PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOLUME 25, NUMBER 4 15 FEBRUARY 1982

Geodesic instability and internal time in relativistic cosmology

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(Received 5 October 1981)

The concept of "internal time" is applied to a cosmological model having spatial hypersurfaces of negative curvature. It is then possible to ascribe an irreversible evolution to the expanding universe without resorting to any "coarse graining" or "loss of information." The key observation which enables this description to be used is that geodesic flow on a four-manifold can be reduced to geodesic flow on a three-manifold when the Robertson-Walker metric is used. If the three-surface is compactified in such a way as not to change the metric, and if it has negative curvature, the geodesic system is a Bernoulli flow—a dynamical system which has the highest degree of instability. We draw various conclusions about mixing in the system pertinent to the microwave background, the observational consequences of negative curvature for objects moving with respect to the galaxies, and we show that the requirement of negative curvature always leads to a particle horizon, a conclusion which has some bearing on the physical spectrum of the internal time operator and on the possibility of removing the cosmological singularity to the infinite past.

I. INTRODUCTION

Classical mechanics starts by idealizing dynamical evolution as being a deterministic and (time) reversible motion of phase points along the phase-space trajectories. Implicit in this idealization is the supposition that initial conditions (phase points) can be determined with infinite precision. This idealization is unobjectionable provided one considers systems (such as periodic planetary motion) for which the phase-space trajectories depend on the initial conditions in a continuous manner. Recent interest has shifted, however, to the study of more complex systems exhibiting instabilities of phase-space trajectories. It has become increasingly evident that, for such systems, the very concept of deterministic motion along phase-space trajectories is an unphysical idealization and a probabilistic description of the system is more appropriate.

Our recent work[1-5] has given a precise formulation of this idea. It has been shown that for a class of dynamical systems, the so-called K-flows,[6] the dynamical motion is indeed "equivalent" (similar), in a well-defined sense, to the stochastic evolution of Markov processes. This "similarity" also permits the construction of an H function or Lyapounov variable associated with the dynamical motion of K-flows. These results demonstrate the intrinsically stochastic and irreversible character of dynamical systems that satisfy the K-flow condition.

Another interesting feature of such systems, which is closely related to the above-mentioned property of intrinsic randomness and irreversibility, is the existence of an *internal time operator.*[2] Briefly, this is an operator acting on the distribution functions of the phase space and which is canonically conjugate to the generator of motion (Liouvillian). Its existence permits one to attribute internal time (or age) to distribution functions in such a manner that advance in internal age corresponds to decrease in an H function of the system (increase of entropy). The internal time operator may thus serve as a microscopic model of thermodynamic time. (For a discussion of thermodynamic time see Ref. 7.)

The main purpose of this paper is to point out that in certain cosmological models all of the above-mentioned features (trajectory instability, equivalence with the Markov process, the existence of a Lyapounov variable, and an internal time operator) are exhibited by the simplest and most fundamental of dynamical motions: the free geodesic motion of test particles. This work illustrates a possibility of how stochasticity and irreversibility could be linked with cosmology.

These considerations are also found to be of interest in connection with the conceptual issues raised by the singularity prediction of Einstein's theory. As is well known,[8] under very general assumptions, Einstein's theory of general relativity predicts the occurrence of a singularity in the spacetime structure which may be regarded in some sense as the "beginning" of the universe. What is disconcerting about this conclusion is that this cosmological singularity ("big bang") is predicted to have occurred at a *finite* proper time in the past. The singularity prediction of general relativity thus confronts us with the unpleasant possibility of a complete breakdown of *all* physical concepts and laws at a *finite* proper time in the past. Naturally, various attempts have been made to avoid this unpleasant implication of Einstein's theory. Most attempts involve modifications of Einstein's classical theory of gravitation, such as the inclusion of quantum gravity effects. However, the outcome of such modifications on the singularity problem have so far remained inconclusive.

Another possible approach to avoid the unpleasant implications of the singularity is to try to remove it to the infinitely distant past. This would be achieved if one could demonstrate the existence of a time scale *that has well-defined physical significance* and is such that, in this scale, the time interval that has elapsed since the cosmological singularity is infinite.

Let us mention in this connection that in a cosmological context the physical significance of the concept of proper time is far from straightforward. The point is, as expressed by Misner, Thorne, and Wheeler,[9] "proper time near the singularity is not a direct counting of simple and actual physical phenomena but an elaborate mathematical extrapolation . . . . Since no single clock (because of its finite size and strength) is conceivable all the way back to the singularity, a statement about the proper time since the singularity involves the concept of an infinite sequence of successively smaller and sturdier clocks with their ticks then discounted and added. 'Finite proper time,' then, need not imply that any finite sequence of events was possible. It may describe a necessarily infinite number of events ('ticks') in any physically conceivable history, converted by mathematics into a finite sum by the action of a nonlocal convergence factor, the 'discount' applied to convert 'ticks' into 'proper time.' "

In a cosmological context the proper time loses its preeminent position as the physically significant time variable. It seems more appropriate to introduce a time concept which, unlike the proper time, does not involve the time measured by an external clock carried by the observer or particle, but is related to the intrinsic properties of the motion of the particle itself. In other words, the new time concept we seek should refer in some suitable sense to the *"internal time"* associated with the particle's motion.

As mentioned before, the possibility of defining such a concept of internal time exists for dynamical systems satisfying the K-flow condition.[2–4] In this paper we point out that the geodesic motion of test particles in a universe having compact three-surfaces of constant negative spatial curvature can be regarded as K-flows. As a consequence, one can associate an internal time operator with the free motion of test particles in such a universe. The time variable $\lambda(t)$ associated with this internal time is found to be a nonlinear function of the cosmological time parameter t.

The existence of a new time scale associated with internal time naturally raises the question of whether the cosmological singularity could be removed to the infinitely remote past in the new scale. A simple general argument shows that this can be done only if the pressure of the cosmological fluid is allowed to take negative values. Since pressure is usually considered to be a positive quantity this seems to lead to the negative conclusion that the singularity cannot be removed to the infinitely remote past by the introduction of the internal time scale. Nevertheless, it should be stressed that the concept of pressure near the singularity is far from being unambiguous and the meaning and possibility of "negative pressure" needs to be examined further. We hope to come back to this question soon in connection with a reexamination of the meaning of the second law in general relativity.

II. INTERNAL TIME OPERATOR

We begin with a brief review of the properties of the internal time operator associated with suitably unstable dynamical evolutions.[2–5]

Let us consider an abstract dynamical system (Ω, B, μ, S_t). Here Ω denotes the phase space of the system which is equipped with σ-algebra B of (measurable) sets, S_t is the one-parameter evolution group mapping Ω onto itself, and μ a measure invariant under S_t: $\mu(S_t\Delta)=\mu(\Delta)$. We shall further suppose that the measure μ is finite and is normalized for convenience by $\mu(\Omega)=1$. Finite classical-mechanical systems give rise to such a structure when Ω is taken to be a constant energy surface of the system, the group S_t describing dynamical motions of phase points and μ being the Liouville measure on Ω. But other systems such as geodesic flows on compact Riemannian manifolds can also be viewed as abstract dynamical systems.

The evolution group S_t on Ω induces a unitary group U_t on $L_\mu{}^2(\Omega)$ which describes the evolution of distribution functions by the relation

$$U_t\rho(\omega)=\rho(S_{-t}\omega) . \tag{2.1}$$

Now, let $H_{-\infty}$ denote the one-dimensional subspace of constant functions on Ω, $P_{-\infty}$ the projections onto $H_{-\infty}$, and $H^{\perp}_{-\infty}$ the subspace that is orthogonal to $H_{-\infty}$. By an internal time operator T of the abstract dynamical system we mean a self-adjoint operator T and $H^{\perp}_{-\infty}$ having the following properties:

(a) $U_t^* T U_t = T + tI$.

(b) Let $\{E_\lambda\}$ denote the spectral projections of T: $T=\int_{-\infty}^{\infty}\lambda dE_\lambda$. Then the projection $P_\lambda = E_\lambda + P_{-\infty}$ of $L_\mu{}^2$ preserves the positivity of functions: i.e., $\rho \geq 0$ a.e. implies $P_\lambda \rho \geq 0$ a.e.

Existence of an internal time operator in the above sense places important restrictions on the dynamical system. Specifically, a dynamical system admits an internal time operator if and only if it is a K-flow.[2–5]

For a precise definition of K-flow, see, for example, Ref. 6. Let us only mention that K-flows have the characteristic property that the knowledge about the past history of the system obtained from an infinite repetition of *any* measurement of finite precision (corresponding to a partition of the phase space into a finite number of disjoint cells) is insufficient to predict the future outcome of the same measurement. The mathematical expression of this fact is that K-flows have strictly positive Kolmogorov entropy.[6] The observed evolution of a K-flow is thus nondeterministic in character, the physical origin of which stems from the high degree of trajectory instability.

Many systems of physical interest, e.g., a system of hard spheres in a box, and more importantly for our present purpose the geodesic flow on a compact manifold of negative curvature, are known to be K-flows.[10,11]

If the system admits an internal time operator T one can interpret the quantity $\langle T\rangle_\rho$ given by

$$\langle T\rangle_\rho = \int_\Omega \bar{\rho}(T\bar{\rho})d\mu = \langle \bar{\rho}, T\bar{\rho}\rangle \tag{2.2}$$

(with $\bar{\rho}=\rho-1$ being the departure from microcanonical equilibrium ensemble $\rho_{eq}=1$), as the average internal time or "age" of the (Gibbsian) ensemble ρ. Property (a) of T then expresses the desirable consistency requirement that in the course of dynamical evolution the system's *average age* advances in step with the increase in the time parameter t labeling the dynamical evolution: i.e., if $\rho_t = U_t\rho_0$ then $\langle T\rangle_{\rho_t} = \langle T\rangle_{\rho_0} + t$.

We shall not discuss here in detail the significance of property (b) of T. The interested reader may see the references cited in the beginning of this section. Let us, however, mention that if an internal time operator T satisfying both (a) and (b) exists then one can construct an *invertible* transformation Λ, mapping states (i.e., non-negative distributions ρ with $\int_\Omega \rho d\mu = 1$) to states and such that $\Lambda U_t \Lambda^{-1} = W_t^*$ $(t \geq 0)$ is the semigroup induced from a probabilistic Markov process on Ω. Λ is constructed as a suitable operator function of T: $\Lambda = h(T) + P_{-\infty}$. The existence of an internal time operator with the stated properties implies that the given system is *intrinsically random* in the sense that for such systems the dynamical evolution can be transformed into the stochastic evolution of a Markov process by a change of representation $\rho_t \rightarrow \Lambda\rho_t$ which (owing to the invertibility of Λ) does not involve any loss of information.

Moreover, the condition (b) also implies that the "projected evolution" $P_0\ U_t P_0$ $(P_0 = E_0 + P_{-\infty})$ (for $t \geq 0$) is itself the semigroup of a stochastic Markov process and that the functional $\Omega(\rho_t) = \int_\Omega P_0\rho_t \ln(P_0\rho_t)d\mu$, where $\rho_t \equiv U_t\rho_0$ and $P_{-\infty}\rho = 1$ is an H function for the system, i.e., $\Omega(\rho_t)$ decreases monotonically to the equilibrium value $\Omega(1)=0$ as $t \rightarrow +\infty$ and increases to the fine-grained value $\int_\Omega \rho_0 \ln\rho_0 d\mu$ as $t \rightarrow -\infty$. To summarize, the possibility of attributing (average) internal time or age to ensembles is associated, on the one hand, with a strong form of instability of phase-space trajectories (K-flow condition), while on the other, it is associated with a representation of the evolution by a Markov process. It should also be emphasized that although the (average) internal time or age keeps step with the dynamical time parameter t, the internal time is a property of the ensemble or state of the system and it is not identical with the time measured by an external clock. In fact, we shall see in the next sections that in certain cosmological models one can define an internal time scale for freely moving test particles which differs from that of the external "cosmic time" of the model.

Finally, let us briefly recall the concept of geodesic flow. Let M be a Riemannian manifold which is geodesically complete. By a *line element* on M is meant some point of M together with a specified direction or, equivalently, a geodesic passing through the point. Geodesic flow on M is an abstract dynamical system for which the phase space Ω is the set of all line elements of M. The geodesic flow S_t is a mapping which carries each line element along the corresponding geodesic at a given *constant speed,* and the invariant measure $d\mu$ under this motion is $d\mu = dV\,d\theta$, where dV is the volume element of the Riemannian space M and $d\theta$ the "angle differential" corresponding to the directions of the geodesics. As mentioned before, geodesic flows on compact Riemannian manifolds of constant negative curvature are known to be K-flows. For later purposes, the important point is to note that the "time parameter" labeling motion of geodesic flows are the affine parameters λ of geodesics, and the time scale determined by average internal age of distributions is the scale defined by λ.

III. REDUCTION OF GEODESIC MOTION IN SPACETIME TO GEODESIC FLOW IN THREE-SPACE

The particular cosmological model we have in mind is a Friedmann universe with negative spatial curvature whose three-dimensional hypersurfaces of simultaneity are homogeneous and are compactified in an appropriate manner. Generally, cosmological models with negative spatial curvature are regarded as "open" or noncompact. However, in the case of negative- (or zero) curvature three-space, the local metric structure does not uniquely determine the *global* topology, and it is possible to compactify the three-dimensional hypersurfaces without changing the metric structure.[12,13]

We need not discuss here the method of compactification in detail. Let us only mention that it involves identication of points that are carried into each other by a suitably chosen discrete subgroup Γ of the group G of all transitive motions that preserve the local metric structure of the hypersurface. In the case of an open three-dimensional manifold of negative curvature the group G of motions is isomorphic to the Lorentz group: $\mathrm{PSL}(2,C) = \mathrm{SL}(2,C)/\{\pm 1\}$. The covering space $\hat{\Omega}$ of the compactified phase space Ω is isomorphic to $\mathrm{PSL}\ (2,C)\ /\ \mathrm{U}(1)$ so that all rotations about the tangent vector to a geodesic are to be identified. We do this because the phase space is the set of unit tangent vectors (line elements) and *not* the set of all three-frames on the three-manifold.[14] Compatification of three-space would thus involve, in this case, identification of points modulo a suitable discrete subgroup Γ of the Lorentz group. Such subgroups are analogous to Fuchsian subgroups arising in the theory of two-dimensional surfaces of negative curvature.[15] The phase space Ω of our dynamical system is then isomorphic to $\Gamma \backslash \mathrm{PSL}(2,C)/\mathrm{U}(1)$.

Compatification of a negative curvature three-space results in a multiply connected three-geometry. Moreover, the resulting spacetime can no longer be globally isotropic, although *local* isotropy is preserved. For these reasons the particular cosmological model studied in this paper is termed a "nonstandard cosmology." It is interesting to note that the existence of a time operator for the cosmological model forces the model to be nonstandard in the above sense. Both negative spatial curvature and compactification of three-dimensional hypersurfaces are required to ensure the K-flow property, which in turn is essential for the existence of a time operator.

We now proceed to show how the geodesic motion of test particles in four-dimensional spacetime reduces to geodesic flow on a fixed spacelike hypersurface of simultaneity. An important consequence of this "projection" from four-dimensional spacetime to a three-dimensional hypersurface is that the affine parameter λ for geodesic flow on three-space becomes a *nonlinear function* of the affine parameter of spacetime geodesics as well as of the cosmic time.

If we use comoving coordinates $(x^0 = t, x^1, x^2, x^3)$ in a homogeneous and isotropic spacetime then the four-metric is the well-known Robertson-Walker metric $g_{\mu\nu}$:

$$\begin{aligned} g_{00} &= -1, \quad g_{ij} = R^2(t)\gamma_{ij}\,, \\ \gamma_{ij} &= \{1 + \tfrac{1}{4}k[(x^1)^2 + (x^2)^2 + (x^3)^2]\}^{-2}\delta_{ij}\,. \end{aligned} \tag{3.1}$$

Here, as elsewhere in this paper, Latin indices i,j, etc. take the values 1,2,3 only.

The comoving coordinates are defined such that the spatial coordinates x^i of cosmological fluid elements (the typical galaxies) do not change in the course of time. The proper distances between the galaxies change, however, due to the presence of a time-dependent scale factor $R(t)$ in the metric. The time coordinate $x^0=t$ represents the cosmic time. It is the proper time measured by clocks moving with the cosmological fluid elements (i.e., galaxies). The constant k appearing in the metric determines the geometry of three-dimensional hypersurfaces. It takes the value $+1$, -1, or 0 depending on whether the spatial curvature is positive, negative, or zero, respectively.

Let us now consider freely moving test particles in a Roberston-Walker universe. The coordinates $x^\mu(\sigma)$ (as functions of the affine parameter σ of spacetime geodesics) satisfy the geodesic equation

$$\frac{d^2x^\mu}{d\sigma^2}+\Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu dx^\lambda}{d\sigma\, d\sigma}=0 \tag{3.2}$$

together with the condition

$$\begin{aligned} g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}&=0 \text{ (massless particle)}\\ &=-1 \text{ (massive particle)} . \end{aligned} \tag{3.3}$$

Here $\Gamma^\mu{}_{\nu\lambda}$ denotes the affine connection.

We now verify that the three spatial coordinates $x^i(\sigma)$ of freely moving test particles follow geodesic motion (with respect to a new affine parameter λ) in the fixed three-dimensional hypersurface having the metric γ_{ij}. To this end, it will be useful to rewrite the geodesic equation in terms of the cosmic time variable t. Now,

$$\begin{aligned}\frac{d^2x^i}{dt^2}&=\left[\frac{dt}{d\sigma}\right]^{-1}\frac{d}{d\sigma}\left[\left[\frac{dt}{d\sigma}\right]^{-1}\frac{dx^i}{d\sigma}\right]\\ &=\left[\frac{dt}{d\sigma}\right]^{-2}\frac{d^2x^i}{d\sigma^2}-\left[\frac{dt}{d\sigma}\right]^{-3}\frac{d^2t}{d\sigma^2}\frac{dx^i}{d\sigma} . \end{aligned} \tag{3.4}$$

Substituting for $d^2x^i/d\sigma^2$ and $d^2t/d\sigma^2 \equiv d^2x^0/d\sigma^2$ from the geodesic equation (3.2) we find

$$\frac{d^2x^i}{dt^2}+\Gamma^i{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}-\Gamma^0{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}\frac{dx^i}{dt}=0 . \tag{3.5}$$

Using the values of $\Gamma^\mu{}_{\nu\rho}$ for the Roberston-Walker metric we finally obtain

$$\ddot{x}^i+\tilde{\Gamma}^i{}_{jk}\dot{x}^j\dot{x}^k+\left[2\frac{\dot{R}}{R}-\frac{\dot{R}}{R}g_{jk}\dot{x}^j\dot{x}^k\right]\dot{x}^i=0 . \tag{3.6}$$

Here and elsewhere $\tilde{\Gamma}^i{}_{jk}$ denotes the affine connection of the *three-space* metric γ_{ij}, and a dot represents the derivative with respect to t. Upon a change of variable $t\rightarrow\lambda(t)$ Eq. (3.6) becomes

$$\left[\frac{d^2x^i}{d\lambda^2}+\tilde{\Gamma}^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\right]\dot{\lambda}^2 +\left[\ddot{\lambda}+\frac{\dot{R}}{R}(2-g_{jk}\dot{x}^j\dot{x}^k)\dot{\lambda}\right]\frac{dx^i}{d\lambda}=0 . \tag{3.7}$$

Equation (3.6) is thus equivalent to the geodesic equation in three-space provided $\lambda(t)$ is chosen to be a solution of the equation

$$\ddot{\lambda}+\frac{\dot{R}}{R}(2-g_{jk}\dot{x}^j\dot{x}^k)\dot{\lambda}=0 . \tag{3.8}$$

To determine the form of $\lambda(t)$ let us solve (3.8). For massless test particles $g_{jk}\dot{x}^j\dot{x}^k=c^2=1$ (in the units adopted here) and Eq. (3.8) reduces to the equation

$$\frac{d^2\lambda}{dt^2}+\frac{\dot{R}}{R}\frac{d\lambda}{dt}=0 . \tag{3.9}$$

[It is understood here, of course, that $R(t)$ is a known function of t determined by Einstein's equations in conjunction with some assumed equation of state an equation for conservation of energy.] From (3.9) it follows that

$$\frac{d\lambda}{dt}=\frac{A}{R(t)}$$

and (3.10)

$$\lambda(t)-\lambda(t_0)=A\int_{t_0}^{t}\frac{ds}{R(s)}$$

with A a constant. Similarly, for massive test particles,

$$g_{jk}\dot{x}^j\dot{x}k=v^2(t) ,$$

where $v(t)$ is the speed of the particle in the comoving frame at time t. On the other hand, it is known (cf., Ref. 16) that for freely moving particles in an expanding universe

$$\frac{R(t)v(t)}{[1-v^2(t)]^{1/2}}=\text{constant, say } \alpha .$$

Thus the equation (3.8) reduces to

$$\frac{d^2\lambda}{dt^2}+\frac{\dot{R}}{R}\left[1+\frac{R^2(t)}{R^2(t)+\alpha^2}\right]\frac{d\lambda}{dt}=0 . \tag{3.11}$$

As before it can be integrated to yield

$$\frac{d\lambda}{dt}=\frac{A}{R(t)[\alpha^2+R^2(t)]^{1/2}}$$

or (3.12)

$$\lambda(t)-\lambda(t_0)=\int_{t_0}^{t}\frac{A\,ds}{R(s)[\alpha^2+R^2(s)]^{1/2}} .$$

Let us note now that in a nonstatic universe ($\dot{R}\neq 0$) the affine parameter $\lambda(t)$ of the spatial geodesics traced by the spatial coordinates of freely moving test particles (that are not at rest with respect to the cosmological fluid) are essentially different from the cosmic time parameter t; i.e., $\lambda(t)$ is *necessarily* a *nonlinear function of t.* The speed $v(t)$ varies as a function of t; however, as a function of the affine parameter λ it is constant, i.e.,

$$\gamma_{mn}\frac{dx^m dx^n}{d\lambda\, d\lambda}=\text{constant} .$$

IV. TIME OPERATOR AND GEODESIC INSTABILITY IN A NONSTANDARD COSMOLOGY

In the preceding section we have seen that the projection of geodesic motion in spacetime to a fixed three-dimensional hypersurface of simultaneity gives rise to geodesic flows in three-space. This result holds in all cosmological models with Robertson-Walker metrics, independent of the sign of the spatial curvature. But it is only in the case of negative spatial curvature that this possibility of reducing geodesic motion in spacetime to geodesic flow in three-space has interesting implications. The present section is devoted to discussing them.

As is well known, an important feature of the geometry of manifolds with constant negative curvature is the instability of their geodesics: the distance between neighboring geodesics increases *exponentially,* at least in one direction.[17] Thus, in a universe with negative spatial curvature, this exponential instability, in principle, will show up in the divergence rate of spacetime geodesics that are not stationary with respect to the cosmological fluid elements. Let $L(t)=[g_{ij}\Delta x^i(t)\Delta x^j(t)]^{1/2}$ be the spatial distance between two neighboring spacetime geodesics of test particles that are not stationary relative to the cosmological fluid. The distance $L_0(t)$ between the projected geodesics in the fixed three-dimensional hypersurface is then given by $L_0(t)=[\gamma_{ij}\Delta x^i(t)\Delta x^j(t)]^{1/2}=L(t)/R(t)$. As L_0 increases exponentially with respect to change in the affine parameter λ of spatial geodesics we have $dL_0/d\lambda=cL_0$, with c being a positive constant. Thus,

$$\frac{dL}{dt}=\frac{d}{dt}[R(t)L_0]=\frac{\dot{R}}{R}L+(c\dot{\lambda})L . \quad (4.1)$$

Substituting the value of $\dot{\lambda}$ found in the previous section we obtain the divergence rates

$$\dot{L}=\frac{\dot{R}}{R}L+\frac{c}{R}L \text{ (null geodesics)} ,$$

(4.2)

$$\dot{L}=\frac{\dot{R}}{R}L+\frac{cL}{R[\alpha^2+R^2]^{1/2}}\text{(timelike geodesics)} .$$

In the case that the three-space is compact, the average $\bar{c}$ of c over the three-space is essentially equal to the K-entropy of the geodesic flow.[10] Thus $\bar{c}$ can be estimated in terms of the volume V of the three-space [at the moment when $R(t)=1$] and the speed of the geodesic flow. Explicitly,

$$\bar{c}\simeq V^{-1/3}\log_2 e \text{ (null geodesics)} ,$$
$$\bar{c}\simeq \alpha V^{-1/3}\log_2 e \text{ (timelike geodesics)} . \quad (4.3)$$

The first term (Hubble term) on the right of each equation (4.2) is due to the expansion of the universe, and it obviously is present, irrespective of the sign of the spatial curvature, for motion which is either fixed or stationary with respect to the galaxies. The second term, on the other hand, is a specific feature of negative spatial curvature. Its origin is the geodesic instability that is characteristic of negative curvature. Observational evidence with regard to the presence or the absence of such an additional term in the divergence of the neighboring geodesic motion of test particles that are *not* stationary with respect to the galaxies would thus help determine the sign of the spatial curvature of the universe.

If the cosmology is nonstandard in the sense that the three-dimensional hypersurfaces, in addition to being manifolds of constant negative curvature, are also compact, then further interesting conclusions follow. In this case the three-dimensional geodesic flow is known to be ergodic, mixing, and in fact, K-flow.[9,10,17]

In such a universe arbitrary initial beams of test particles distributed in space and direction (or more precisely, beams corresponding to square-integrable distribution functions on the space of line elements of three-space) will tend toward the uniform (microcanonical) distribution as time progresses. There is thus a natural mechanism leading to homogeneity and (local) isotropy in such a universe. In particular, photons from different regions of the early universe would form the isotropic background radiation after a sufficient lapse of time.

Distributions of massive test particles, on the other hand, would behave somewhat differently. Though the "mixing mechanism" is present in this case also, an initial distribution of massive particles may not be able to reach the uniform distribution. This difference in behavior between photons and massive test particles comes from the fact that the parameter $\lambda(t)$ for massive test particles, in contrast to that of photons, stays bounded even as $t\rightarrow\infty$: The physically admissible values of $\lambda(t)$ cannot be made arbitrarily large. The mixing property of three-dimensional geodesic flow would lead to a uniform distribution in general only in the asymptotic limit $\lambda\rightarrow\infty$.

Let us now turn to the internal time operator T associated with the geodesic motion of test particles in the nonstandard cosmological model under consideration. We shall not discuss here its explicit construction. (The interested reader may see forthcoming publications.[18]) But its existence is, of course, assured by the previously noted fact that in the nonstandard cosmology the spacetime geodesic motion of test particles, when projected to a fixed three-dimensional hypersurface of simultaneity, gives rise to a K-flow. The internal time operator T under discussion will thus be an operator acting on the distribution functions on the space of line elements of the three-dimensional hypersurface and will satisfy the relation

$$[T,U_\lambda]=\lambda U_\lambda \ .$$

Here U_λ denotes, of course, the unitary group induced by the projected three-dimensional flow.

The important point to note is that the time parameter of the projected flow is not the cosmic time parameter t but is a nonlinear function $\lambda(t)$ given in the preceding section, and it is $\lambda(t)$ rather than t that must occur in the defining relation of T. As a result, the (average) age of distribution functions on the space of line elements changes as the test particles move freely, keeping step not with t, but with $\lambda(t)$. The time scale defined by internal time is thus distinct from the cosmic time scale and corresponds to that of $\lambda(t)$. It is interesting that (see the expression for $d\lambda/dt$ given in the previous section) in very early epochs of the universe the internal time flows more rapidly compared with the cosmic time: $d\lambda/dt\rightarrow 1/R(t)\rightarrow+\infty$ as $t\rightarrow 0$. Similarly, as the universe ages the internal time scale gets dilated relative to the cosmic time: $d\lambda/dt\rightarrow 0$ as $t\rightarrow+\infty$. The physical meaning of these relative rates of flow of internal time and cosmic time is that the mixing rate (i.e., rate of approach to equilbrium) *with respect to change in t* approaches ∞ as one nears the singularity: whereas there is practically no mixing in a sufficiently aged universe.

The existence of a new physically significant internal time scale in the nonstandard cosmology naturally raises the question of whether the cosmological singularity could be removed to the infinitely distant past in this time scale. In view of the expressions (3.10) and (3.11) the cosmological singularity can be removed to the infinite past if and only if the integral

$$\int_{t_0}^{t}\frac{ds}{R(s)}$$

diverges as $t_0\rightarrow 0$ for any finite positive t.

Now if one assumes a linear equation of state $p=(\gamma-1)\rho$ connecting the pressure p and the energy density ρ of the cosmological fluid then it is well known that Einstein's field equations imply the following behavior for $R(t)$: $R(t)\simeq(t)^{2/3\gamma}$ near the singularity $t=0$. The integral would thus diverge only if $\gamma\leq\frac{2}{3}$, i.e., only if the pressure p becomes negative. A simple general argument given in the Appendix shows that this conclusion is independent of any specific assumption about the equation of state.

Since pressure is usually a non-negative quantity the above considerations seem to lead to the negative conclusion that the cosmological singularity cannot be removed to the infinitely distant past even in the internal time scale. Nevertheless, it seems worthwhile to stress that the concept of pressure near the singularity is far from being unambiguous. Although no generally accepted definition of gravitational pressure exists, in the presence of a strong gravitational field the concept of pressure is bound to be modified. In such a situation pressure may as well be negative without involving any physical absurdity. However, we shall not pursue this idea in this paper.

Quite apart from the question of the singularity, the existence of an internal time operator T in the nonstandard cosmology has other important implications. As mentioned in Sec. II, it allows one to associate a Lyapounov variable or H function with the geodesic motion of test particles. Moreover, the existence of T implies the intrinsic randomness of geodesic motion in the sense of Sec. II. This illustrates how irreversibility and randomness could emerge as essential features of dynamical systems embedded in a suitable cosmological model.

We can make an additional comment about the evolution of distribution functions in the early universe provided we assume the existence of an initial singularity (in the λ time.) The commutation relation $[T,L]=iI$ holds where L is defined by $U_\lambda \equiv e^{-i\lambda L}$. As a result, for any distribution function ρ we have the uncertainty relation

$$(\Delta T)_\rho (\Delta L)_\rho \geq \tfrac{1}{2} \ .$$

In the early universe we must have $(\Delta T)\rho$ very small which implies that the dispersion $(\Delta L)\rho$ in the frequencies is very large. This implies that the evolution of such ρ will be rather chaotic just after the initial singularity.

It is amusing to recall that Einstein cherished the belief that "God does not play dice." A serious challenge to this point of view comes from quantum mechanics. In our opinion, an equally important challenge comes from the recent studies of classical systems exhibiting strong forms of trajectory instability. As was said before, such systems are intrinsically random, and we find here that Einstein's own theory allows cosmological models in which the simplest and most fundamental of motions, the geodesic motion of photons and test particles, exhibits this nondeterministic feature.

APPENDIX: PARTICLE HORIZON AND NEGATIVE PRESSURE

We show here that the integral $\int_0 dt/R(t)$ would diverge only if the pressure of cosmological fluid takes negative values. As discussed in the text, this means that the price to be paid for being able to remove the cosmological singularity to the *infinitely* remote past (in the internal time scale) is to allow the pressure p to take negative values. The divergence of the above integral also implies the *absence* of *particle horizons*.[19] It may be recalled here that *absence* of particle horizon means that an observer at any given spacetime point can, in principle, receive signals emitted at sufficiently earlier times from *any* point of the three-dimensional spatial hypersurface of the universe. The argument given here thus shows the existence of particle horizons can be avoided only at the cost of allowing negative pressure.

The proof given below is quite general. In particular, it is independent of any specific assumption about the equation of state or about the behavior (such as the assumption of power-law behavior) of $R(t)$ near the singularity. We do assume that the spatial curvature is negative, however.

Einstein's field equations yield the following first-order differential equation for $R(t)$:

$$\dot{R}^2 = 1 + \frac{8\pi G}{3}\rho R^2 \ . \tag{A1}$$

In addition we have the equation of energy conservation

$$\dot{p} R^3 = \frac{d}{dt}[R^3(p+\rho)] \ . \tag{A2}$$

Using (A1) and (A2) one can easily express the pressure p entirely in terms of R and its derivatives:

$$p = \frac{1}{4\pi G R^2}\left(\tfrac{1}{2} - R\ddot{R} - \tfrac{1}{2}\dot{R}^2\right) \ . \tag{A3}$$

Positivity of pressure is thus equivalent to the condition

$$R\ddot{R} + \tfrac{1}{2}\dot{R}^2 \leq \tfrac{1}{2} \ . \tag{A4}$$

Now behavior of R near the singularity conforms with either of the following two alternatives: (i) $\dot{R}$ stays bounded in a neighborhood of $t=0$, and (ii) $\dot{R}(t) \to \infty$ as $t \to 0$. We first verify that the first case can occur only if pressure p is negative. Indeed, boundedness of $\dot{R}$ means also that ρR^2 remains bounded near the singularity [see (A1)]. As a consequence $\rho R^3 \to 0$ as $t \to 0$. On the other hand, integrating both sides of (A2) from 0 to t we obtain

$$-3\int_0^t p R^2 \dot{R} = [R^3 \rho]_0^t = R^3(t)\rho(t) > 0 \ . \tag{A5}$$

Since R^2 and $\dot{R}$ are positive quantities it follows that p must assume negative values.

Having disposed of case (i), let us now show that, in the case of alternative (ii), positivity of pressure [condition (A4)] implies the convergence of the integral $\int_0 [1/R(t)]dt$.

First, let us note that the conditon (A4) can be rewritten to read

$$\frac{d}{dt}\left[\frac{R}{\dot{R}}\right] \geq \frac{3}{2} - \frac{1}{2\dot{R}^2} \ . \tag{A6}$$

Integrating both sides from 0 to t and noting that $\lim_{t\to 0} R(t)/\dot{R}(t) = 0$, we obtain

$$\frac{R(t)}{\dot{R}(t)} \geq \frac{3t}{2} - \frac{1}{2}\int_0^t \frac{ds}{\dot{R}^2(s)} \ . \tag{A7}$$

Since $\dot{R}^2(t) \to \infty$ as $t \to 0$, there exits a t_0 such that $1/\dot{R}^2(t) < c$ with $c < 1$ for $t \leq t_0$. Thus

$$\int_0^t \frac{1}{\dot{R}^2(s)} ds < ct \ . \tag{A8}$$

The inequalities (A7) and (A8) imply the inequality

$$\frac{R}{\dot{R}} \geq \frac{(3-c)t}{2} \text{ with } c<1, \text{ or}$$
$$\frac{\dot{R}}{R} \leq \frac{2}{3-c} t^{-1} \ . \tag{A9}$$

Integrating both sides of this inequality between t_1 and t_2 (with $0 < t_1 < t_2 < t_0$) one obtains

$$\ln \frac{R(t_2)}{R(t_1)} \leq \ln \left[\frac{t_2}{t_1} \right]^{2/(3-c)} \ . \tag{A10}$$

Thus, near the singularity $1/R(t_1) \leq a t_1^{-2/(3-c)}$. As c is strictly less than 1, it follows that $1/R(t)$ is integrable near $t=0$. This completes the proof that nonnegativity of pressure p necessarily implies the convergence of $\int_0 [1/R(t)] dt$ or equivalently the existence of a particle horizon.

ACKNOWLEDGMENTS

The authors would like to thank Professor L. Shepley of The University of Texas at Austin for several helpful discussions. The Robert A. Welch Foundation and the Institute Internationale de Physique et Chimie (Solvay) provided financial support for this work.

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