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Colliding gravitational waves in expanding cosmologies

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We discuss the singularity behavior and causal structure of expanding vacuum plane-symmetric cosmologies containing colliding gravitational waves. Penrose's theorem excluding global Cauchy hypersurfaces in plane-wave spacetimes does not hold for our expanding cosmologies. The singularity in our solutions occurs in the past and can be identified with the big-bang singularity. Analytical estimates based on a Green's-function analysis near the collision provide an explicit example which is verified by numerical computation.

I. INTRODUCTION

The idea that a collision of plane gravitational waves inevitably produces singularities in the future of the interaction has become part of the folklore of general relativity. Kahn and Penrose,[1] Szekeres,[2,3] and Nutku and Halil[4] have all given examples of colliding plane-wave vacuum solutions in which the waves start out traveling towards each other in flat space, collide, and move apart again, leaving behind a nonflat interaction region in which a singularity inevitably develops. These solutions, although highly specialized, have led to the belief that colliding plane waves always focus to a future singularity. This notion can lead to great confusion when considering the behavior of strong gravity waves during a chaotic phase in the early universe.

Certainly, there is no proof that general, nonplanar gravitational waves focus to singularities, and good reason to believe that they do not.[5] Recently, Tipler[5] has proved that the collision of any *plane* waves, gravitational or electromagnetic, requires a singularity either in the past or future of the interaction. We[6,7] have studied the collision of plane gravitational waves in an expanding vacuum cosmology and find that in general the waves do not focus to a singularity in the future. This result is in keeping with Tipler's theorem because we do have the cosmological singularity in the past of the collision. In other words, the universal expansion forces our plane waves to share in the singularity of the big bang, and frees them from a singularity in the future.

In the scenarios of Kahn and Penrose and of Szekeres the initial data for colliding waves were set on flat space. Prior to collision, the convergence and shear of the null rays lying in the wave fronts vanished. The interaction of the waves induced shear in the null congruence, which then induced convergence and resulted in a singularity a finite time in the future. Szekeres's[3] solution makes this behavior especially clear, since he explicitly solves the field equations using a Green's-function technique and demonstrates that at a finite time in the future of the collision the Green's function becomes singular. Consequently, certain Riemann tensor invariants diverge and thus physically inescapable tidal singularities occur in the future.

In our study of colliding plane waves in cosmological models we set data in a particular way[6] on an expanding three-dimensional spacelike hypersurface. The waves thus share the expansion of the background. After the collision their expansion is still positive and no singular focusing occurs in the future. The wave disturbances die down and eventually become just small fluctuations in an ever-expanding universe. The nonsingular behavior of these waves has led us to examine their interaction more closely using the optical scalars.[8,9,10] We present here some explicit analytical and numerical results concerning colliding plane waves in expanding vacuum cosmologies in the hope of gaining insight into the influence of more general gravity waves on the evolution of the early universe.

II. PLANE-SYMMETRIC COSMOLOGIES

Previous work on colliding plane waves set data on null planes matched to flat space.[1,2,3] We evolve our data numerically using the "3 + 1" formalism[11] of numerical relativity, and therefore require data on a three-dimensional spacelike slice. Our particular initial data set represents slices from two different anisotropic, homogeneous cosmologies abutted at a two-plane. The technique[6] gives data with three-metric $\gamma_{ij} = \delta_{ij}$ everywhere on the slice, and, in the vacuum case, second fundamental form K_{ij} as appropriate for a Kasner[12] anisotropic cosmology at each point. For a consistent data set, the momentum constraint imposes a particular continuity requirement across the join surface. Within limits, one may pick the age and shape of the anisotropy of one of the cosmologies, and the age, say, of the other. The momentum constraint then determines the shape of the anisotropy of the other model, and the initial data set is complete. A discontinuous join is not mandatory; the age, for example, may be varied continuously. For analytical work we suppose the join is discontinuous in order to study the collision of gravitational shock waves. In our numerical analysis, we smooth the join over several grid zones, but keep it steep to approximate the analytical situation.

Analytical work is carried out in a metric of the form[13]

$$ds^2 = e^{\gamma-\psi}(-dT^2 + dZ^2) + (e^{\psi}dX^2 + e^{-\psi}T^2dY^2)\,, \tag{2.1}$$

where ψ and γ depend only on Z and T. The only dynamical field equation in vacuum is

$$\ddot{\psi} + \dot{\psi}/T - \psi'' = 0\,, \tag{2.2}$$

where a dot denotes a partial derivative with respect to time T and a prime denotes a partial derivative with respect to spatial coordinate Z. The remaining field equations yield γ by a direct quadrature once ψ is known,

$$\dot{\gamma} = \frac{T}{2}(\dot{\psi}^2 + \psi'^2)\,, \tag{2.3a}$$

$$\gamma' = T\dot{\psi}\psi'. \tag{2.3b}$$

In specifying our data, it is convenient to use a parametrization which emphasizes the age and anisotropy of a Kasner model. We take a typical four-dimensional Kasner model in the form

$$ds^2 = -dt^2 + \left(\frac{t}{t_0}\right)^{2p_1} dx^2 + \left(\frac{t}{t_0}\right)^{2p_2} dy^2 + \left(\frac{t}{t_0}\right)^{2p_3} dz^2\,, \tag{2.4}$$

where

$$p_1 + p_2 + p_3 = 1\,, \tag{2.5a}$$

$$p_1^2 + p_2^2 + p_3^2 = 1\,, \tag{2.5b}$$

and each p_i has a possible range $-\frac{1}{3} \le p_i \le 1$. Equations (2.5) imply that only one of the p_i is unconstrained, and we take this to be p_3.

Setting our initial data corresponds to specifying the value of t_0 and p_3 at one point. The constraint equations permit us to specify, for example, $t_0(z)$ and then determine $p_3(z)$. This data set corresponds to a $T = T_0 =$ constant slice in the variables of Eq. (2.1).

Consider now data that are set by abutting two different Kasner cosmologies (i.e., different t_0 and p_3) at a join surface $z = 0$. Because the three-metric is $\gamma_{ij} = \delta_{ij}$, $z = Z$ in the slice $T = T_0$. We will refer to quantities specified at $z < 0$ with the subscript A and those specified at $z > 0$ with the subscript B. On this constant T_0 slice we have[6]

$$\dot{\psi}\,|_{T=T_0} = \frac{2p_1(z)}{T_0}\,\frac{1}{1 - p_3(z)}\,. \tag{2.6}$$

Note that the quantity $1 - p_3(z)$ cannot vanish; if it did, the Kasner model at that point would be Minkowski space and the constraints would then require the solution to be Minkowski space everywhere. Hence, $\dot{\psi}$ is finite, a fact which we will use repeatedly below. On the other hand, because the data set is discontinuous, the derivatives $\dot{\psi}$ and ψ' will be discontinuous and δ-function singularities will appear as gravitational shocks in the Riemann tensor, traveling at the speed of light out of the join surface.

Now consider the time evolution of this initial data set. Equation (2.2) is of Euler-Darboux form and a Green's-function analysis[6] similar to that given by Szekeres[3] gives the value of the metric function ψ at any field point $(\tilde{Z}, \tilde{T})$:

$$\psi(\tilde{Z}, \tilde{T}) = \int_{D_3}^{D_4} (\tfrac{1}{2})\,\dot{\psi} R\, dZ\,. \tag{2.7}$$

Here, the integral is taken over a region $[D_3, D_4]$ on the initial slice $T = T_0$. The parameters D_3 and D_4 are defined by

$$D_3 = \tilde{Z} - (\tilde{T} - T_0)\,, \tag{2.8a}$$

$$D_4 = \tilde{Z} + (\tilde{T} - T_0)\,. \tag{2.8b}$$

This situation is diagrammed in Fig. 1.

The Green's function R is the hypergeometric function

$$R = \left(\frac{T_0}{\tilde{T}}\right)^{1/2} F(\tfrac{1}{2}, \tfrac{1}{2}; 1; -q)\,, \tag{2.9}$$

where

$$q = -\frac{(Z - D_4)(Z - D_3)}{4T_0\tilde{T}} > 0 \tag{2.10}$$

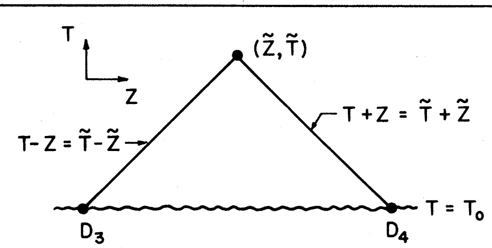

FIG. 1. The metric at the field point $(\tilde{Z}, \tilde{T})$ depends on an integral over the region $[D_3D_4]$ on the initial slice $T = T_0$. The lines $\tilde{T} + \tilde{Z} =$ constant and $\tilde{T} - \tilde{Z} =$ constant are null rays. This figure shows the field point to the future of T_0; for $\tilde{T} < \tilde{T}_0$, the positions of D_3 and D_4 are interchanged.

in the region of integration. The Green's function has a singularity as $\tilde{T} \to 0$, corresponding to the big-bang singularity. Since R is well behaved in the future, and since $\dot{\psi}$ is finite on the original slice, one obtains $\psi \sim \ln\tilde{T}$, a form typical of an expanding cosmology, in the far future. Thus, no future singularity occurs; the gravitational shocks do not focus to a singularity in the future of their collision. These statements about the behavior of the Green's function and the behavior of the solution are justified in Sec. IV.

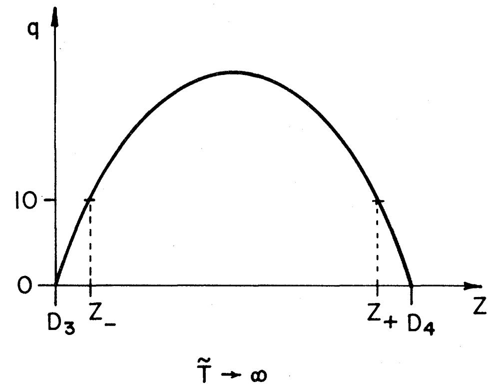

FIG. 2. The edge regions D_3, Z_- and Z_+, D_4 contribute only a finite piece, which vanishes as $\tilde{T} \to \infty$ and $\tilde{T} \to 0$, to the integral over the Green's function. The situation shown here is for $\tilde{T} \to \infty$; for $\tilde{T} \to 0$, the positions of D_3 and D_4 are interchanged.

III. INTERACTION OF WAVES IN PLANE GEOMETRIES

Penrose has proved a theorem[10,14] stating that a *plane-wave spacetime* cannot contain a global Cauchy surface. A plane-wave spacetime is one in which the metric depends only on one null coordinate, and wave fronts traveling along the surfaces where this coordinate is constant have vanishing expansion. In contrast, in our cosmologies null planes of the form $T \pm Z =$ constant have positive expansion [cf. Eq. (5.25)].

In its simplest form, Penrose's theorem refers to weak sandwich waves in a plane-wave spacetime. A sandwich wave is described by a flat region followed by a region in which plane waves propagate past in one direction, in turn followed by another flat region. Penrose's theorem says that, except for one exceptional null ray, every generator of the future null cone of a point in the past of the passing plane wave intersects a single spacelike line in the future of the wave passage. If the wave is electromagnetic rather than purely gravitational, the refocusing is to a precise point. This refocusing of the null rays completely changes the causal structure of the spacetime and prevents it from admitting a global Cauchy surface

We may summarize Penrose's proof as follows. For simplicity, consider the case of pure electromagnetic waves. (Penrose[10] gives the proof for the case of plane gravitational waves also.) Electromagnetic waves contain energy and thus, by Raychaudhuri's theorem,[15] tend to increase the convergence of a null congruence passing through them. If a test null cone expanding from a specific event in an initially flat region encounters a plane (nonexpanding) electromagnetic wave, the expansion of the null cone may be sufficiently small that it is overcome and a net convergence is induced by the encounter with the wave. If the plane wave has fixed amplitude, we may consider encounters occurring further and further from the source event of the cone and eventually this net convergence will result. This situation is shown in Fig. 2 of Ref. 14. Penrose shows that the null cone, except for one exceptional ray, emanating from a single event is refocused to a single event. (For gravitational plane waves the refocusing is less accurate and the null rays reintersect along a spacelike line.) This curious causal structure is sufficiently different from the usual flat-space paradigm that no Cauchy surface can exist.[10,15] Because the effect of the plane waves is always to induce convergence, the initial restriction to sandwich plane waves in an otherwise flat universe is unnecessary.

An important point of Penrose's proof is that any plane wave, no matter how weak, induces convergence in a null cone emanating from *some* earlier event. For very weak plane waves, such an event must be very far away and hence in the distant past of the encounter. But, there are many such events in the flat part of the spacetime ahead of the wave. Hence, Penrose's theorem states that an arbitrarily weak plane wave prevents the existence of a global Cauchy surface in a plane-wave spacetime.

Consider now two colliding plane waves moving with flat space between them. As the colliding waves initially travel towards one another they both have zero expansion. The collision then induces convergence in each of these plane waves via Raychaudhuri's theorem. Since these are real waves, and not just test waves as in the null-cone discussion above, the eventual refocusing leads to singular Riemann tensor behavior, and there will always be a singularity in the future. The causality anomalies on which Penrose's theorem depends become curvature singularities in this plane-wave spacetime.

How is the situation different for waves propagating in expanding plane cosmologies? Briefly put, there is a net overall expansion in the universe which tends to counteract any convergence. To illustrate this point, consider a test null cone emanating from a specific event in an expanding Robertson-Walker cosmology with metric[16]

$$ds^2 = a^2(\eta)[-d\eta^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{3.1}$$

The geodesic equation for spherically outgoing null geodesics $k^\mu (k^\eta = k^r)$ is

$$(\ln k^\eta)_{,\eta} + (\ln a^2)_{,\eta} = 0\,, \tag{3.2}$$

which is equivalent to the well-known red-shift result[11]

$$k_r = \text{constant}. \tag{3.3}$$

The rotation and shear of the expanding congruence of null geodesics

$$k_\eta = -k_r = \text{constant} \tag{3.4}$$

both vanish. The expansion is

$$\theta = \tfrac{1}{2} k^\alpha{}_{;\alpha} = \frac{k_r}{a}\left(\frac{a_{,t}}{a} + \frac{1}{ar}\right), \tag{3.5}$$

where the comma indicates ordinary partial differentiation, $dt = a\, d\eta$, and $D = ar$ is the proper distance the ray has propagated.

Note that in flat space, the metric function a = constant and so a null cone has expansion θ ~ constant/(distance to source). In contrast, in an expanding cosmology the expansion of the universe adds a positive contribution to θ. Hence it is not merely how distant the source event is from a plane wave, but also how rapidly the whole universe is expanding at the time a collision with a plane wave occurs, that determines whether refocusing can occur. Notice in particular that if $r = \infty$ (a plane wave) the expansion of the null geodesics is nonzero, in contrast to the case of plane-fronted waves in flat space.

Consider now an expanding plane-symmetric cosmology containing plane waves. Does Penrose's theorem that there is no global Cauchy surface hold in this case? If there is a cosmological expansion as described by Einstein's equations, with a cosmological singularity in the past, the theorem does *not* hold in general.

Because there is a cosmological singularity a finite time in the past, it is not possible for an event to be further in the past of a plane wave than a light travel time equal to the age of the universe. Arbitrarily weak plane waves will thus be unable to induce convergence in the null cone expanding from that event, and so Penrose's proof fails in this case.

The future null cone of a more distant event will intersect the plane wave at a later time in the evolution of the universe. As before, for simplicity, we consider plane *electromagnetic* waves. From Eqs. (3.3) and (3.5) we see that the expansion of the plane electromagnetic wave in an expanding cosmology will induce a corresponding decrease in its energy density, as calculated in geometrical optics.[11] For a plane electromagnetic wave propagating in a given flat Robertson-Walker cosmology we have

$$(H^2 k^\mu)_{;\mu} = 0\,.$$

This implies that

$$H^2 a^2 = \text{constant}\,,$$

where H is the amplitude of the vector potential wave and thus $H^2 k_t{}^2$ is proportional to its energy density. From Eq. (3.3), $k_t \propto a^{-1}$. The focusing accomplished by the expanding plane wave is proportional to its energy density; by this analysis the energy density is proportional to a^{-4}.

In the Robertson-Walker model considered above, the plane-wave limiting form of the expansion of the test null geodesics in Eq. (3.5) is proportional to $a^{-2}a_{,t}$ and so will eventually be overcome by the focusing induced by a plane wave if $a_{,t}$ falls off faster than a^{-2}. Now,

$$a_{,t} \propto a^{-2-3\epsilon}, \quad \epsilon > 0$$

implies

$$a \sim t^{(1-\epsilon)/3}$$

for focusing. In other words, a version of Penrose's theorem holds if the universal expansion is slow enough.

In a dust $(p = 0)$ Robertson-Walker model we have $a \propto t^{2/3}$ and a radiation $(p = \frac{1}{3}\rho c^2)$ model has $a \propto t^{1/2}$; a hard fluid $(p = \rho c^2)$ model has $a \propto t^{1/3}$, which is also the behavior of the average expansion in a Kasner (vacuum, anisotropy dominated) model. In all of these models, Penrose's theorem fails for arbitrarily weak plane waves.

A similar type of argument shows that, in expanding plane-symmetric cosmologies, sufficiently weak colliding plane waves do not focus in the future. These plane waves have a positive expansion due to the cosmology. If they are weak enough, no convergence occurs after their collision. Cauchy surfaces do exist in these plane-symmetric cosmologies; numerical relativity is directly applicable to them.[17] The ubiquitous focusing of plane waves moving in flat space occurs because the waves themselves have zero expansion initially, and an arbitrarily weak induced convergence will lead to eventual focusing of the waves in the future.

IV. GREEN'S FUNCTION ANALYSIS

We now justify the statements made in Sec. II concerning the singularity behavior of the Green's function (2.8) and the fact that gravitational shocks emanating from the join surface on the matched initial data slice do not focus to a singularity in the future.

Recall from Sec. II that the Green's-function analysis of the dynamical field equation (2.2) gives the metric function $\psi(\tilde{Z},\tilde{T})$ as an integral of the Green's function R multiplied by a derivative of ψ over a region $[D_3, D_4]$ of the initial slice. Because of the finiteness of the data, singularities in the solution can arise only where the Green's function has a singularity, or perhaps for $\tilde{T}$ in the far future when the integration interval is infinite.

The Green's function R as defined in Eq. (2.9) contains the hypergeometric function

$$F(\tfrac{1}{2},\tfrac{1}{2};1;-q)=\frac{2}{\pi}K(-q) \tag{4.1a}$$

$$=\frac{2}{\pi}\,\frac{1}{(1+q)^{1/2}}K\left(\frac{q}{1+q}\right), \tag{4.1b}$$

where $K(m)$ is the complete elliptic integral of parameter m, in the notation of Ref. 18. The only singularity[3] of $K(m)$ is at $m=1$.

The parameter q as defined in (2.10) is zero at the end points of the integration region and positive inside. The maximum value of q occurs at $Z=\frac{1}{2}(D_3+D_4)$ and takes the limiting forms

$$q_{\max}\approx\frac{\tilde{T}}{4T_0},\quad \tilde{T}\gg T_0 \tag{4.2a}$$

$$q_{\max}\approx\frac{T_0}{4\tilde{T}},\quad \tilde{T}\ll T_0\,. \tag{4.2b}$$

The maximum value of q on the initial slice thus gets very large as the field point $(\tilde{Z},\tilde{T})$ moves towards the far future $\tilde{T}\gg T_0$ and towards the big-bang singularity at $\tilde{T}=0$. For large q, the elliptic integral (4.1) is dominated by a logarithmic term:

$$\frac{1}{(1+q)^{1/2}}K\left(\frac{q}{1+q}\right)\approx q^{-1/2}K\left(1-\frac{1}{q}\right)$$
$$\sim\tfrac{1}{2}q^{-1/2}\ln q\,. \tag{4.3}$$

This limiting form for the Green's function is valid when q is larger than, say, 10. From (2.10) we have $q=10$ at the points

$$Z_\pm=\tfrac{1}{2}\{D_3+D_4\pm[(D_3-D_4)^2-160T_0\tilde{T}]^{1/2}\} \tag{4.4}$$

on the initial slice. Note that $q\geq 10$ only for values of $\tilde{T}$ sufficiently larger or smaller than T_0 to make $(D_3-D_4)^2-160T_0\tilde{T}\geq 0$. Thus, as $\tilde{T}\to\infty$ and as $\tilde{T}\to 0$, there is a neighborhood around each of the points D_3 and D_4 in which $q\leq 10$. This situation is diagrammed in Fig. 2. Using Eqs. (4.4) and (2.8), we find the size of these edge regions, for $\tilde{T}\to\infty$,

$$\begin{aligned}&D_4-Z_+\sim 20T_0\,,\\&Z_--D_3\sim 20T_0\,,\end{aligned} \tag{4.5}$$

and for $\tilde{T}\to 0$,

$$\begin{aligned}&D_3-Z_+\sim 20\tilde{T}\,,\\&Z_--D_4\sim 20\tilde{T}\,.\end{aligned} \tag{4.6}$$

To find the metric function $\psi(\tilde{Z},\tilde{T})$ we use the integral over the Green's function times $\dot{\psi}(T_0)$ given in Eq. (2.7). Recall from Sec. II that $\dot{\psi}(T_0)$ is finite and piecewise constant on the matched initial data slice. We thus suffer no loss of generality in assuming $\dot{\psi}=$ constant at T_0 in the rest of this section. To keep track of dimensional quantities we will however carry it along as an explicit constant $\dot{\psi}(T_0)$. Putting all these facts together we have

$$\psi(\tilde{Z},\tilde{T})\sim\frac{\dot{\psi}(T_0)}{2}\int_{\text{edges}}R\,dZ$$
$$+\frac{\dot{\psi}(T_0)}{2}\int_{\text{center}}R\,dZ\,.$$

First consider the integral over the edge regions, which gives

$$\psi_{\text{edge}}\sim\frac{\dot{\psi}(T_0)T_0^{1/2}}{\pi\tilde{T}^{1/2}}\int_{\text{edges}}\frac{1}{(1+q)^{1/2}}K\left(\frac{q}{1+q}\right)dZ\,. \tag{4.7}$$

Note that the elliptic integral K is positive and bounded above in this region ($0\leq q\leq 10$, say), as is the function $1/\sqrt{1+q}$. As $\tilde{T}\to\infty$, the size of the integration region remains constant by (4.5) and thus the edge contributions go to zero as $\tilde{T}^{-1/2}$. For the case in which $\tilde{T}\to 0$, the size of the integration region falls off like $\tilde{T}$ by (4.6). The edge contributions near the big bang thus vanish as $\tilde{T}^{1/2}$. In both cases, the edge contributions vanish and do not contribute a singular piece to ψ.

To derive the singular properties of these solutions we need then only look at the contributions to ψ from the central region where q is large. We have

$$\psi_{\text{center}}\sim\frac{\dot{\psi}(T_0)}{2\pi}\,\frac{T_0^{1/2}}{\tilde{T}^{1/2}}\int_{\text{center}}q^{-1/2}\ln q\,dZ\,, \tag{4.8}$$

using (2.9) and (4.3). This gives

$$\psi \sim \frac{T_0\dot{\psi}(T_0)}{2\pi} \int_{\text{center}} \frac{dZ}{[(D_4 - Z)(Z - D_3)]^{1/2}}$$
$$\times \left[\ln\left(\frac{D_4 - Z}{2T_0}\right) + \ln\left(\frac{Z - D_3}{2T_0}\right) - \ln\frac{\tilde{T}}{T_0}\right] \tag{4.9a}$$

$$\equiv \frac{T_0\dot{\psi}(T_0)}{2\pi} (\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3). \tag{4.9b}$$

We will estimate $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ by assuming that the integration extends all the way to the limits D_3 and D_4. Thus for purposes of estimation we assume the large-q asymptotic form holds even near the edges where q is small. We will shortly show that the errors this introduces do not affect the asymptotic behavior of ψ.

Rewriting the integrals in dimensionless form yields (if the limits are extended to D_4 and D_3)

$$\mathcal{I}_1 = \mathcal{I}_2 = 2\ln\left(\frac{D_4 - D_3}{2T_0}\right) \int_0^1 \frac{dv}{(1-v^2)^{1/2}} + 4\int_0^1 \frac{dv\,\ln v}{(1-v^2)^{1/2}}, \tag{4.10}$$

$$\mathcal{I}_3 = -2\ln\frac{\tilde{T}}{T_0} \int_0^1 \frac{dv}{(1-v^2)^{1/2}}. \tag{4.11}$$

Now

$$\int_0^1 \frac{dv}{(1-v^2)^{1/2}} = \frac{\pi}{2} \quad \text{(Ref. 19, p. 54, Sec. 2.01, No. 17)}, \tag{4.12}$$

$$\int_0^1 \frac{dv\,\ln v}{(1-v^2)^{1/2}} = -\frac{\pi}{2}\ln 2 \quad \text{(Ref. 19, p. 535, Sec. 4.24, No. 6)}. \tag{4.13}$$

Notice that the integrand of Eq. (4.13) is negative throughout the range of integration, so our estimate based on integration over the entire range $[D_3, D_4]$ slightly overestimates the absolute magnitude of this term. However this integral multiplies quantities which are perfectly regular for all $\tilde{T}$, so we can in any case ignore this term.

For $\tilde{T} \gg T_0$, $D_4 - D_3 \sim 2\tilde{T}$, and

$$\psi \sim \frac{T_0\dot{\psi}(T_0)}{2} \ln\frac{\tilde{T}}{T_0}. \tag{4.14}$$

In this situation one may trace through the substitutions that lead to (4.10) and (4.11), to find the integration variable $v \sim [(Z - D_3)/(D_4 - D_3)]^{1/2}$ or $[(D_4 - Z)/(D_4 - D_3)]^{1/2}$.

The edge zones via (4.5) then correspond to an interval in v proportional to $(T_0/\tilde{T})^{1/2}$, so we make vanishingly small errors estimating the singular part of ψ in the far future (cf. Ref. 6).

If $\tilde{T} \ll T_0$, $D_4 - D_3 \sim 2T_0$ and

$$\psi \sim -\frac{T_0\dot{\psi}(T_0)}{2} \ln\frac{\tilde{T}}{T_0}. \tag{4.15}$$

In this case the edge interval in v is proportional to $(\tilde{T}/T_0)^{1/2}$ and again the edge contribution to the singular behavior can be ignored.

For both $\tilde{T}/T_0 \gg 1$ and $\tilde{T}/T_0 \ll 1$ the solution approaches a solution like an average of the Kasner's specified in the data integration interval.

For $T \gg T_0$, Eq. (4.14) shows that typical Riemann tensor components fall off like T^{-2} (since they are constructed by two $\tilde{T}$ derivatives of ψ; cf. Refs. 3 and 6). Hence we have in the future regular behavior similar to that found in homogeneous cosmologies.

As Szekeres has shown, the singularity in the Green's function for $\tilde{T} \to 0$ again produces a Riemann tensor $\sim T^{-2}$. Thus at $\tilde{T} \to 0$ physical singularities arise in the tidal Riemann tensor. But $\tilde{T} = 0$ is just the big-bang singularity, and so the waves in our solutions focus only at the cosmological singularity.

V. COLLISION OF GRAVITATIONAL SHOCKS: ANALYTIC TREATMENT

Substantial insight into the behavior of colliding gravitational waves in an expanding universe can be obtained by an analytical approximation in which the waves are generated by step-discontinuous data, which represent δ-function Riemann-tensor waves colliding at the instant $T = T_0$. Evolved forward, these data show the waves moving apart; evolved backward, they show the model prior to the wave collision. If we concentrate on an interval ΔT in T sufficiently small that $\Delta T \ll T_0$, we can simplify many of the formulas of Secs. III and IV, and give an explicit analytic solution which shows the behavior of the waves as they collide.

It must be emphasized that an assumption of physically discontinuous metric derivatives, leading to δ-function Riemann-tensor behavior, is only a mathematical approximation. In real physics we expect to see only very large, very narrow Riemann-tensor pulses. Because it simplifies the analysis, it is very useful here to admit these Riemann-tensor δ functions, which means we are dealing with a piecewise C^1 metric. A slight amount of smoothing on our data (which in fact we do for our numerical work in Sec. VI) returns us to a C^2 solution.[5] We will show that in our case the piecewise C^1 situation in this section is legitimately a limiting case of a C^2 situation with sharp Riemann-tensor pulses.

We consider it imperative that a putative result obtained in a piecewise C^1 situation be stable to slight smoothing to become C^2, as ours is. We have found no indication that any property of plane-symmetric cosmologies (or plane-wave spacetimes) is not stable in this sense; one should be very suspicious of any such alleged property.

First consider the expression we obtain if the Green's function (2.9) is expanded out to first order in its argument, $O(\Delta T)$. Using the expansion of the Green's function from Ref. 6, Eqs. (6.7) and (6.8), we have

$$\psi(\tilde{Z},\tilde{T}) = \frac{1}{(T_0\tilde{T})^{1/2}}\int_{D_3}^{D_4}\frac{p_1}{1-p_3}\left(1-\frac{(D_4-Z)(Z-D_3)}{16T_0^{\,2}}\right)dZ\,. \tag{5.1}$$

Because the integration interval $D_4 - D_3 \sim O(\Delta T)$, the term proportional to T_0^{-2} in the integrand which arises from the Green's function is already second order, and so will be dropped. The fact that $\tilde{T}^{-1/2}$ appears multiplying the integral suggests an expansion of $\tilde{T}$ in ΔT; this term leads to only lower-order contributions to ψ and to the Riemann tensor, so we drop it also. Thus,

$$\psi(\tilde{Z},\tilde{T}) \simeq \frac{1}{T_0}\int_{D_3}^{D_4}\frac{p_1}{1-p_3}\,dZ\,, \tag{5.2}$$

and for $\Delta T>0$, and for initial data which are discontinuous at $Z=0$ (dropping the tilde indicating the field point),

$$\begin{aligned}
&\psi \simeq 2\frac{\Delta T}{T_0}A, \quad Z<-\Delta T<0\\
&\psi \simeq \frac{\Delta T}{T_0}\left(A\frac{\Delta T - Z}{\Delta T}+B\frac{\Delta T+Z}{\Delta T}\right), \quad -\Delta T\leqslant Z\leqslant \Delta T\\
&\psi \simeq 2\frac{\Delta T}{T_0}B, \quad Z>\Delta T>0
\end{aligned} \tag{5.3}$$

where the "approximate equality" signs denote the fact that we are keeping only the largest terms in the expression. Also

$$A = \left.\frac{p_1}{1-p_3}\right|_A \tag{5.4a}$$

and

$$B = \left.\frac{p_1}{1-p_3}\right|_B \tag{5.4b}$$

are the constants associated with model A and model B, respectively. In more compact notation,

$$\begin{aligned}
\psi = \frac{1}{T_0}\{&2A\Delta T\Theta(-Z-\Delta T)+2B\Delta T\Theta(Z-\Delta T)\\
&+[A(\Delta T - Z)\\
&\quad + B(\Delta T + Z)]\Theta(Z+\Delta T)\Theta(\Delta T - Z)\},\\
&\Delta T>0
\end{aligned} \tag{5.5}$$

where Θ is the Heaviside step function: $\Theta(x)=0$ for $x<0$ and $\Theta(x)=1$ for $x>0$.

It is straightforward to compute

$$\psi_{,Z} = \frac{1}{T_0}(B-A)\Theta(Z+\Delta T)\Theta(\Delta T - Z), \quad \Delta T>0 \tag{5.6}$$

which as anticipated vanishes in the homogeneous regions away from the causal influence of the discontinuity in the initial data.

We also have

$$\begin{aligned}
\psi_{,T} = \frac{1}{T_0}\,&[2A\Theta(-Z-\Delta T)+2B\Theta(Z-\Delta T)\\
&+(A+B)\Theta(Z+\Delta T)\Theta(\Delta T - Z)], \quad \Delta T>0
\end{aligned} \tag{5.7}$$

which also takes its homogeneous value outside the causal influence of the initial discontinuity.

We will also need

$$\psi_{,TT} = \frac{B-A}{T_0}[\delta(Z+\Delta T)-\delta(Z-\Delta T)], \quad \Delta T>0 \tag{5.8}$$

$$\psi_{,ZZ} = \frac{B-A}{T_0}[\delta(Z+\Delta T)-\delta(Z-\Delta T], \quad \Delta T>0 \tag{5.9}$$

$$\psi_{,ZT} = \psi_{,TZ} = \frac{B-A}{T_0}[\delta(Z+\Delta T)+\delta(Z-\Delta T)], \quad \Delta T>0 \tag{5.10}$$

which produce the interesting result

$$\psi_{,ZZ}+\psi_{,TT}+2\psi_{,TZ} = 4\frac{B-A}{T_0}\delta(Z+\Delta T), \quad \Delta T>0\,. \tag{5.11}$$

For $\Delta T<0$, we rewrite as

$$\begin{aligned}
\simeq \frac{1}{T_0}\{&2A\Delta T\Theta(-Z+\Delta T)+2B\Delta T\Theta(Z+\Delta T)\\
&+[A(\Delta T + Z)+B(\Delta T - Z)]\\
&\times\Theta(Z-\Delta T)\Theta(-\Delta T - Z)\}, \quad \Delta T<0\,.
\end{aligned} \tag{5.12}$$

Then

$$\psi_{,Z}=\frac{A-B}{T_0}\,\Theta(Z-\Delta T)\Theta(-\Delta T-Z),\quad \Delta T<0 \tag{5.13}$$

$$\psi_{,T}=\frac{1}{T_0}\,[2A\Theta(-Z+\Delta T)+2B\Theta(Z+\Delta T) + (A+B)\Theta(Z-\Delta T)\Theta(-\Delta T-Z)],\quad \Delta T<0 \tag{5.14}$$

$$\psi_{,TT}=\frac{A-B}{T_0}\,[\delta(Z-\Delta T)-\delta(Z+\Delta T)],\quad \Delta T<0 \tag{5.15}$$

$$\psi_{,ZZ}=\frac{A-B}{T_0}\,[\delta(Z-\Delta T)-\delta(Z+\Delta T)],\quad \Delta T<0 \tag{5.16}$$

$$\psi_{,TZ}=\psi_{,ZT}=\frac{B-A}{T_0}\,[\delta(Z-\Delta T)+\delta(Z+\Delta T)],\quad \Delta T<0 \tag{5.17}$$

whence

$$\psi_{,ZZ}+\psi_{,TT}+2\psi_{,TZ}=4\frac{B-A}{T_0}\,\delta(Z+\Delta T),\quad \Delta T<0\,. \tag{5.18}$$

As checks on our formulation, we note that $\psi_{,T}$ has the same sign for $\Delta T=-\epsilon<0$ as for $\Delta T=\epsilon>0$, while the spatial derivative changes sign, as expected because the quantity ψ passes through zero at $T=T_0$ (i.e., $\Delta T=0$). Furthermore, the quantity

$$\psi_{4(S)}=\tfrac{1}{2}\,(\psi_{,ZZ}+\psi_{,TT}+2\psi_{,TZ}) \tag{5.19}$$

is identical for $\Delta T=\pm\epsilon$. This latter quantity is the most singular part (near $\Delta T=0$) of the Weyl-tensor component[3]:

$$\Psi_4=C_{\alpha\beta\gamma\delta}N^\alpha\bar{M}^\beta N^\gamma\bar{M}^\delta\,, \tag{5.20}$$

where N^μ is a null vector, which has values

$$N^\mu\partial_\mu=\partial_T+\partial_Z \tag{5.21}$$

at $\Delta T=0$, and M^μ is a unit complex spacelike vector; in our case

$$M^\mu=2^{-1/2}\left(e^{-\psi/2}\partial_X+i\frac{e^{\psi/2}}{T}\,\partial_Y\right) \tag{5.22}$$

and $\bar{M}^\mu$ is the complex conjugate of M^μ. If N^μ is extended by requiring it be tangent to the null ray $Z-T=$const, affinely parametrized, then $\bar{M}^\mu$ is parallel propagated along the null geodesic.

Ψ_4 defined above, via Eqs. (5.10), (5.18), and (5.19), is large on the line $Z=-\Delta T$; it is the gravitational strength of the shock traveling along rays with tangent L^μ which at the time $T=0$ can be taken as

$$L^\mu\partial_\mu=\partial_T-\partial_Z\,. \tag{5.23}$$

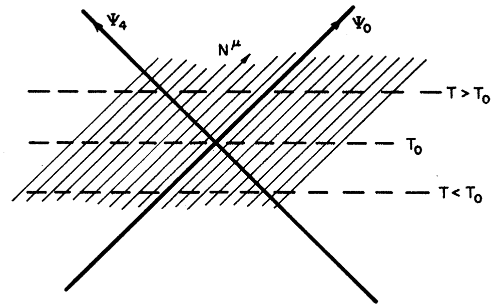

FIG. 3. Ψ_4 is a source of the shear in the null congruence N^μ [cf. Eq. (5.32)] which is tangent to the direction of propagation of the pulse in Ψ_0. Thus we expect the shear associated with N^μ to be continuous on a line $T=$constant except at the location of the Ψ_4 wavefront. The step in σ thus propagates leftward with Ψ_4. This is shown in Figs. 5 below, where the step is seen smoothed in the C^2 data evolved numerically.

If we considered the object $\Psi_0=C_{\alpha\beta\gamma\delta}L^\alpha M^\beta L^\gamma M^\delta$, we would have computed the strength of the other, "rightward"-traveling shock. Entirely symmetrical formulas would result. We would, of course, find that N^μ lay tangent to this rightward-traveling shock, as shown in Fig. 3.

We are here principally interested in the interaction of the shocks as they collide. We wonder, for instance, what the focusing effect of one on the other will do to the convergence or the shear associated with N^μ, say. (For our simple models the rotation vanishes identically.) We calculate some of the properties of the congruence N^μ defined by Eq. (5.21) at $\Delta T=0$, and by the demand that it be an affinely parametrized null tangent for $T\neq T_0$.

First of all, notice that our metric form (2.1) guarantees $N^T=N^Z$ for a null vector. The geodesic equation is simply

$$0=\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial Z}\right)N^T+[(\gamma-\psi)_{,T}+(\gamma-\psi)_{,Z}]N^Z\,, \tag{5.24a}$$

i.e.,

$$0=\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial Z}\right)(e^{\gamma-\psi}N^T)\,. \tag{5.24b}$$

The expansion[7] of the geodesic congruence N^μ is very simple:

$$\theta=\tfrac{1}{2}N^\mu{}_{;\mu}=\frac{N^T}{2T}\,. \tag{5.25}$$

We note at this point that unless the spacetime is very pathological, θ cannot go to zero (a requirement for the appearance of a singularity) unless $T\to\infty$. Now $N^T=dT/d\lambda$ along the ray, where λ is an affine parameter. Hence the spacetime must have a finite T point which is the limit for all geodesics of this form, in order for convergence to occur.

Our Green's-function analysis showed that in the far future $\psi \sim \ln\tilde{T}$. Further, both γ and ψ vanish on the surface $T = T_0$. From Eq. (2.3a) we have

$$\dot{\gamma} \propto \tilde{T}/\tilde{T}^2 \propto \tilde{T}^{-1}.$$

Hence

$$N^T = e^{\psi-\gamma} = t^w, \tag{5.26}$$

where $|w| < \infty$. Thus N^T can vanish only at $t = \infty$, and reconvergence cannot occur.

We use Penrose's[9] definition of the shear:

$$\begin{aligned}\sigma &= N_{\mu;\nu}\bar{M}^\mu \bar{M}^\nu \\ &= \frac{N^T}{2}\ \psi_{,T} + \psi_{,z} - \frac{1}{T}\ .\end{aligned} \tag{5.27}$$

Now the important point of consistency between the optical scalar σ and the Weyl tensor Ψ_4 is the equation

$$N^\mu \sigma_{,\mu} = -2\sigma\theta + \Psi_4, \tag{5.28}$$

i.e., near $\Delta T = 0$

$$\partial_u\, \sigma = \sigma_{,T} + \sigma_{,Z} \simeq \Psi_4, \tag{5.29}$$

where

$$\partial_u = \partial_T + \partial_Z \tag{5.30}$$

introduces a null coordinate u and where we keep only the most singular terms on the right side. But then, near $\Delta T = 0$, from (5.27)

$$\begin{aligned}\sigma_{T,} + \sigma_{,Z} &\simeq \tfrac{1}{2}(\psi_{,TT} + \psi_{,ZT} + \psi_{,TZ} + \psi_{,ZZ}) \\ &\simeq \Psi_4\end{aligned} \tag{5.31}$$

and so we have a consistent scheme.

Rewriting Eq. (5.29), using the Ψ_4 we found near $\Delta T = 0$ [i.e., Eqs. (5.18) and (5.19)] we have

$$\partial_u \sigma = 2\frac{B-A}{T_0}\ \delta(2u)\,; \tag{5.32}$$

the expected jump in σ for N^μ (the rightward-traveling tangent) due to the collision is thus

$$\sigma_{\text{after}} - \sigma_{\text{before}} = \frac{B-A}{T_0}. \tag{5.33}$$

We may verify that Eq. (5.27) evaluated using the expressions (5.6), (5.7) and (5.13), (5.14) is

$$\sigma_{\text{before}} \cong \frac{A}{T_0} - \frac{1}{2T_0} = \frac{1}{2T_0}\left(\frac{p_1 - p_2}{1 - p_3}\right)_A, \tag{5.34}$$

$$\sigma_{\text{after}} \cong \frac{B}{T_0} - \frac{1}{2T_0} = \frac{1}{2T_0}\left(\frac{p_1 - p_2}{1 - p_3}\right)_B, \tag{5.35}$$

which obviously satisfy (5.33). In evaluating the shear as in (5.31) and (5.32) we assumed a congruence parallel to the shock, both ahead and behind the shock, with tangent normalized to $N^\mu \partial_\mu = \partial_T + \partial_Z$ on $\Delta T = 0$.

We see, via Eqs. (5.33)–(5.35) that in a small region near $Z = \Delta T = 0$, the null ray which lies just to the past of $Z = \Delta T$ (for $\Delta T < 0$) has the same value of shear as the null ray which lies just to the future of $Z = \Delta T$ (for $\Delta T < 0$); both these rays suffer the same jump [given by (5.33)] on crossing the other shock. But the ray which lies in the future of $Z = \Delta T$ is in a homogeneous region; the ray just to the past is in an inhomogeneous region of the spacetime. The properties of the rays parallel to the plane-wave front are continuous across the wave front.

VI. COLLISION OF GRAVITATIONAL SHOCKS: NUMERICAL EVOLUTION[20]

We now consider the numerical evolution of an initial ($T = T_0$) slice that describes two Kasner models A and B matched together at a join surface. The future evolution of this data set shows[7] gravitational shocks Ψ_0 and Ψ_4 traveling to the right ($+Z$ direction) and left ($-Z$ direction), respectively, out of the join surface. Since we are interested in the collision of Ψ_0 and Ψ_4, we first run the code backwards in time until the shocks are separated. Using this earlier time slice as our new "initial" data slice, we then evolve forward in time, through the collision at $T = T_0$, and on until the shocks are separated in the future. The situation is shown in Figs. 4(a)–4(c). We then compare our computations in the region $\Delta T \ll T_0$ around the $T = T_0$ slice with the analytic estimates derived in Sec. V.

Note that to evolve a slice backwards in time, we set the time interval $dT \to -dT$ in the code. This procedure will not be valid whenever we expect irreversible physical processes to be present. We checked that no errors were introduced by this technique by verifying the behavior of the metric as the evolution passed through $T = T_0$. There, the three-metric should be flat and we find that $\gamma_{ij} = \delta_{ij} \pm 10^{-5}$. We also remark that the code uses periodic boundary conditions; we thus have two join surfaces, with model A matched to B, and then B matched to A to achieve periodicity. The figures show only the left-hand side of this evolution for clarity.

Our numerical work uses a metric of the form

$$ds^2 = -dt^2 + \gamma_{xx}\,dx^2 + \gamma_{yy}\,dy^2 + \gamma_{zz}\,dz^2, \tag{6.1}$$

where the γ_{ij} are functions of z and t only. Since $\gamma_{ij} = \delta_{ij}$ on the initial slice, and since the analytical field variables ψ and γ both vanish at $T = T_0$, the coordinates (z, t) are the same as (Z, T) in a suf-

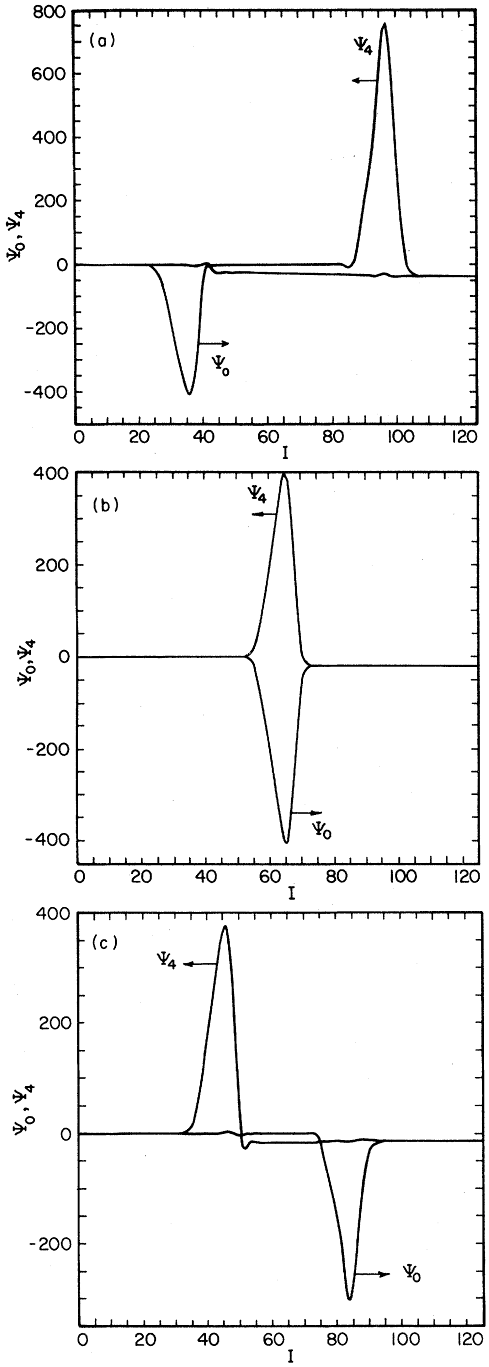

FIG. 4. (a) The shocks Ψ_0 and Ψ_4 are traveling towards each other and will collide in the future at $T = T_0$. This figure shows the situation $\Delta t = 3\times10^{-2}$ to the past of the "initial slice" T_0. The code was evolved backwards to this earlier slice to separate the shocks, and then run forward to follow the collision at T_0. Arrows show the direction the shocks are traveling and the horizontal axis labels zone number I. (b) The shocks Ψ_0 and Ψ_4 collide at $T = T_0$. (c) The shocks Ψ_0 and Ψ_4 have collided and are now moving apart. This figure shows the situation at $\Delta t = 2\times10^{-2}$ to the future of T_0.

ficiently small region $\Delta T \ll T_0$ around that slice. Thus, we can readily compare analytical and numerical results in that region.

To calculate Ψ_4 in the code we use the expression $\Psi_4 = C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$. These null vectors are

$$n^\mu = A(1, 0, 0, (\gamma_{zz})^{-1/2}), \tag{6.2a}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}} (0, (\gamma^{xx})^{1/2}, -i(\gamma^{yy})^{1/2}, 0). \tag{6.2b}$$

The rightward-traveling vector n^μ is geodesic, with the amplitude A satisfying the geodesic equation

$$(\gamma_{zz})^{1/2} \frac{\partial A}{\partial t} + \tfrac{1}{2} A (\gamma_{zz})^{-1/2} \frac{d\gamma_{zz}}{dt} + \frac{dA}{dz} = 0. \tag{6.3}$$

The leftward-traveling shock Ψ_4 is a source of the expansion and shear of this rightward-traveling null congruence n^μ. We are free to give the amplitude of A at one point on each ray. We set $A = 1$ on the slice $T = T_0$, consistent with the analytic treatment in the previous section. To compute Ψ_0, we use an entirely analogous procedure; $\Psi_0 = C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta$, where now l^μ is a leftward-traveling geodesic congruence.

We are now ready to compare analytical and numerical results. Our input to the code is

$$t_{0A} = 0.900, \tag{6.5a}$$

$$p_{3A} = 0.100, \tag{6.5b}$$

and

$$t_{0B} = 0.180. \tag{6.6a}$$

As discussed in Ref. 6, the constraint equations then give

$$p_{3B} = 0.820. \tag{6.6b}$$

Thus, both models are expanding in the z direction; model B is younger than model A, however, and thus is expanding more rapidly. Equations (2.5) give

$$p_{1A} = 0.991, \tag{6.7a}$$

$$p_{2A} = -0.091, \tag{6.7b}$$

and

$$p_{1B} = 0.484, \tag{6.8a}$$

$$p_{2B} = -0.304. \tag{6.8b}$$

The quantities A and B in Eqs. (5.4) are then

$$A = 1.101, \tag{6.9a}$$

$$B = 2.689. \tag{6.9b}$$

Refer now to Fig. 4(b). Equations (5.18) and

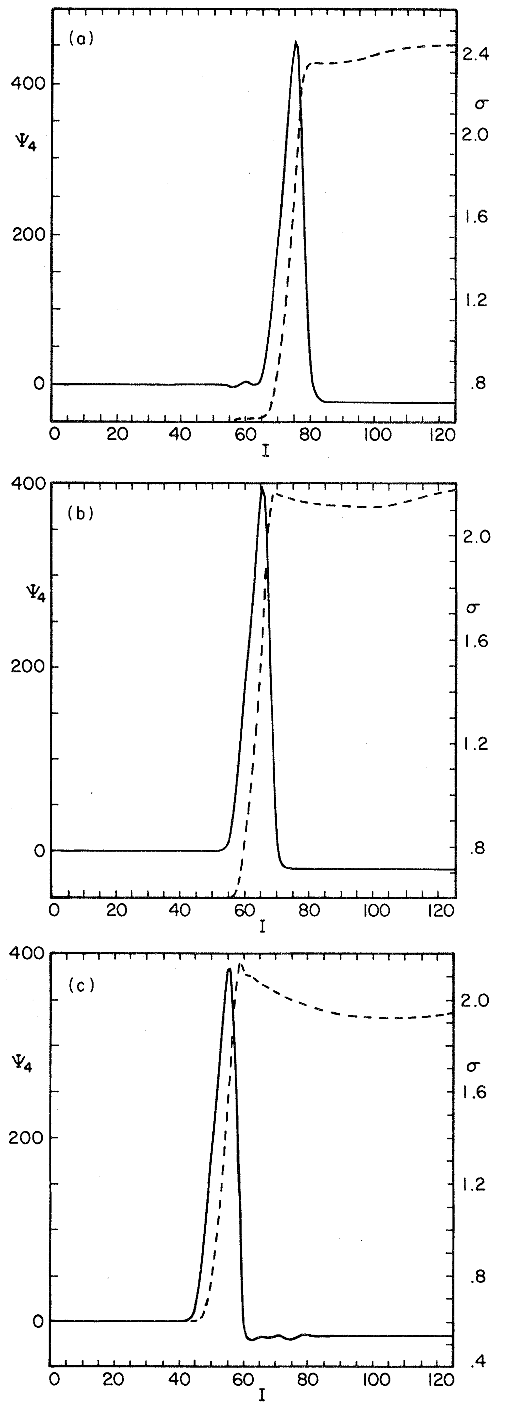

FIG. 5 (a) The shear σ of the congruence N^{μ} tangent to Ψ_4 suffers a jump only at the position of Ψ_4. This graph shows the situation at $\Delta t = 1\times 10^{-2}$ to the past of the collision at T_0. The dashed line shows σ which is measured on the right-hand scale; the solid line shows Ψ_4 which is measured on the left-hand scale. (b) This graph shows the situation at T_0 as Ψ_0 and Ψ_4 collide. (c) This graph shows the situation at $\Delta t = 1\times 10^{-2}$ to the future of the collision at T_0.

(5.19) say that

$$\Psi_{4(s)} = 2\frac{(B-A)}{T_0}\,\delta(Z+\Delta T)\,. \tag{6.10}$$

Thus, the area under the curve Ψ_4 in Fig. 4(b) should be $2(B-A)/T_0$. Since[6]

$$T_0 = 1\,, \tag{6.11}$$

we get

$$\text{area under } \Psi_{4(s)}\big|_{\text{analytic}} = 3.176\,.$$

Now, using the fact that the spatial mesh size $\Delta z = 10^{-3}$ and approximating Ψ_4 by a triangle, we find

$$\text{area under } \Psi_4\big|_{\text{numerical}} \sim 3\,,$$

in good agreement with the analytical estimate.

Focus now on the behavior of the shear. As discussed in Sec. V, the shear of the N^{μ} congruence suffers a jump as it crosses Ψ_4 on each slice. There is no jump in the shear across Ψ_0 since the properties of the rays N^{μ} are continuous across that parallel wave front. From Eqs. (5.34) and (5.35) we find, for the shear before crossing Ψ_4,

$$\sigma_{\text{before}} \sim 0.601\,, \tag{6.12a}$$

and for the shear after crossing Ψ_4,

$$\sigma_{\text{after}} \sim 2.189\,. \tag{6.12b}$$

Figures 5(a)–5(c) show that our numerical evoluation has produced good agreement with these analytical estimates. We consider a region 50 time steps to the past and future of T_0. Since the time-step size is $\delta t = 2\times 10^{-4}$, this produces an interval of duration $\sim 1.0\times 10^{-2}$ centered on T_0. Figure 5(a) shows the situation before collision, when Ψ_0 and Ψ_4 are still separated; Fig. 5(b) shows the collision at T_0, and Fig. 5(c) shows the situation in the future, when the shocks are once again separated. In all three cases, we see that the jump in the shear occurs at the position of Ψ_4. The numerical values for σ_{before} and σ_{after} are both close to the analytic estimates (6.12). We take this as justification that our piecewise C^1 analysis of Sec. V is an accurate predictor of the behavior of large, sharp Riemann-tensor pulses in these cosmologies.

VII. SUMMARY

We have discussed the singularity behavior and causal structure of expanding vacuum plane-symmetric cosmologies containing colliding gravitational waves. The singularity in our solutions occurs in the past and can be identified with the big-bang singularity. Penrose's[14] theorem excluding global Cauchy hypersurfaces in "plane-wave spacetimes" (which contain nonexpanding wave fronts traveling along a single null direction) does not hold for our expanding cosmologies.

Analytical estimates based on a Green's-function analysis near the shock collision provide an explicit example which is verified by numerical computation.

The expansion of the wave fronts in plane-symmetric cosmologies is always positive [cf. Eq. (5.25)]. This provides a direct proof that the supposed "nonsingular vacuum solution" of Stoyanov[21] is in fact no solution at all. It has rays lying in initial wave fronts ("colliding waves") with vanishing initial expansion. After the collision, Stoyanov's solution may be verified to be an expanding Kasner solution. Hence, no matter how the individual rays are connected up, the invariantly defined congruence traveling toward increasing Z (say) has zero expansion before the collision and *positive* expansion afterward [cf. Eq. (5.25)]. From the equation for optical scalars,[8] if there is any coordinate system in which the null rays can be followed across the collision front, they must have encountered *negative* matter-energy there (since the solutions have vanishing rotation everywhere).

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