12\pi. \tag{4.16}$$

But the acceleration of the top of the box is less than that of the bottom, implying that the box must have lost energy in the lowering process.

Thus, energy is indeed transferred out of a box when it is lowered toward a black hole. Similarly, *negative* energy flows out of the box when it is returned to infinity. This explains the energy-transfer mechanism from the inertial viewpoint.

Let us now demonstrate that the energy flow out of the box together with the true vacuum pressure are just sufficient to account for the reduced string tension predicted by the ‘‘accelerating-observer viewpoint.’’ We take as two-dimensional black-hole model spacetime the metric

$$\begin{aligned} ds^2 &= (1 - 2M/r) dt^2 - dr^2 / (1 - 2M/r) \\ &= \frac{2Me^{-r/2M}}{r} dU dV, \end{aligned} \tag{4.17}$$

where U, V are Krushal coordinates. We will now calculate the net force which the rope must supply to hold up the box. This will be equal to the weight of the box (reduced by the flow of energy out of the box due to its changing acceleration) minus the buoyant force due to the residual pressures on the top and bottom faces of the box produced by the quantum field outside the box.

In order to calculate the pressure of the quantum field on the box we must calculate the stress-energy tensor of the quantum field in the state in which the black hole is in thermal equilibrium. In our two-dimensional model, this corresponds to choosing appropriate null coordinates u and v' on the two sides of the box. As was shown by Hartle and Hawking¹⁶ the appropriate choice is to use the Kruskal-type coordinates throughout. The appropriate conformal factor to use in Eqs. (4.2) to (4.4) is

$$\Omega = (2Me^{-r/2M}/r)^{-1/2} \tag{4.18}$$

with u, v, u', v' being the coordinates U, V of (4.17). The appropriate stress-energy tensor in t, r coordinates is given by

$$T_{tt} = \frac{1}{24\pi} \left[\frac{7M^2}{r^4} - \frac{4M}{r^3} + \frac{1}{16M^2} \right], \tag{4.19}$$

$$T_{rr} = -\frac{1}{24\pi} \frac{1}{(1 - 2M/r)^2} \left[\frac{M^2}{r^4} - \frac{1}{16M^2} \right], \tag{4.20}$$

$$T_{rr}=0. \quad (4.21)$$

The pressure on a wall of the box located at radius r is given by $T_{r'r'}$ where r' is the proper radius

$$r' = \int \frac{dr}{(1-2M/r)^{1/2}}. \quad (4.22)$$

The net force which the rope must supply is then given by

$$F = (E_0 - \delta E)a - \Delta T_{r'r'} = E_0 a - \Delta \left[\frac{1}{24\pi} a^2 - \frac{1}{24\pi} \frac{1}{(1-2M/r)} \left[\frac{M^2}{r^4} - \frac{1}{16M^2} \right] \right], \quad (4.23)$$

where Δ denotes taking the difference between the bottom and top of the box of the term in square brackets. However, we have

$$a^2 = \frac{1}{(1-2M/r)} \left[\frac{M}{r^2} \right] \quad (4.24)$$

giving us

$$\begin{aligned} F &= E_0 a - \Delta \left[\frac{\pi}{6} \frac{1}{(1-2M/r)} \frac{1}{(8\pi M)^2} \right] \\ &= E_0 a - \Delta \left[\frac{\pi}{6} T^2 \right], \end{aligned} \quad (4.25)$$

where T is defined in Eq. (2.8) and E_0 is the initial energy of the box. But the term $\frac{1}{6}\pi T^2$ is just equal to the pressure of thermal radiation of temperature T for a massless free scalar field in two dimensions. This expression is therefore exactly the same as we obtained from the accelerated observers point of view. The energy flow out of the box together with the true quantum stress energy does, indeed, properly account for the reduced string tension.

For completeness, we also examine the stress energy of the material enclosed within the box. We will show that the energy inside the box is the sum of the Rindler vacuum energy, the Casimir energy, and particle energy terms. Again we will examine the behavior of a free massless scalar field within a box in two-dimensional spacetime. Because of the multiple reflection of any change of state of the field from the two walls of the box, it is very difficult to analyze the behavior of the field within the box in nonstationary situations. We will therefore calculate the stress-energy tensor for the enclosed field for a stationary box. We assume that the metric within the box is of the form

$$ds^2 = \Omega^{-2}(x)(dt^2 - dx^2) \quad (4.26)$$

with the walls of the box located at $x=0$ and $x=H$. The solution for the modes within the box is given by

$$\phi_n(t, x) = \frac{e^{-in\omega_0 t} \sin n\omega_0 x}{(n\omega_0 H)^{1/2}}, \quad (4.27)$$

where ω_0 is π/H . The quantum field Φ in the box can be expanded as

$$\Phi(t, x) = \sum_n [a_n \phi_n(t, x) + a_n^\dagger \phi_n^*(t, x)].$$

Let us now assume that the field is in a state such that the mean number of particles in any state n is given by $N(n)$ —i.e.,

$$\langle a_n^\dagger a_{n'} \rangle = N(n) \delta_{nn'}. \quad (4.28)$$

Now, the expectation value for the stress-energy tensor can be written as

$$\begin{aligned} \langle T_{\mu\nu} \rangle &= \langle 0 | T_{\mu\nu} | 0 \rangle \\ &+ \sum_n N(n) T_{\mu\nu}^{(n)}, \end{aligned} \quad (4.29)$$

where $T_{\mu\nu}^{(n)}$ is the stress energy of the classical field ϕ_n and where the state $\langle 0 |$ is defined by

$$a_n | 0 \rangle = 0. \quad (4.30)$$

The first term of (4.29) is divergent and must be regularized. Using point-splitting regularization in the same way as was done in obtaining Eqs. (4.2) to (4.4) in DFU (Ref. 15) we find

$$\langle 0 | T_{tt} | 0 \rangle = \left[-\frac{\Omega_{,x,x}}{12\pi\Omega} + \frac{(\Omega_{,x})^2}{24\pi\Omega^2} \right] - \frac{\pi}{24H^2}, \quad (4.31)$$

$$\langle 0 | T_{tx} | 0 \rangle = [0], \quad (4.32)$$

$$\langle 0 | T_{xx} | 0 \rangle = \langle 0 | T_{tt} | 0 \rangle - [\mathcal{R}/24\pi\Omega^2]. \quad (4.33)$$

The terms in square brackets are identical to the results in free space given in Eqs. (4.2) to (4.4) where u, v are given by $(t-x)$ and $(t+x)$, respectively. The additional term proportional to H^{-2} is the so-called Casimir energy. Assuming the box to be small so that $\Omega(0) \approx \Omega(H)$, we find the proper Casimir energy E_C is given by

$$E_C = \Omega \int_0^H T^t, \Omega^{-2} dx = -\frac{\pi}{24L}, \quad (4.34)$$

where L is the proper length of the box, H/Ω . This energy can be associated with the box itself; it is part of the energy necessary to construct the box and remains constant at all heights.

Similarly, if the box is lowered sufficiently slowly, $N(n)$ will remain constant during the lowering process. The contribution of these quanta to the total proper energy

$$\sum N(n)n\omega_0\Omega = \sum_n N(n)\frac{n\pi}{L} \quad (4.35)$$

also remains constant as the box is lowered. However, the contribution due to the vacuum polarization does not remain constant, but rather decreases as the box is lowered nearer to the horizon of the black hole. The term in square brackets in (4.31) gives a contribution E_v to the total energy of the field within the box of

$$\begin{aligned} E_v &\approx \left[\frac{\Omega_{,x}^2}{2\Omega^2} - \frac{\Omega_{,x,x}}{\Omega} \right] \frac{L}{24\pi} \\ &= \left[\frac{7M^2}{r^4} - \frac{4M}{r^3} \right] \frac{L}{24(1-2M/r)}, \end{aligned} \quad (4.36)$$

where the second expression is the first as evaluated in a black-hole metric. From the inertial observer's viewpoint, the box therefore grows lighter as it is lowered nearer the black hole because the vacuum energy density of the field within the box decreases.

As mentioned earlier in this section, the accelerated observer takes the field energy of the empty box as his reference, thus neglecting the terms in square brackets in (4.31). For him, the mass of the box and of the radiation within the box remain constant as the box is lowered toward the horizon of the black hole. Since the neglected terms are just a part of a conserved energy-momentum tensor, the accelerated observer's viewpoint is completely consistent.

V. ACCELERATING BOXES IN FLAT SPACETIME

The close mathematical analogy between the exterior Schwarzschild solution ($r > 2M$) and the Rindler spacetime

$$ds^2 = x^2 dt^2 - dx^2 - dy^2 - dz^2, \quad (5.1)$$

which corresponds to a wedge of Minkowski spacetime is well known and, indeed, has been used to gain insight into the quantum processes occurring near black holes.⁷ Thus, one would expect that effects analogous to those of boxes lowered toward a black hole should occur for accelerating boxes in Minkowski spacetime. The purpose of this section is to explore these analogous effects.

Consider a rectangular box with perfect reflecting walls in Minkowski spacetime which undergoes rigid acceleration, i.e., the box follows the orbit of a boost Killing field. Then each face of the box will react exactly as if placed in a thermal bath of temperature⁷

$$T = a/2\pi, \quad (5.2)$$

where a is the local acceleration. However, the local acceleration is given by

$$a = \frac{1}{\chi} \frac{d\chi}{dx} = \frac{1}{x}, \quad (5.3)$$

where the "red-shift factor" χ is now the norm of the boost Killing field $\partial/\partial t$, so that $\chi = x$. Hence, the "top face" of the box (where "upward" is taken to be the direction of the acceleration, i.e., the direction of increasing x) will undergo a smaller acceleration than the bottom face. Thus, just as in the black-hole case, the box will feel an effective buoyancy force from the acceleration radiation. This buoyancy force will reduce the applied force required to keep the box in uniform acceleration. Again, for some sufficiently large acceleration, the box would float at which acceleration it would continue to uniformly accelerate without any external force. If accelerated beyond the floating acceleration the buoyancy force would be so large that the external force needed to keep it in uniform acceleration would be opposite the direction of acceleration.

How can the existence of such a self-accelerating box in Minkowski spacetime be consistent with conservation of energy? The answer is best understood from the "inertial viewpoint" developed in the previous section. In the inertial viewpoint, as we quasistatically increase the acceleration of an empty box, the interior remains in its ground state, energy flows out of the box, and the energy of the field inside the box becomes negative. As we shall show below, at the floating acceleration, the total energy content of the box vanishes; the negative

energy of the Rindler vacuum inside the box cancels the positive mass of the box walls. Thus, the box can “self-accelerate” without violating conservation of energy in the same way as can a system composed of a positive mass m connected to a negative mass $-m$ by a spring. If the box could be accelerated beyond its floating acceleration, its total energy would become negative.

We derive now the stress conditions required of the box in order to achieve its floating acceleration. At the floating acceleration no external forces are necessary to maintain the box at that acceleration. Thus, in the inertial viewpoint, the stress energy density can be nonzero only within the box. From the conservation of stress energy in the Rindler coordinates of (5.1) we find

$$0 = T^{x\mu}{}_{;\mu} = \frac{1}{x} \frac{\partial}{\partial x^\mu} (x T^{x\mu}) + \Gamma_{\mu\nu}^x T^{\mu\nu} = \frac{1}{x} \frac{\partial}{\partial x^\mu} x T^{x\mu} - x T'' = \left(\frac{\partial}{\partial t} T^{xt} + \sum_{i=1}^3 \frac{1}{x} \frac{\partial}{\partial x^i} x T^{xi} \right) - x T'' . \quad (5.4)$$

Integrating with respect to $x \, dx \, dy \, dz$ over all of a $t = \text{const}$ spacelike slice, we find that the first term is zero because the box remains static in this coordinate system, and the second term is zero by an integration by parts since $T^{\mu\nu}$ is zero outside the box. We have therefore

$$\begin{aligned} 0 &= - \int T'' x^2 dx \, dy \, dz \\ &= - \int T^t{}_{,t} dx \, dy \, dz . \end{aligned} \quad (5.5)$$

This can be rewritten as

$$\int T^t{}_{,t} dX \, dy \, dz = 0 , \quad (5.6)$$

where t' and X are Minkowski coordinates such that a $t = \text{const}$ hypersurface is also a $t' = \text{const}$ surface. This justifies our previous contention that the total energy is zero. However, because the left-hand side of (5.6) is independent of hypersurface and because the above is true for all $t = \text{const}$ hypersurfaces, we find that t', X and X, X components also integrate to zero. Using the stress-energy conservation we finally obtain

$$\int T^{\mu\nu} dX \, dy \, dz = 0 \quad (5.7)$$

for all components of $T^{\mu\nu}$ in Minkowski coordinates. Thus, the integrated stress-energy components over the walls of the box must equal the negative of the integrated stress energy over the interior of the box. However, if we neglect Casimir terms, the stress energy inside the box is just the negative of the stress energy of blackbody radiation. Thus, we find

$$\int_{\text{walls}} T_{\mu\nu} d^3x = \int T_{\mu\nu}^{\text{BB}} d^3x , \quad (5.8)$$

where $T_{\mu\nu}^{\text{BB}}$ denotes a stress tensor appropriate to

blackbody radiation. Now, unless the number of species of elementary particles increases rapidly at large masses, thermal radiation should be dominated by massless particles and particles whose mass can be neglected compared with their thermal energy. Hence, to a good approximation, the trace of a thermal stress tensor should vanish, $T^{\text{BB}} \approx 0$.

Therefore, if we model a box wall as having mass per unit area μ , and tension \mathcal{T} in the two directions tangent its surface, but take all other stress-energy components to vanish, then according to Eq. (5.8) the tension must be negative (i.e., a pressure) and must, on average, satisfy

$$|\mathcal{T}| = \frac{1}{2} \mu . \quad (5.9)$$

Although this does not violate the dominant energy condition, there is no known physical matter which satisfies Eq. (5.8). Thus, it does not appear that one could construct a box that could withstand the stresses required to achieve its floating acceleration (or, in the black-hole case, the stresses required to be lowered to its floating point). In reaching this conclusion that “floating boxes” (or negative total energy boxes) cannot be produced physically, we have not even taken into account the physical limitations on producing perfect reflecting walls or the possibility that the acceleration radiation would cause the box walls to melt.

Although it may already seem surprising that the analogy between black holes and Rindler spacetime carries over as far as the analysis of buoyancy effects on boxes, the analogy can be carried much further. Flat spacetime can be mined by accelerating boxes. If we accelerate a box, open and close the box door while it is accelerating, decelerate the box, and bring it back to our laboratory, we will

find that it is filled with thermal radiation.

Where does the energy come from when we “mine” flat spacetime? Since total energy must be conserved, the energy source for the mining of flat spacetime must be the external agent responsible for accelerating the box. For a quasistatic mining process, the forces such an agent must exert on the box can be divided into three categories: (1) the ordinary inertial force needed to accelerate the energy contained in the box, (2) the force needed to counteract the buoyancy force of the acceleration radiation (which, in the inertial viewpoint, would be considered part of the first force), and (3) the force needed to counteract the radiation reaction force. The radiation reaction force provides the key to understanding energy balance in the mining of flat spacetime, and, to gain more insight into the nature of it, we return to our two-dimensional model of Sec. IV.

When the acceleration of a mirror (such as a reflecting wall of the box) changes with time, the mirror radiates and a radiation reaction force acts on the mirror. This radiation reaction force is given by

$$F_R^\mu = T^\mu{}_\nu \eta^\nu|_+ - T^\mu{}_\nu \eta^\nu|_-, \quad (5.10)$$

where $T^\mu{}_\nu$ is the stress tensor of the quantum field, η^ν is the outward normal, and the plus and minus refer to the two sides of the mirror. A calculation similar to that which gave (4.13) yields

$$\begin{aligned} F_R^\mu &= \frac{1}{6\pi} \frac{da}{d\tau} \frac{a^\mu}{a} \\ &= \frac{1}{6\pi} \left[\frac{da^\mu}{d\tau} - V^\mu V_\nu \frac{da^\nu}{d\tau} \right]. \end{aligned} \quad (5.11)$$

Note that this quantum radiation reaction force on a moving mirror is identical in form to the radiation reaction force on a point charge in classical electrodynamics. Thus runaway solutions will exist for moving mirrors even though the renormalized stress energy is finite everywhere, i.e., even though there is no infinite self-energy as in the case of a classical point charge. Energy is conserved because the local energy density in the field is not necessarily positive, and the mirror will radiate negative energy as it self-accelerates. Thus moving mirrors provide an interesting example of radiation reaction dynamics¹⁴ and show furthermore that quantum mechanics of itself does not eliminate runaway-solution problems.

It should be emphasized that the radiation reac-

tion force, Eq. (5.11), is completely distinct from the buoyancy force we have previously analyzed.

The buoyancy force vanishes, of course, for a single mirror as opposed to a box. On the other hand, the radiation reaction force vanishes for uniform acceleration, and thus can be neglected in the quasistatic analyses of the previous sections.

Thus, if we vary the acceleration of a moving mirror we must exert an extra force, $F^\mu = -F_R^\mu$, to compensate for the radiation reaction force. Using Newton's law, $dp^\mu/d\tau = F^\mu$, we find that this force will do work W_R given by

$$\begin{aligned} W_R = \Delta p^0 &= \int F^0 d\tau = - \int F_R^0 d\tau \\ &= - \frac{1}{6\pi} \int \gamma v \frac{da}{d\tau} d\tau, \end{aligned} \quad (5.12)$$

where v is the velocity of the mirror (taken positive if in the direction of the acceleration) and $\gamma = (1 - v^2)^{-1/2}$. This energy W_R will go into the radiation emitted by the moving mirror. Note that if the mirror is initially inertial, undergoes a period of acceleration, and returns to a final inertial state, the boundary terms in the integral for W_R will disappear when we integrate by parts and we obtain

$$W_R = \frac{1}{6\pi} \int a^2 \gamma d\tau. \quad (5.13)$$

Apart from constant factors, this agrees with the formula for total energy radiated by a point particle in classical electrodynamics, as could have been foreseen by the agreement of the radiation reaction forces. Note, however, that the nature of the emission process differs greatly in the two cases: The mirror radiates only when its acceleration changes, while the point particle also radiates while it accelerates uniformly.

Returning to the mining process, we may divide our mining operation into three regimes: (i) The box is started from rest and is slowly brought to a constant acceleration a . (ii) The box door is opened, allowing the box to fill with thermal energy, and then closed. (iii) The box is slowly decelerated and brought back to rest in the laboratory or allowed to fly off with constant velocity. Assuming for simplicity that the differential acceleration across the box is small compared with a in step (ii), we find that at the end of the process the box will be filled with thermal radiation at temperature $T = a/2\pi$. Since the energy density of thermal radiation in two dimensions is $\frac{1}{6}\pi T^2$, the box will have energy

$$\gamma E = \gamma \frac{\pi}{6} T^2 L = \frac{1}{24\pi} a^2 L, \quad (5.14)$$

where L is the length of the box, and $\gamma = (1 - v_f^2)^{-1/2}$ where v_f is the final velocity of the box. The ordinary inertial force (1) on the radiation in the box in steps (ii) and (iii) accounts for the energy ($E - E_0$) where

$$E_0 = \gamma_0 \frac{1}{24\pi} a^2 L \quad (5.15)$$

with $\gamma_0 = (1 - v_0^2)^{-1/2}$ where v_0 is the velocity of the box during the time the box filled with radiation. Thus, the buoyancy force (2) and the radiation reaction force (3) must account for the energy E_0 . We calculate now their contributions.

According to (5.13), the work done against the radiation reaction force is

$$W_R = \frac{1}{6\pi} \left[\int a_t^2 \gamma_t d\tau_t + \int a_b^2 \gamma_b d\tau_b \right], \quad (5.16)$$

where “ t ” and “ b ” denote the top and bottom faces of the box. On the other hand, since the “pressure” (i.e., force) of thermal radiation in two dimensions is $\frac{1}{6}\pi T^2$ the buoyancy force is given by

$$F_B = \frac{1}{24\pi} (a_t^2 - a_b^2) \quad (5.17)$$

and thus the work done against the buoyancy force is

$$W_B = \frac{1}{24\pi} \left[\int a_b^2 \gamma_b v_b d\tau_b - \int a_t^2 \gamma_t v_t d\tau_t \right]. \quad (5.18)$$

Comparison of (5.16) and (5.18) shows that W_R always dominates W_B . In particular, if the differential acceleration is small compared with the average acceleration as we have assumed here for simplicity, we have $W_B \ll W_R$.

We show now that W_R is sufficient to account for E_0 . Namely, in order to fill the box with thermal radiation we must accelerate it with acceleration a in step (ii) for a time $\Delta\tau$ at least as great as the light travel time L across the box. In fact, for the mining procedure to be quasistatic, we need it to accelerate with acceleration close to a for $\Delta\tau \gg L$. Thus, from (5.13) we find

$$W_R \gg \gamma_0 a^2 L \sim E_0. \quad (5.19)$$

Thus, W_R indeed can account for energy balance in the “mining” process.

Thus, our picture of the process of “mining” flat spacetime is the following. During regimes (i) and (iii) above when the acceleration of the box is changing, the front and back faces of the box emit radiation. The energy for this radiation comes from the extra work done by the external agent in combating the radiation reaction force. While the acceleration is held constant and the box door is opened and closed, the empty box fills with thermal radiation according to the accelerated observer. From the inertial observer’s point of view, the box is full of negative energy density before the door is opened and has zero energy content when the door is shut again. Thus, during the time the door is open the negative energy in the box escapes as a negative-energy flux to infinity. The total energy radiated to infinity by the moving walls of the box in the entire process must always be positive as the vacuum state of the field is a minimum energy state. Thus, the mechanism by which the opening of the box has converted part of the total work of the accelerating force to thermal energy within the box rather than to energy emitted to infinity is rather surprising. It does this not by trapping part of the radiation which would have escaped to infinity, for during the period of constant acceleration there would have been no energy flux produced by the box walls if the door had not been opened. Rather, it does this by radiating a negative-energy flux which subtracts from the larger positive-energy flux emitted during the periods when the acceleration changed.

The energy source of flat-spacetime “mining” contrasts sharply with the “black-hole mining” discussed in Sec. III. In the black-hole case, the velocity v of the box is unrelated to the local acceleration a of the box. Consequently, although the local radiation reaction force on a mirror in our two-dimensional model is still $(1/6\pi)da/d\tau$, no analog of (5.13) exists and the work done by the force can be made negligible by lowering the box sufficiently slowly. Thus, in the black-hole case, true mining occurs; the energy comes from the black hole, not from the forces combating radiation reaction. It is interesting that the processes of “mining” flat spacetime and mining a black hole have sharply differing global interpretations even though the local descriptions of these two phenomena near the box are virtually identical.

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