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# Singularities in quantum-chromodynamic perturbation theory for $x \sim 1$

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The QCD perturbation series has been observed to converge poorly near phase-space boundaries; for example, as $x \to 1$ in structure functions. Methods have been proposed for improving the convergence by including kinematical corrections. We compare these methods, and apply them to the calculation of the photon structure function.

## I. INTRODUCTION

Many authors have noted that nonleading terms in the QCD perturbation series can become large near the boundary of phase space.[1] For example, in deep-inelastic lepton scattering the QCD series for structure functions contains nonleading terms of order $\alpha_s \ln^2(1-x)$, which are singular as $x \to 1$. The total mass squared $s$ of the hadronic system produced is just $s = Q^2(1-x)/x$; as $x \to 1$ for fixed $Q^2$, $s$ approaches threshold. The standard lowest-order term in QCD does not incorporate the proper threshold behavior. In trying to recover the threshold behavior the higher-order terms must become large as $x \to 1$. Thus it is reasonable to hope that one can improve the convergence of the perturbation expansion by including correct kinematics at each order. Analyses of these kinematical corrections have been given by Brodsky and Lepage[2] (BL), and by Amati, Bassetto, Ciafaloni, Marchesini, and Veneziano[3] (ABCMV). There is no proof in the literature that either method correctly resums the perturbation expansion, in the sense that they include *all* terms which are singular as $x \to 1$.

In this paper we test the two methods for kinematical improvement of the QCD perturbation expansion by (i) comparing them with each other, (ii) comparing them with second-order perturbative expansions, and (iii) applying them to the calculation of the photon structure function for $x \to 1$. Our principal results are as follows: (i) Although the ABCMV and BL methods agree in the leading correction terms of order $\alpha_s \ln^2(1-x)$, they differ in higher orders in $\alpha_s$. (ii) The leading correction term agrees with second-order perturbation theory. Perturbation theory, however, contains terms of the form $\alpha_s \ln(1-x)$, which are not present in the expansion of either of the kinematically corrected methods. (iii) Neither method correctly accounts for the $x \to 1$ behavior of the photon structure function, but this turns out to be an exceptionally severe test. The leading-order terms cancel, and the pointlike coupling of the photon gives rise to $x \to 1$ singularities peculiar to the photon problem.

## II. HADRONIC STRUCTURE FUNCTIONS FOR $x \sim 1$

### A. Brodsky-Lepage method

We shall first review the derivation of the Brodsky-Lepage method, going into some detail because no treatment yet exists in the published literature. Retaining BL's notation,[2] we define the quark distribution function $\tilde{q}(z,Q^2)$ to be the sum of the ladder-graph series shown in Fig. 1 integrated over $\vec{k}_\perp^{\,2} < Q^2$, including full wave-function renormalization of both quark legs. The leading-logarithmic approximation yields the following integral equation for the ladder sum in a physical gauge:

$$\tilde{q}(z,Q^2) = \tilde{q}(z,Q_0^2) + 2C_F \int_{Q_0^2}^{Q^2} \frac{d\vec{k}_\perp^{\,2}}{\vec{k}_\perp^{\,2}} \frac{\alpha_s(\vec{k}_\perp^{\,2})}{4\pi} \int_z^1 \frac{dy}{y} \frac{1+y^2}{1-y} \tilde{q}\left[\frac{z}{y}, \vec{k}_\perp^{\,2}\right], \tag{2.1}$$

where

$$C_F = \frac{n_c^{\,2}-1}{2n_c} = \tfrac{4}{3}.$$

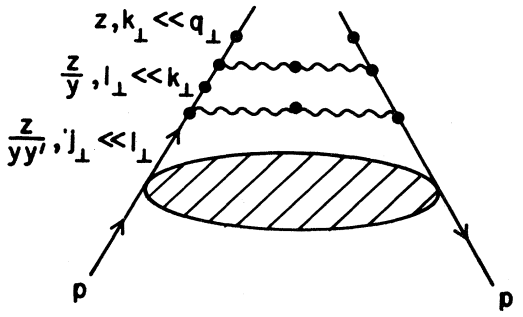


FIG. 1. Ladder-graph contributions to the quark distribution function $\tilde{q}(z,Q^2)$.

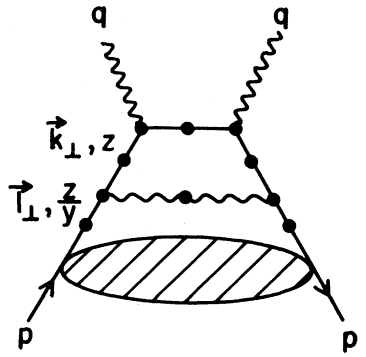


FIG. 2. Ladder-graph contributions to structure functions.

This equation can easily be derived using light-cone variables. Defining $p_\pm \equiv p_0 \pm p_z$ and choosing the target momentum in the $z$ direction,

$$p_\mu \equiv (p_+, p_-, \vec{p}_\perp) = (p_+, M^2/p_+, \vec{0}) , \tag{2.2}$$

we parametrize each constituent's four-momentum (see Fig. 2) by

$$k_\mu \equiv (k_+, k_-, \vec{k}_\perp) = \left[ z p_+, \frac{\vec{k}_\perp^{\,2} + m^2}{z p_+}, \vec{k}_\perp \right] . \tag{2.3}$$

One essential point to note about Eq. (2.1) is that the argument of $\alpha_s$ is $\vec{k}_\perp^{\,2} = -k^2(1-y)$. There is general agreement on this point; the justification can be found in Refs. 2 and 3.

A quantity which is more closely related to observable structure functions is $q(z,Q^2)$ defined by

$$q(z,Q^2) = d_q^{-1}(Q^2)\tilde{q}(z,Q^2) , \tag{2.4}$$

where $d_q(k^2)$ is the quark propagator function:

$$S(k) = d_q(k^2)\not{k}/k^2 + \text{gauge terms} . \tag{2.5}$$

The distribution function $q(z,Q^2)$ can easily be shown to satisfy an evolution equation of the Altarelli-Parisi form,[4]

$$\frac{\partial}{\partial \ln Q^2} q(z,Q^2) = \frac{\alpha_s(Q)^2}{2\pi} \int_z^1 \frac{dy}{y} P_{q/q}(y) q\left( \frac{z}{y}, Q^2 \right) , \tag{2.6a}$$

where

$$P_{q/q}(y) = C_F \left[ \frac{1+y^2}{1-y} \right]_+ \equiv C_F \left[ \frac{1+y^2}{1-y} - \delta(1-y) \int_0^1 dx \frac{1+x^2}{1-x} \right] . \tag{2.6b}$$

Of course, gluonic contributions will also need to be included when one considers the singlet distribution, but we confine our attention here to the nonsinglet case. Not only does the inclusion of the singlet introduce no new principles, but it also turns out that the gluon contribution is negligible in the $z \sim 1$ region.

To analyze the $z \sim 1$ region it is convenient to rewrite (2.6) in the form[2]

$$\frac{\partial}{\partial \ln Q^2} q(z,Q^2) = -\frac{C_F \alpha_s(Q^2)}{2\pi} q(z,Q^2) \int_0^z dy \frac{1+y^2}{1-y} + \frac{C_F \alpha_s(Q^2)}{2\pi} \int_z^1 dy \frac{1+y^2}{1-y} \left[ \frac{q(z/y,Q^2)}{y} - q(z,Q^2) \right] . \tag{2.7}$$

As $z \sim 1$ the first term dominates:

$$\frac{\partial q(z,Q^2)}{\partial \ln Q^2} \sim -P(z,Q^2) q(z,Q^2) , \tag{2.8}$$

where

$$P(z,Q^2)=\frac{C_F\alpha_s(Q^2)}{2\pi}\int_0^z dy\frac{1+y^2}{1-y}\sim\frac{a(z)}{\ln(Q^2/\Lambda^2)}\ , \tag{2.9a}$$

and where

$$a(z)=-\frac{4C_F}{\beta_0}[\ln(1-z)+\tfrac{3}{4}]\ . \tag{2.10}$$

Thus one finds for $z\sim 1$

$$q(z,Q^2)\sim q(z,{Q_0}^2)\exp\left[-a(z)\ln\frac{\ln(Q^2/\Lambda^2)}{\ln({Q_0}^2/\Lambda^2)}\right]\ . \tag{2.11}$$

Now to leading order in $\ln Q^2$ the nonsinglet structure function $F_2$ is essentially equal to the quark distribution function $q(x,Q^2)$:

$$F_2(x,Q^2)\approx x\sum e_i^2 q_i(x,Q^2)\ . \tag{2.12}$$

If we wish to capture nonleading terms which become singular as $x\to 1$ by incorporating correct threshold behavior, then the structure-function ladder sum, shown in Fig. 2, differs from the quark distribution function $q(x,Q^2)$. As noted by Brodsky and Lepage,[2] the effect of threshold kinematics is significant only at the top rung of the ladder in Fig. 2. The result is to change the upper limit of the $\vec{k}_\perp^{\,2}$ integration in (2.1) from $Q^2$ to $Q^2(1-y)$. Equivalently, an extra piece must be added to the right-hand side of (2.12) so that it reads

$$F_2(x,Q^2)=x\sum e_i^2[q_i(x,Q^2)+\delta q_i(x,Q^2)]\ , \tag{2.13}$$

where

$$\delta q_i(x,Q^2)=-2C_F\int_0^1 dy\frac{1+y^2}{1-y}\int_{(1-y)Q^2}^{Q^2}\frac{d\vec{k}_\perp^{\,2}}{\vec{k}_\perp^{\,2}}\frac{\alpha_s(\vec{k}_\perp^{\,2})}{4\pi}\left\{q_i\left[\frac{x}{y},\vec{k}_\perp^{\,2}\right]\frac{\theta(y>x)}{y}-q_i(x,\vec{k}_\perp^{\,2})\right\}\ . \tag{2.14}$$

For $x\sim 1$ one finds by using the same technique as in (2.7) that

$$\delta q(x,Q^2)\cong 2C_F\int_0^x dy\frac{1+y^2}{1-y}\int_{(1-y)Q^2}^{Q^2}\frac{d\vec{k}_\perp^{\,2}}{\vec{k}_\perp^{\,2}}\frac{\alpha_s(\vec{k}_\perp^{\,2})}{4\pi}q(x,\vec{k}_\perp^{\,2})\ . \tag{2.15}$$

To evaluate this correction we use (2.11) and perform the $\vec{k}_\perp^{\,2}$ integration to obtain

$$\delta q(x,Q^2)=-\frac{2C_F q(x,Q^2)}{\beta_0 a(x)}\int_0^x dy\frac{1+y^2}{1-y}\left\{1-\exp\left[-a(x)\ln\frac{\ln Q^2(1-y)/\Lambda^2}{\ln Q^2/\Lambda^2}\right]\right\}\ . \tag{2.16}$$

Expanding the exponential one finds the result

$$\frac{\delta q(x,Q^2)}{q(x,Q^2)}\sim\frac{2C_F}{\beta_0}\left[\frac{\ln^2(1-x)}{\ln(Q^2/\Lambda^2)}+\frac{4}{3}\frac{C_F}{\beta_0}\frac{\ln^4(1-x)}{\ln^2(Q^2/\Lambda^2)}+\cdots\right]\ . \tag{2.17}$$

### B. Amati-Bassetto-Ciafaloni-Marchesini-Veneziano method

The Brodsky-Lepage method for taking $x\sim 1$ threshold singularities into account involves solving the ordinary Altarelli-Parisi equation (2.6) for the quark distribution function $q(x,Q^2)$, then adding an extra contribution $\delta q$ which corrects the kinematics at the top rung in the ladder sum. By contrast, the method of Amati, Bassetto, Ciafaloni, Marchesini, and Veneziano (ABCMV) involves solving a modified evolution equation for the (nonsinglet) structure function itself,[3]

$$\frac{\partial}{\partial\ln Q^2}F(x,Q^2)=\frac{1}{2\pi}\int_x^1\frac{dy}{y}[\alpha_s(Q^2(1-y))P_{q/q}(y)]_+F\left[\frac{x}{y},Q^2\right]\ . \tag{2.18}$$

The original arguments for this equation were only heuristic, with *a posteriori* justification coming from consideration of certain sums of diagrams.[5] We shall not repeat these arguments here, turning our attention instead to a comparison of the BL and the ABCMV methods. We find that they are not equivalent, agreeing with each other only on the leading term in (2.17).

Again using the method of Eq. (2.7) we find that the leading behavior of (2.18) for $x \sim 1$ is given by

$$\frac{\partial}{\partial \ln Q^2} F(x,Q^2) = -\frac{C_F}{2\pi} F(x,Q^2) \int_0^x dy \frac{1+y^2}{1-y} \alpha_s(Q^2(1-y)) \,. \tag{2.19}$$

Using the lowest-order form for $\alpha_s$ and expanding (for $x$ not too close to unity)

$$\begin{aligned} \alpha_s(Q^2(1-y)) &= \frac{4\pi}{\beta_0 \ln[Q^2(1-y)/\Lambda^2]} \\ &= \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)} \left[ 1 - \frac{\ln(1-y)}{\ln(Q^2/\Lambda^2)} + \frac{\ln^2(1-y)}{\ln^2(Q^2/\Lambda^2)} + \cdots \right] . \end{aligned} \tag{2.20}$$

Substituting this expansion in (2.19) and doing a bit of algebra one finds

$$\frac{\delta F(x,Q^2)}{F_0(x,Q^2)} \sim \frac{2C_F}{\beta_0} \left[ \frac{\ln^2(1-x)}{\ln(Q^2/\Lambda^2)} + \frac{C_F}{\beta_0} \frac{\ln^4(1-x)}{\ln^2(Q^2/\Lambda^2)} + \cdots \right] , \tag{2.21}$$

where $F_0(x,Q^2)$ is the uncorrected leading-order form

$$F_0(x,Q^2) = F(x,{Q_0}^2) \exp\left[ -a(x) \ln \frac{\ln(Q^2/\Lambda^2)}{\ln({Q_0}^2/\Lambda^2)} \right] . \tag{2.22}$$

Comparing (2.17) and (2.21) one sees that they agree to leading order, but disagree at the next order.

### C. Higher-order corrections to evolution equation

Further insight into the difference between the BL and ABCMV methods for dealing with the $x \to 1$ singularities can be obtained by examining the higher-order corrections to the Altarelli-Parisi evolution equations. Recently Herrod, Wada, and Webber have discussed two methods for incorporating higher-order corrections which closely parallel the two methods for $x \to 1$ corrections.[6] The first method, which they call the two-step method, consists of first modifying the equation for the quark distribution $q(x,Q^2)$ as follows:

$$\frac{\partial}{\partial \ln Q^2} q(x,Q^2) = \int_x^1 \frac{dy}{y} \left\{ \frac{\alpha_s(Q^2)}{2\pi} P_{qq}^{(0)}(y) + \left[ \frac{\alpha_s(Q^2)}{2\pi} \right]^2 P_{qq}^{(1)}(y) \right\} q\left[ \frac{x}{y}, Q^2 \right] . \tag{2.23}$$

Again, we are confining our attention to the nonsinglet case. The second step is to convolute the solution with the inverse Mellin transforms of the coefficient functions to obtain a structure function:

$$F(x,Q^2) \propto q(x,Q^2) + \frac{\alpha_s(Q^2)}{4\pi} \int_x^1 \frac{dy}{y} C_q^{(1)}(y) q\left[ \frac{x}{y}, Q^2 \right] . \tag{2.24}$$

Note the similarity to the Brodsky-Lepage method for incorporating $x \sim 1$ singularities, as given by Eqs. (2.13) to (2.15).

Herrod, Wada, and Webber also introduce a "one-step" method. They point out that $F(x,Q^2)$ as given by the two-step satisfies a differential equation[6]

$$\frac{\partial F(x,Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dy}{y} \left\{ \frac{\alpha_s(Q^2)}{2\pi} P_{qq}^{(0)}(y) + \left[ \frac{\alpha_s(Q^2)}{2\pi} \right]^2 \left[ P_{qq}^{(1)}(y) - \frac{\beta_0}{4} C_q^{(1)}(y) \right] \right\} F\left[ \frac{x}{y}, Q^2 \right] + O({\alpha_s}^3) \tag{2.25}$$

up to corrections of order ${\alpha_s}^3$. We shall show that this equation reduces to the ABCMV equation in the ap-

propriate limit. To understand what is the appropriate limit let us recall what we are trying to achieve in modifying the Altarelli-Parisi equations to take account of $x\sim 1$ singularities: We are trying to improve the convergence of the perturbation expansion by writing down an equation that is lowest order in $\alpha_s(Q^2)$ except for including high-order terms which are more singular as $x\to 1$ than the corresponding lowest-order terms. For example, $P_{qq}^{(0)}$ behaves like $1/(1-x)_+$. Terms in $P_{qq}^{(1)}$ with this behavior we discard, because they are higher order in $\alpha_s(Q^2)$ without being any more singular as $x\sim 1$. Since there are no terms in $P_{qq}^{(1)}$ which are any more singular, we can discard it entirely. On the other hand, $C_q^{(1)}$ does contain singular terms:

$$C_q^{(1)}(x)\sim 2C_F\left\{\left[(1+x^2)\frac{\ln(1-x)}{1-x}\right]_+-\frac{3}{4}\left[\frac{1+x^2}{1-x}\right]_+\right\}. \tag{2.26}$$

If we now look at (2.23) we see that the evolution equation for $q(x,Q^2)$ is unmodified, whereas the relation between the structure function and $q(x,Q^2)$ contains an extra term. The extra term in (2.24) yields

$$\delta q=\frac{\alpha_s(Q^2)}{4\pi}\int_x^1\frac{dy}{y}C_q^{(1)}(y)q\left[\frac{x}{y},Q^2\right],$$

$$\frac{\delta q}{q}\sim-\frac{\alpha_s(Q^2)}{4\pi}C_F\int_0^x dy\left[\frac{4\ln(1-y)}{1-y}-\frac{3}{1-y}\right]=\frac{2C_F}{\beta_0}\left[\frac{\ln^2(1-x)}{\ln(Q^2/\Lambda^2)}-\frac{3}{2}\frac{\ln(1-x)}{\ln(Q^2/\Lambda^2)}\right]. \tag{2.27}$$

The most singular term, proportional to $\ln^2(1-x)$, agrees with the expansion of the BL or ABCMV kinematical corrections. Note, however, the presence of a term proportional to $\ln(1-x)$, which is not obtained by either kinematical correction method.

To derive the ABCMV form from the one-step second-order equation (2.25), we again discard the higher-order terms which are no more singular as $x\to 1$. The resulting equation is

$$\frac{\partial F(x,Q^2)}{\partial\ln Q^2}=\int_x^1\frac{dy}{y}\frac{\alpha_s(Q^2)}{2\pi}\left[P_{qq}^{(0)}(y)-\frac{\alpha_s(Q^2)}{2\pi}\frac{\beta_0}{4}C_q^{(1)}(y)\right]F\left[\frac{x}{y},Q^2\right], \tag{2.28a}$$

and using (2.26),

$$\frac{\partial F(x,Q^2)}{\partial\ln Q^2}\approx C_F\int_x^1\frac{dy}{y}\frac{\alpha_s(Q^2)}{2\pi}\left\{\frac{1+y^2}{1-y}\left[1-\frac{\ln(1-y)}{\ln(Q^2/\Lambda^2)}\right]\right\}_+F\left[\frac{x}{y},Q^2\right] \tag{2.28b}$$

$$\approx\frac{C_F}{2\pi}\int_x^1\frac{dy}{y}\left[\alpha_s(Q^2(1-y))\frac{1+y^2}{1-y}\right]_+F\left[\frac{x}{y},Q^2\right], \tag{2.28c}$$

which is just the ABCMV modified form of the Altarelli-Parisi equation.

To summarize, we have in this section compared the Brodsky-Lepage and the ABCMV methods for taking account of kinematic singularities encountered for $x\sim 1$. We have reviewed the Brodsky-Lepage derivation, which shows that their method sums ladder graphs with correct kinematics. The ABCMV method is equivalent only for the leading corrections, but disagrees beyond leading order. On the other hand, the ABCMV equation is simpler, and for most purposes the leading-order correction may be adequate. Moreover, there is no proof that either method accomplishes the goal of including all nonleading terms which are singular for $x\to 1$. In fact, neither method succeeds in obtaining the corrections of form $\alpha_s\ln(1-x)$ which are present in the perturbation expansion.

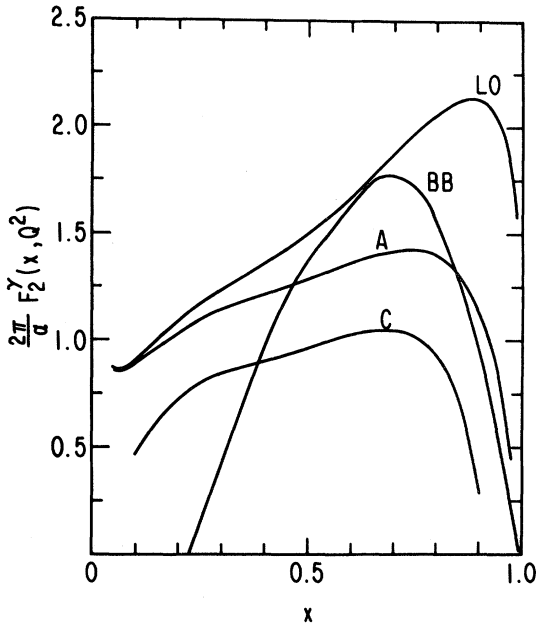


FIG. 3. Calculation of photon structure function $F_2$: comparison of leading order (LO); leading order plus second order (BB); Chase's formula (C), Eq. (3.8); approximate formula (A), Eq. (3.11); all at $Q^2=3$ GeV$^2$ and $\Lambda=0.5$ GeV.

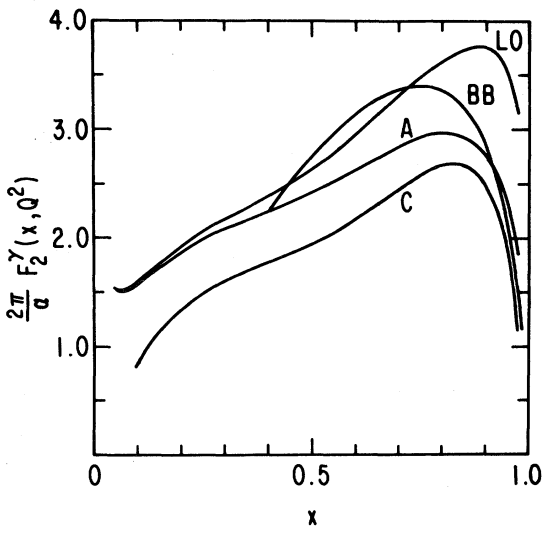


FIG. 4. Same as Fig. 3, but at $Q^2=20$ GeV$^2$.

## III. PHOTON STRUCTURE FUNCTION FOR $x\sim 1$

The photon structure functions, which are measurable in deep-inelastic lepton-lepton scattering, have been discussed as an especially interesting testing ground of QCD.[7] They contain anomalous pieces, arising from the pointlike fragmentation of the target photon into quarks, which are completely calculable in QCD. The additional hadronlike pieces, arising from vector-meson-dominated fragmentation of the target photon, are not completely calculable but are negligibly small except at small $x$.

The QCD calculation of the anomalous pieces of the photon's structure functions has been carried to next-to-leading (hereafter, second) order in $\alpha_s$ by Bardeen and Buras.[8] They find reasonably small corrections at moderate values of $x$, but find quite large corrections at large and small $x$ (compare curves marked LO and BB in Figs. 3 and 4).[9] Complete understanding of the corrections at small $x$ is difficult, in that it presumably involves disentangling the pointlike piece from the large vector-meson-dominance background. At any rate, this problem is distinct from our concern in this paper with the $x\sim 1$ threshold region.

It would be quite desirable to have a method of summing those higher-order terms which become singular as $x\to 1$. As we saw in the previous section, the Brodsky-Lepage and the ABCMV methods are partially successful in summing such terms for hadron structure functions. As we shall see, the extension of these methods to the photon structure functions involves additional difficulties.

### A. Leading-order calculation corrected for threshold kinematics

The transverse photon structure functions are, in leading order in $\alpha_s$, related to the quark distribution functions of the photon, $q_i(x,Q^2)=G_{q_i/\gamma}(x,Q^2)$, as follows:

$$F^\gamma(x,Q^2)=6\sum_{i=1}^{f} e_i^2 q_i(x,Q^2)\ , \tag{3.1}$$

where $F^\gamma$ is $F_2^\gamma/x$ (or in leading order $2F_1^\gamma$). Now $q_i(x,Q^2)$ satisfies the usual evolution equation (2.6), except that not only quark and gluon but also photon intermediate states must be included[10]:

$$\frac{\partial}{\partial \ln Q^2} G_{q/\gamma}(x,Q^2)=\frac{1}{2\pi}\alpha_{\rm EM}\int_x^1 \frac{dy}{y} P_{q\gamma}\left[\frac{x}{y}\right] G_{\gamma/\gamma}(y,Q^2) + \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y}\left\{P_{qq}\left[\frac{x}{y}\right] G_{q/\gamma}(y,Q^2)+P_{qG}\left[\frac{x}{y}\right] G_{G/\gamma}(y,Q^2)\right\}. \tag{3.2}$$

But to lowest order in $\alpha_{\rm EM}$ the distribution of photons in a photon target is just

$$G_{\gamma/\gamma}(y,Q^2)=\delta(y-1)+O(\alpha_{\rm EM}) \tag{3.3}$$

which reduces the first term on the right-hand side of (3.2) to simply

$$\frac{1}{2\pi}\alpha_{\rm EM}P_{q\gamma}(x) ,$$

where, in leading order,

$$P_{q_i\gamma}(x)\approx P^{(0)}_{q_i\gamma}(x)=e_i{}^2[x^2+(1-x)^2] . \tag{3.4}$$

In examining the region $x\sim 1$ we can neglect the gluon contribution, as can easily be seen from the nonsingular character of

$$P^{(0)}_{qG}(x)=\tfrac{1}{2}[x^2+(1-x)^2]$$

as contrasted to the singular $P_{qq}$ given by (2.6b). To incorporate the $x\sim 1$ kinematics we could use either the BL or ABCMV methods, since we shall concentrate on order-$\alpha_s$ corrections, where the two methods are equivalent. Choosing the ABCMV method we have finally for a lowest-order kinematically corrected equation

$$\frac{\partial}{\partial \ln Q^2}q_i(x,Q^2)=\frac{1}{2\pi}\alpha_{\rm EM}P_{q\gamma}(x)+\frac{1}{2\pi}\int_x^1\frac{dy}{y}[\alpha_s(Q^2(1-y))P_{qq}(y)]_+ q_i\left[\frac{x}{y},Q^2\right] . \tag{3.5}$$

Again using the method of Eq. (2.7) one finds for the leading $x\to 1$ behavior

$$\frac{\partial}{\partial \ln Q^2}q(x,Q^2)\approx\frac{1}{2\pi}\alpha_{\rm EM}P_{q\gamma}(x)-\tilde{P}(x,Q^2)q(x,Q^2) , \tag{3.6}$$

where, in analogy to (2.9), we have defined

$$\tilde{P}(x,Q^2)=\frac{C_F}{2\pi}\int_0^x dy\frac{1+y^2}{1-y}\alpha_s[Q^2(1-y)] . \tag{3.7}$$

One can solve (3.6) to obtain

$$F^\gamma(x,Q^2)=h_B(x)\int_{Q_0{}^2}^{Q^2}\frac{dt}{t}\exp\left[\int_{Q^2}^t\frac{dk^2}{k^2}\tilde{P}(x,k^2)\right] , \tag{3.8a}$$

where we have defined

$$h_B(x)=\frac{\alpha_{\rm EM}}{2\pi}\sum_{i=1}^f e_i{}^4\left[x^2+(1-x)^2\right] , \tag{3.8b}$$

and where, to retain the correct threshold,

$$Q_0{}^2=\frac{\langle m_q{}^2\rangle}{1-x} . \tag{3.8c}$$

Equation (3.8) was first obtained by Chase, by directly summing graphs.[11] It is plotted in Figs. 3 and 4 in comparison to the Bardeen-Buras results. Although it is numerically rather similar, we shall find that it does not in fact have the same perturbation expansion in powers of $\alpha_s(Q^2)$. The perturbation expansion of Chase's formula can be found by expanding

$$\alpha_s[Q^2(1-y)]=\frac{4\pi}{\beta_0[\ln(Q^2/\Lambda^2)+\ln(1-y)]}\approx\alpha_s(Q^2)\left[1-\frac{\ln(1-y)}{\ln(Q^2/\Lambda^2)}\right] . \tag{3.9}$$

Using this expansion and performing the integrations in (3.8) one finds

$$F^\gamma(x,Q^2)=h(x)\ln\frac{Q^2}{\Lambda^2}\times[1+\tfrac{1}{2}\epsilon+O(\epsilon^2)+O(1-x)] , \tag{3.10a}$$

where

$$\epsilon=\frac{\ln(1-x)}{\ln(Q^2/\Lambda^2)} , \tag{3.10b}$$

and where

$$h(x)=\frac{h_B(x)}{1+d(x)}\ , \tag{3.10c}$$

$$d(x)=-\frac{C_F}{\beta_0}[4\ln(1-x)+3]\ . \tag{3.10d}$$

Note that $F^\gamma=h(x)\ln(Q^2/\Lambda^2)$ is the large-$x$ form of the leading-order result. The factor in brackets in (3.10a) reflects the kinematic correction. It is interesting to note that (3.10) can be rewritten as

$$F^\gamma(x,Q^2)\approx h(x)\ln\left[\frac{Q^2}{\Lambda^2}(1-x)^{1/2}\right] \tag{3.11}$$

This approximation to Chase's formula, valid for $\epsilon<1$, is also plotted in Figs. 3 and 4. Several authors have conjectured that the kinematic corrections could be included by modifying the argument of the logarithm,[12] but they made the reasonable guess that the corrected argument would be $s=Q^2(1-x)/x$. It was not anticipated that the factor $1-x$ would appear instead to the power $\frac{1}{2}$.

### B. Higher-order corrections to the photon form factor for $x\sim 1$

Bardeen and Buras give their analytic result in terms of moments:

$$\int_0^1 dx\, x^{n-2}F_2^\gamma(x,Q^2)=F_{2,n}^\gamma(Q^2)=a_n\ln\frac{Q^2}{\Lambda^2}+b_n+\tilde{a}_n\ln\ln\frac{Q^2}{\Lambda^2}\ . \tag{3.12}$$

Since large $n$ probes large $x$, we are interested in the leading behavior of the moments for large $n$. We find

$$\frac{b_n}{a_n}\approx-\left[\frac{1}{2}-\frac{8}{3\beta_0}\right]\ln n+O(1)\ . \tag{3.13}$$

There are terms going like $\ln^2 n$, but they precisely cancel. On the other hand, our simple formula (3.10), when translated into moments, also has no $\ln^2 n$ term. In fact, the $\ln n$ term is just

$$b_n/a_n\sim-\tfrac{1}{2}\ln n\ . \tag{3.14}$$

Thus we have reproduced only a portion of the $\ln n$ term by analyzing kinematical corrections, although the $\frac{1}{2}$ happens to be numerically larger than the $2C_F/\beta_0$. The $2C_F/\beta_0$ term appears to be an essentially (non-kinematical) higher-order effect.

In order to see the origin of the various terms in the Bardeen-Buras result it is helpful to analyze second-order corrections to the evolution equations for $F^\gamma(x,Q^2)$. Instead of the one-step formulation of Herrod, Wada, and Webber, we follow a "two-step" formulation analogous to (2.3) and (2.24):

$$F^\gamma(x,Q^2)=6\sum_{i=1}^{f}e_i^2\int_x^1\frac{dy}{y}C_i(y)G_{q_i/\gamma}\left[\frac{x}{y},Q^2\right]+\text{gluons}+\int_x^1\frac{dy}{y}C_\gamma(y)G_{\gamma/\gamma}\left[\frac{x}{y},Q^2\right]\ . \tag{3.15}$$

Using (3.3) one reduces the second term to just $C_\gamma(x)$, which can be obtained from the box diagram

$$C_\gamma(x)=3\frac{\alpha_{\rm EM}}{\pi}\sum_{i=1}^{f}e_i^4\left\{[x^2+(1-x)^2]\ln\frac{1-x}{x}-1+8x(1-x)\right\}\ . \tag{3.16}$$

Moreover,

$$C_i(y)=C_i^{(0)}(y)+\frac{\alpha_s}{4\pi}C_i^{(1)}(y)\ , \tag{3.17a}$$

where

$$C_i^{(0)}(y)=\delta(y-1)\ , \tag{3.17b}$$

and where $C_i^{(1)}(y)$ is given by (2.26). For large $x$, where the gluon contribution is negligible, (3.15) becomes

$$F^\gamma(x,Q^2)=C_\gamma(x)+6\sum_{i=1}^{f}e_i^2q_i(x,Q^2)+6\frac{\alpha_s(Q^2)}{4\pi}\sum_{i=1}^{f}e_i^2\int_x^1\frac{dy}{y}C_q^{(1)}(y)q_i\left[\frac{x}{y},Q^2\right]\ . \tag{3.18}$$

Now $q_i(x,Q^2)$ satisfies the evolution equation

$$\frac{\partial}{\partial \ln Q^2} q_i(x,Q^2)=\frac{1}{2\pi}\alpha_{\rm EM}P_{q\gamma}(x)+\frac{1}{2\pi}\alpha_s(Q^2)\int_x^1 \frac{dy}{y}P_{qq}(y)q_i\left[\frac{x}{y},Q^2\right], \tag{3.19}$$

and for our goal of finding singular terms as $x\to 1$ it suffices to take $P_{qq}=P_{qq}^{(0)}$, as discussed in Sec. II C. On the other hand,

$$P_{q_i\gamma}=P_{q_i\gamma}^{(0)}+\frac{\alpha_s}{2\pi}P_{q_i\gamma}^{(1)}, \tag{3.20}$$

where $P_{q_i\gamma}^{(0)}$ is given by (3.4), and where $P_{q_i\gamma}^{(1)}$ can be found from expressions given in the literature for $P_{qG}^{(1)}$ by looking for the part proportional to $C_F T_R$, removing the factor $T_R$, and multiplying by an appropriate charge factor. In the notation of Herrod and Wada,[6]

$$P_{q_i\gamma}^{(1)}=\tfrac{1}{2}e_i^2 C_F F_{qG}^2 \underset{x\to 1}{\sim} e_i^2 C_F[x^2+(1-x)^2]\ln^2(1-x) \tag{3.21}$$

and, from (2.26),

$$C_q^{(1)}(x)\sim C_F\left\{4\left[\frac{\ln(1-x)}{1-x}\right]_+-\frac{3}{(1-x)_+}\right\}. \tag{3.22}$$

Now one can solve (3.19) perturbatively by expanding

$$q(x,Q^2)=\frac{1}{\alpha_s(Q^2)}q^{(0)}(x)+q^{(1)}(x)+\cdots. \tag{3.23}$$

Substituting the expansion in (3.19) and collecting terms one finds the usual lowest-order solution

$$q^{(0)}(x)=\frac{(2\alpha_{\rm EM}/\beta_0)P_{q\gamma}^{(0)}(x)}{1+d(x)}, \tag{3.24}$$

where $d(x)$ is given in (3.10d). The higher-order equation yields

$$q^{(1)}(x)=\frac{(\alpha_{\rm EM}/2\pi)P_{q\gamma}^{(1)}(x)}{\frac{\beta_0}{2}d(x)}. \tag{3.25}$$

Expanding the photon form factor

$$F^\gamma(x,Q^2)=F_\gamma^{(0)}(x)\ln\frac{Q^2}{\Lambda^2}+F_\gamma^{(1)}(x)+\cdots \tag{3.26}$$

one finds

$$F_\gamma^{(0)}(x)=\frac{6(\alpha_{\rm EM}/2\pi)f\langle e^4\rangle}{1+d(x)} \tag{3.27a}$$

$$\sim -\frac{3}{2}\frac{\alpha_{\rm EM}}{2\pi}f\langle e^4\rangle\frac{\beta_0}{C_F\ln(1-x)}. \tag{3.27b}$$

For the next-to-leading order one finds three terms,

$$F_\gamma^{(1)}\underset{x\to 1}{\sim} C_\gamma+6\sum_{i=1}^f e_i^2 q_i^{(1)}+\frac{6}{4\pi}\sum_{i=1}^f e_i^2 C_q^{(1)}*q_i^{(0)}$$

$$=C_\gamma+12\frac{\alpha_{\rm EM}}{2\pi}\frac{\sum e_i^2 P_{q\gamma}^{(1)}}{\beta_0 d}+6\frac{\alpha_{\rm EM}}{2\pi\beta_0}\frac{\sum e_i^2 C_q^{(1)}*P_{q\gamma}^{(0)}}{1+d}, \tag{3.28}$$

where the asterisk indicates a convolution. Note the one-to-one correspondence between (3.28) and the large-$n$ form of the corresponding result of Bardeen and Buras.[8] In the notation of Duke and Owens[9] it reads

$$b_n \underset{n\to\infty}{\sim} \langle e^4\rangle \left[ 6B_G^n + \frac{K_{\rm NS}^{(1)}}{2\beta_0 d_{\rm NS}} + \frac{B_{\rm NS}K_{\rm NS}^{(0)}}{2\beta_0(1+d_{\rm NS})} \right] \tag{3.29}$$

(NS indicates nonsinglet). Using Eqs. (3.21) to (3.25) to evaluate (3.28) one finds

$$F_\gamma^{(1)} \sim 3\frac{\alpha_{\rm EM}}{2\pi} f\langle e^4\rangle \left[ 1 - \frac{\beta_0}{4C_f} \right] \tag{3.30}$$

or, using (3.27),

$$\frac{F_\gamma^{(1)}}{F_\gamma^{(0)}} \sim -\left[ \frac{2C_F}{\beta_0} - \tfrac{1}{2} \right] \ln(1-x)\,, \tag{3.31}$$

in agreement with (3.13).

All three terms of (3.28) contribute to $F_\gamma^{(1)}$. Only the first and third could be taken into account by a kinematic modification of the leading-order calculation. The first term $C_\gamma$ is just the nonleading part of the box diagram, and the third term is the form of the Brodsky-Lepage correction [see the discussion following Eq. (2.26)]. On the other hand, the second term is essentially higher order. It is proportional to $P_{q\gamma}^{(1)}$, which involves diagrams of the form shown in Fig. 5. The presence of such diagrams can be traced back to the inhomogeneous terms in (3.19), which in turn resulted from the $\delta$-function singularity of $G_{\gamma/\gamma}$ in (3.3). In other words, the photon structure-function singularity as $x\to 1$ is essentially different from that of a hadron, because the photon is itself one of the fundamental fields and because of the photon's pointlike coupling to the quarks.

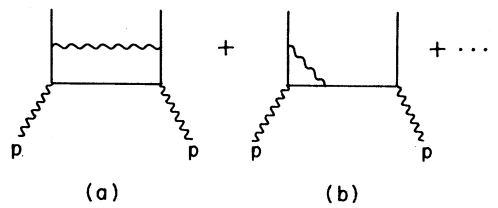


FIG. 5. Diagrams contributing to $P_{q\gamma}^{(1)}$, except that leading-ln$Q^2$ part of (a) is to be excluded because its contribution is included by solving the evolution equation.

A second fact arguing the basic impossibility of a purely kinematical explanation of the $x\to 1$ singularity in the photon structure is that the leading terms of form $\ln^2(1-x)$ in $F_\gamma^{(1)}/F_\gamma^{(0)}$ precisely cancel. The leading nonvanishing terms are then of order $\ln(1-x)$, and we saw in Sec. II that neither the BL nor the ABCMV methods account correctly for such terms, even for hadronic targets.

## IV. CONCLUSIONS

The Brodsky-Lepage and Amati-Bassetto-Ciafaloni-Marchesini-Veneziano methods resum QCD perturbation theory to take into account singular behavior as $x\to 1$. Although both methods correctly account for terms of order $\alpha_s(Q^2)\ln^2(1-x)$ in hadron structure functions, the two methods differ at order $\alpha_s^2$. Moreover, neither method accounts for terms of order $\alpha_s(Q^2)\ln(1-x)$. Nevertheless, the methods do represent some improvement over unmodified QCD, and recent work by Barnett, Schlatter, and Trentadue[13] indicates their value in phenomenology.

The situation is much less encouraging for the photon structure. The behavior of the photon structure functions as $x\to 1$ is essentially more complicated than that of the hadronic structure functions. The leading $\alpha_s(Q^2)\ln^2(1-x)$ terms cancel, leaving $\alpha_s(Q^2)\ln(1-x)$ terms responsible for the $x\to 1$ behavior. Since this order is not correctly predicted by either BL or ABCMV methods in the hadronic case, there is little hope for success for the photon case.

A further difficulty in the photon structure problem is the presence of essentially higher-order diagrams which contribute to the $x\to 1$ singularity. For these reasons it is not surprising that attempts at kinematical analysis of $x\to 1$ singularities have failed the test of agreement with perturbation theory in next-to-leading order in $\alpha_s$ for the photon structure problem. The formulas which have been derived can perhaps be of some phenomenological value, but they do not represent a solution in principle to the problem of resumming perturbation theory to improve its convergence for $x\to 1$. This does not mean that QCD makes no prediction in this region, but it does mean that the convergence of the series is nonuniform. One must go to higher values of $Q^2$ as one takes $x$ closer to unity if one wishes to retain the same level of accuracy of prediction.

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[1]The following is an incomplete sampling of references; the $x \to 1$ problem has been noted and remarked upon by so many workers that a complete citing is impossible. G. Altarelli, R. K. Ellis, and G. Martinelli, Nucl. Phys. B143, 521 (1978); B146, 544 (1978); B157, 461 (1979); J. Kubar-André and F. E. Paige, Phys. Rev. D 19, 221 (1979).

[2]S. J. Brodsky and G. P. Lepage, in *Quantum Chromodynamics,* proceedings of the SLAC Summer Institute on Particle Physics, 1979, edited by Anne Mosher (SLAC, Stanford, 1980), p. 133.

[3]D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini, and G. Veneziano, Nucl. Phys. B173, 429 (1980).

[4]G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977).

[5]Further analyses of the ABCMV method have been carried out which show its correctness for certain sums of diagrams. See M. Ciafaloni, Phys. Lett. 95B, 113 (1980); M. Ciafaloni and G. Curci, *ibid.* 102B, 352 (1981).

[6]R. T. Herrod, S. Wada, and B. R. Webber, Cambridge University Report No. HEP 81/7 (unpublished); R. T. Herrod and S. Wada, Phys. Lett. 96B, 195 (1980).

[7]See, for example, the recent review by W. R. Frazer in *Proceedings of the 4th International Colloquium on Photon-Photon Interactions, Paris, 1981,* edited by Georges W. London (World Scientific, Singapore, 1981).

[8]W. A. Bardeen and A. J. Buras, Phys. Rev. D 20, 166 (1979).

[9]D. W. Duke and J. F. Owens, Phys. Rev. D 22, 2280 (1980).

[10]R. J. DeWitt, L. M. Jones, J. D. Sullivan, D. E. Willen, and H. W. Wyld, Phys. Rev. D 19, 2046 (1979); 20, 1751 (E) (1979).

[11]M. K. Chase, Nucl. Phys. B (to be published).

[12]C. Peterson, T. F. Walsh, and P. Zerwas, Nucl. Phys. B174, 424 (1980); T. DeGrand (private communication).

[13]R. M. Barnett, D. Schlatter, and L. Trentadue, Phys. Rev. Lett. 46, 1659 (1981); G. Parisi, Phys. Lett. 90B, 295 (1980); G. Curci and M. Greco, *ibid.* 92B, 175 (1980).