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Baryon- and lepton-number violation by electroweak instantons

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We make a quantitative study of instanton-induced baryon- and lepton-number-violating processes in an $SU(2)\times U(1)$ electroweak gauge theory at zero and finite temperatures (in the "dilute-instanton-gas" approximation). As an example we consider a simplified model involving only the proton, neutron, electron, and electron neutrino. At zero temperature the total cross sections for $p+n\to\bar{e}+\bar{\nu}$ and eleven other similar reactions are of order $s\times10^{-195}$ cm^2, where s is the total center-of-momentum energy squared in GeV^2. The neutron decays via $n\to\bar{p}+\bar{e}+\bar{\nu}$ with a lifetime of the order 10^{146} years. The cross sections and neutron decay width decrease with temperature because color-electric-charge screening reduces the self-dual-instanton density at finite temperature. At high temperature the cross sections (for a given s) and neutron decay width fall off as $T^{-47/3}$ in this simplified model. It is suggested that correctly treating the instanton gas as very dense (as discussed by Berg, Luscher, and Stehr) and including finite-energy tunneling solutions could increase the predicted reaction rates.

I. INTRODUCTION

Lately various authors have studied baryon- and lepton-number violation generated by superheavy gauge bosons in grand unified theories.[1] There is one baryon- and lepton-number-violating process which is a necessary consequence of relativisitic quantum field theory itself. This is the Adler-Bell-Jackiw (ABJ) anomaly[2] in the chiral limit of negligible fermion masses. Remarkably, the ABJ anomaly, the Belavin *et al.* instanton,[3] and electroweak baryon- and lepton-number violation are related phenomena, as first elucidated by 't Hooft.[4] He recognized that an instanton could be associated with a violation of axial-vector charge equal to that generated by the ABJ anomaly. In the limit of massless fermions in left-handed SU(2) weak-isospin doublets and right-handed singlets, this is equivalent to a violation of fermion number (e.g., baryon and lepton number).

Previously, Christ[5] has used S-matrix theory to study the anomalous creation of weakly interacting fermions by a time-dependent background Yang-Mills field at high energies. Using 't Hooft's effective-Lagrangian method, we undertake here a quantitative calculation of the cross sections for baryon- and lepton-number violation in an SU(2) $\times$U(1) electroweak gauge theory. These instanton-generated processes will be considered at both zero and finite temperature. To do this we will make use of the finite-temperature instanton-density calculation of Gross, Pisarski, and Yaffe.[6]

The outline of this paper is as follows. In Sec. II we briefly review the relation between the ABJ anomaly and instantons. In Sec. III, we review 't Hooft's construction of the effective $2N_f$-fermion interaction generated by unit-winding-number instantons in an SU(2) weak gauge theory and exhibit the (zero-temperature) coupling constant for the $2N_f$-fermion vertex. In Appendix A a general method for constructing effective-Lagrangian functions is illustrated.

In Sec. IV we specialize to an $SU(2)\times U(1)$ electroweak gauge theory with two left-handed fermion doublets: the proton and neutron plus the electron and electron neutrino. We find that the reaction $p+n\to\bar{e}+\bar{\nu}$ and 11 other instanton-mediated baryon-lepton reactions have cross sections of the order $s\times10^{-195}$ cm^2 at zero temperature, where s is the total center-of-momentum energy squared in GeV^2. The neutron decays via $n\to\bar{p}+\bar{e}+\bar{\nu}$ with a lifetime of the order 10^{146} yr in this model. A brief calculational outline for these specific two reactions is given in Appendix B.

In Sec. V we generalize 't Hooft's effective-Lagrangian construction from Sec. III to finite temperature. The form of the $2N_f$-fermion interaction is temperature independent with only the effective vertex coupling becoming a function of temperature. Since the presently known finite-temperature instantons are self-dual, they have zero energy. The classical tunneling amplitude for such a solution will not increase with temperature but will remain constant at $\exp(-8\pi^2|n|/g^2)$. At the one-loop level, the self-dual-instanton density decreases with temperature due to color-electric-charge screening,[6] and consequently the vertex coupling decreases. For the above two-doublet model, we find in Sec. VI that at temperatures above 10^{15} °K (approximately 100 GeV in energy units), the coupling decreases like $T^{-47/6}$, while below 10^{15} °K it is nearly temperature independent. The neutron decay width and the various cross sections thus fall off like $T^{-47/3}$ above 10^{15} °K but are nearly constant below this temperature.

We close with a discussion of our results in Sec. VII. Because the instanton gas is thought to be very dense rather than dilute, we conjecture that our "dilute-gas" reaction rates may be only lower bounds on the actual rates for a dense instanton gas. It is also possible that finite-energy tunneling solutions could increase the finite-temperature tunneling rate.

II. INSTANTONS AND THE ADLER-BELL-JACKIW ANOMALY

For an SU(2) gauge theory described by the Yang-Mills (YM) Lagrangian

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}, \quad a=1,2,3$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu\,, \tag{2.1}$$

the winding number is defined as

$$n = \frac{g^2}{32\pi^2}\int F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}d^4x\,, \tag{2.2}$$

where $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$. For self-dual (instanton) and anti-self-dual (anti-instanton) Euclidean field configurations, n is an integer. Such configurations are automatically solutions to the Euclidean equations of motion.

The unit-winding-number instanton in Euclidean space may be expressed as

$$A^a_\mu(x) = \frac{2}{g}\frac{\eta^a_{\mu\nu}(x-z)_\nu}{(x-z)^2+\rho^2}\,, \tag{2.3}$$

where z and ρ are the instanton position and scale size, respectively, and $\eta^a_{\mu\nu}$ ($\bar{\eta}^a_{\mu\nu}$) is 't Hooft's[4] η symbol which projects out self-dual (anti-self-dual) tensors. The instanton (2.3) is the WKB interpolating field configuration, or "most probable escape path" (MPEP),[7] for a tunneling from one vacuum state to a gauge-rotated vacuum (not obtainable by a series of infinitesimal gauge rotations). The corresponding one-dimensional tunneling barrier in winding-number space is[7]

$$V(q) = (3\pi^2/g^2\rho)U(q) \tag{2.4}$$

with $U(q)$ plotted in Fig. 1. Here q is a continuous winding variable defined such that $q(\tau=-\infty)=0$ and $q(\tau=+\infty)=1$, where τ is the Euclidean time. The WKB tunneling amplitude is $\exp(-8\pi^2/g^2)$.

Now let us add N_f massless fermion doublets coupled to the gauge fields with Lagrangian

$$\mathcal{L}_f = -\sum_{s=1}^{N_f}\bar{\psi}_s D\!\!\!/\,\psi_s\,, \tag{2.5}$$

where

$$D_\mu\psi^\alpha_s = \partial_\mu\psi^\alpha_s - \frac{ig}{2}\tau^a_{\alpha\beta}A^a_\mu\psi^\beta_s, \quad \alpha=1,2\,. \tag{2.6}$$

In general, when a classical Dirac field theory is quantized, the operator current

$$J_\mu(x) = i\bar{\psi}\gamma_\mu Q\psi \tag{2.7}$$

acquires an anomalous divergence[2,5]

$$\partial_\mu J_\mu(x) = -iN_{ab}\frac{g^2}{32\pi^2}F^a_{\mu\nu}(x)\tilde{F}^b_{\mu\nu}(x)\,, \tag{2.8}$$

where $J_\mu(x)$ is defined to be invariant under non-Abelian gauge transformations of the background Yang-Mills field. In (2.8) the matrix N_{ab} is given by

$$N_{ab} = \tfrac{1}{2}\,{\rm Tr}(\gamma_5 Q T^a T^b)\,, \tag{2.9}$$

where T^a are the gauge-group generators.

In the case of (2.5), the vector currents

$$J_{\mu st} = i\bar{\psi}_s\gamma_\mu\psi_t\,, \tag{2.10}$$

and the traceless part of the axial-vector current

$$J^5_{\mu st} = i\bar{\psi}_s\gamma_\mu\gamma_5\psi_t \tag{2.11}$$

are conserved without anomalies. Hence, the theory has an exact global $SU(N_f)_L\times SU(N_f)_R\times U(1)$ chiral symmetry. The axial-vector current

$$J^5_\mu = \sum_s J^5_{\mu ss} \tag{2.12}$$

has an anomaly

$$\partial_\mu J^5_\mu = -i\frac{N_f g^2}{16\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\,. \tag{2.13}$$

Comparison with (2.2) shows that a unit-winding-number instanton can be associated with an axial-vector-charge violation

$$\Delta Q^5 = 2N_f\,. \tag{2.14}$$

Because the $U(N_f)_R\times U(N_f)_L$ symmetry of (2.5) is thus broken down intrinsically by instantons, and not spontaneously, to $SU(N_f)_R\times SU(N_f)_L\times U(1)$, one does not expect a physical massless Goldstone boson to appear. This was 't Hooft's solution to the so-called U(1) problem.[8]

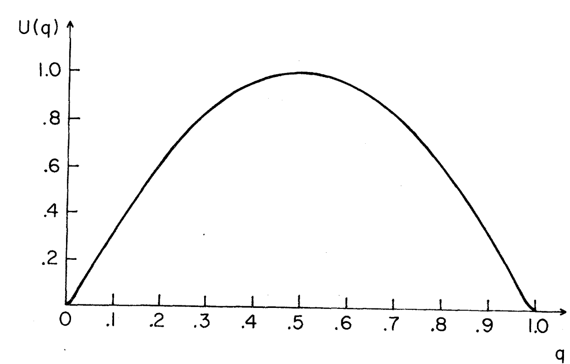

FIG. 1. The potential barrier U as a function of the winding number q for an $n=1$ vacuum tunneling. The winding number q is defined in Eq. (2.4) (taken from the first paper under Ref. 7).

III. EFFECTIVE LAGRANGIAN IN THE CHIRAL LIMIT

We will be discussing an SU(2) weak-isospin gauge theory in the chiral limit with Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - D_\mu\Phi^* D_\mu\Phi - \bar{\psi}\gamma_\mu D_\mu\psi + \bar{\psi}_s J_{st}\psi_t \, . \tag{3.1}$$

The complex scalar fields Φ may contain several multiplets of arbitrary isospin I,

$$D_\mu\Phi = \partial_\mu\Phi - igT^a A^a_\mu \Phi \, , \tag{3.2}$$

$$[T^a, T^b] = i\epsilon_{abc}T^c \, . \tag{3.3}$$

The spinors ψ are in N_f isospin-$\frac{1}{2}$ doublets. The gauge-invariant source term will be used to obtain Green's functions. The indices $s,t=1,\ldots,N_f$ label isospin doublets.

The superficical chiral $U(N_f)\times U(N_f)$ global symmetry in (3.1) is broken down to $SU(N_f)\times SU(N_f) \times U(1)$ by an Adler-Bell-Jackiw anomaly associated with the chiral U(1) current. Using the instanton solutions, 't Hooft[4] has calculated in the one-loop approximation the chiral-U(1)-breaking part of the vacuum-to-vacuum transition amplitude

$$W = {}_{\text{out}}\langle 0|0\rangle_{\text{in}} = \int DA\, D\psi\, D\Phi\, D\phi \exp\left(\int [\mathcal{L} - \tfrac{1}{2}G^2(A) + \mathcal{L}_{\text{ghost}}(\phi)]d^4x\right). \tag{3.4}$$

In (3.4), $G(A)$ and $\mathcal{L}_{\text{ghost}}(\phi)$ are the gauge-fixing and ghost terms, respectively.

If $|0\rangle$ is the vacuum, and $|0'\rangle$ is the gauge-rotated vacuum arrived at via a tunneling, then in the absence of fermions, the total contribution to $\langle 0'|0\rangle$ from all Euclidean paths that have a unit-winding-number instanton at z within d^4z, with scale between ρ and $\rho+d\rho$, is given by

$$dW = n(\rho)d^4z\, d\rho \, . \tag{3.5}$$

Here $n(\rho)$ is the instanton density

$$n(\rho) = [\det(-D^2\delta_{\mu\nu} - 2F_{\mu\nu})_{\text{adj(gauge)}}]^{-1/2}[\det(-D^2)_{\text{adj(ghost)}}]\prod_I [\det(-D^2)_{I(\text{scalar})}]^{-1}\exp[-8\pi^2/g^2(\mu)] \, , \tag{3.6}$$

where $D_\mu = \partial_\mu - igA_\mu$, A_μ is the instanton field, and all fluctuation determinants are understood to be normalized by the corresponding vacuum determinants. Specifically, 't Hooft has shown that

$$n(\rho) = \frac{1}{4\pi^2}\left(\frac{8\pi^2}{g^2(\mu)}\right)^4 \frac{1}{\rho^5}\exp\left[-\frac{8\pi^2}{g^2(\mu)} + \left(\frac{22}{3} - \frac{1}{6}\sum_I N^s(I)C(I)\right)\ln\mu\rho - \alpha(1) - \sum_I N^s(I)\alpha(I)\right] \, , \tag{3.7}$$

where $N^s(I)$ denotes the number of scalar multiplets of isospin I. Each complex scalar multiplet counts as one, and each real multiplet counts as one-half. The coefficients $C(I)$ and $\alpha(I)$ can be found in Table I of the second paper under Ref. 4.

The one-loop contribution to (3.7) from massive isospin-$\frac{1}{2}$ fermions would be

$$\prod_{s=1}^{N_f}[\det(\not{D} + m_s)] = \prod_{s=1}^{N_f}(m_s\rho\det{}'\not{D}) = \prod_{s=1}^{N_f}\{m_s\rho\exp[-\tfrac{2}{3}\ln\mu\rho + 2\alpha(\tfrac{1}{2})]\} \, . \tag{3.8}$$

In the chiral limit of massless fermions, $m\rho$ must be replaced by the lowest eigenvalue of the fermion fluctuation operator when perturbed by a small source J_{st}. In this case the fermion fluctuation determinant is

$$\det M_\psi = \int D\psi\, D\bar{\psi}\exp(\bar{\psi}M_\psi\psi) = \det Z(J)\det{}'\not{D} \, , \tag{3.9}$$

where $M_\psi = \not{D} + J$.

The zero-mode determinant is given by

$$\det Z(J) = \det(\psi^* J\psi) = \det_{st}(\psi^*_s J_{st}\psi_t) \, , \tag{3.10}$$

where the ψ's are the fermion zero-mode wave functions,

$$\psi^\alpha = \left(\frac{2\rho^3}{\pi^2}\right)^{1/2}(\rho^2 + x^2)^{-3/2}u^\alpha \, . \tag{3.11}$$

(The instanton is at the origin here.) The Dirac spinors u ($u^*u=1$) contain an isospin index α ($=1,2$) and are left-handed chiral eigenstates ($\gamma_5 u = -u$). Recall that the propagator for a massless fermion in coordinate space is

$$S_F(x) = \frac{\gamma\cdot x}{2\pi^2(x^2)^2} \, . \tag{3.12}$$

Hence, for large x^2 the zero-mode determinant is seen to have the form of an N_f-point Green's function with each source connected to the instanton position (the origin $z^\mu = 0$) by two fermion lines (Fig. 2).

Because $\det Z(J)$ vanishes if J is set to zero, a unit-winding-number tunneling occurs only when accompanied by a $2N_f$-fermion point interaction at the instanton location. Indeed, one can determine the gauge-invariant effective interaction Lagran-

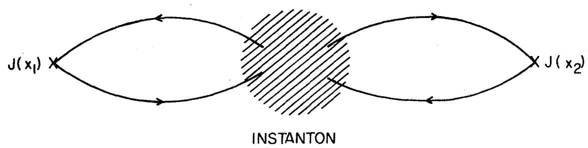

FIG. 2. The instanton-induced Green's function for the case of two massless fermion doublets. The sources J change the sign of the axial-vector charge Q^5. The amplitude goes like J^2.

gian $\mathcal{L}_{\rm eff}(\bar\psi,\psi)$ generated by the instanton. Since the Lagrangian must reproduce the zero-mode behavior (3.10) by an interaction vertex at the instanton location z^μ, it must satisfy

$$\int D\psi\, D\bar\psi\, \mathcal{L}_{\rm eff}(\bar\psi,\psi)\exp(\bar\psi J\psi)=\det(\psi^* J\psi)\,. \qquad (3.13)$$

From (3.13) 't Hooft finds that the effective fermion interaction can be written as

$$\mathcal{L}_{\rm eff}(\bar\psi,\psi)=(8\pi^2\rho^3)^{N_f}\left\langle \prod_{s=1}^{N_f}(\bar\psi_s\omega)(\bar\omega\psi_s)\right\rangle, \qquad (3.14)$$

where ω^α is a parity reflection of the u^α. The angular brackets denote the average over all gauge rotations of ω,

$$\langle \omega_\alpha\bar\omega_\beta\rangle=\tfrac{1}{4}\delta_{\alpha\beta}(1+\gamma_5)\,. \qquad (3.15)$$

For example, if we have two massless fermion doublets, then

$$\mathcal{L}_{\rm eff}(\bar\psi,\psi)=(8\pi^2\rho^3)^2\langle(\bar\psi_1\omega)(\bar\omega\psi_1)(\bar\psi_2\omega)(\bar\omega\psi_2)\rangle$$
$$=\frac{(8\pi^2\rho^3)^2}{6}(2\delta_{\alpha\beta}\delta_{\gamma\sigma}-\delta_{\alpha\sigma}\delta_{\beta\gamma})$$
$$\times(\bar\psi_1^\alpha\Lambda_+\psi_1^\beta\bar\psi_2^\gamma\Lambda_+\psi_2^\sigma-\bar\psi_1^\alpha\Lambda_+\psi_2^\beta\bar\psi_2^\gamma\Lambda_+\psi_1^\sigma)\,, \qquad (3.16)$$

where $\Lambda_+=\frac{1}{2}(1+\gamma_5)$. This result is derived in Appendix A by an equivalent method[9] to 't Hooft's gauge averaging. Anti-instantons, of course, give rise to an effective fermion interaction which is the Hermitian conjugate of (3.14).

As is evident from (3.15), the instanton-generated interaction (3.14) takes right-handed fermion doublet states over to left-handed fermion doublet states. Anti-instantons mediate the reverse reactions. The phenomenology of these interactions will be explored in Sec. IV.

The product of (3.7) and $\det'\not{D}$ in (3.9) determines the effective coupling constant for the vertex (3.14). Since a unit-winding-number instanton can have any scale size, we must also integrate over ρ. This integration over scale size has the usual infrared divergence for $\rho\to\infty$ characteristic of a scale-invariant gauge theory. For a weak-interaction theory, however, the Higgs field will provide a cutoff. The Higgs-field contribution to the classical action has been estimated by 't Hooft to be

$$S_H=-4\pi^2 IF^2\rho^2\,, \qquad (3.17)$$

where I is the isospin of the Higgs multiplet, and F^2 is the Higgs vacuum expectation value.

With $\exp(S_H)$ multiplying (3.7), the full effective Lagrangian, $\mathcal{L}_{\rm eff}(z)$, describing the instanton-generated interaction between massless fermions is

$$\mathcal{L}_{\rm eff}(z)d^4z=ce^{-a}\mu^b\exp\left(\frac{-8\pi^2}{g^2(\mu)}\right)d^4z\int_0^\infty \rho^{-5+3N_f+b}\exp(-4\pi^2 IF^2\rho^2)d\rho\left\langle\prod_{s=1}^{N_f}(\bar\psi_s\omega)(\bar\omega\psi_s)\right\rangle+\text{H.c.}\,, \qquad (3.18)$$

where

$$a=\alpha(1)-2N_f\alpha(\tfrac{1}{2})+\sum_I N^s(I)\alpha(I)\,, \qquad (3.19)$$

$$b=\tfrac{22}{3}-\tfrac{2}{3}N_f-\tfrac{1}{6}\sum_I N^s(I)C(I)\,, \qquad (3.20)$$

$$c=\frac{(8\pi^2)^{N_f}}{4\pi^2}\left(\frac{8\pi^2}{g^2(\mu)}\right)^4. \qquad (3.21)$$

The integral over the instanton position z_μ corresponds here to the usual integration in coordinate space over the vertex variable. Performing the ρ integration in (3.18) yields 't Hooft's result

$$\mathcal{L}_{\rm eff}(z)=\mathcal{G}\left\langle\prod_{s=1}^{N_f}(\bar\psi_s\omega)(\bar\omega\psi_s)\right\rangle+\text{H.c.}\,, \qquad (3.22)$$

where the effective coupling constant is given by

$$\mathcal{G}=\tfrac{1}{2}ce^{-a}\mu^b(4\pi^2 IF^2)^{(4-3N_f-b)/2}$$
$$\times\Gamma\left(\frac{-4+3N_f+b}{2}\right)\exp\left(\frac{-8\pi^2}{g^2(\mu)}\right). \qquad (3.23)$$

IV. ZERO-TEMPERATURE CROSS SECTIONS

Since the interaction (3.22) couples left- and right-handed fermion doublets, it can lead to a violation of fermion number in a gauge theory containing left-handed doublets and right-handed singlets. In this section we specialize to a very simple electroweak gauge theory in which such violations occur.

The theory we consider is an SU(2)×U(1) electroweak gauge theory in the chiral limit with one complex Higgs scalar doublet and two left-handed fermion doublets: the proton and neutron plus the electron and the electron neutrino,

$$\psi_{1,L}=\begin{pmatrix}\psi_1^1\\ \psi_1^2\end{pmatrix}_L=\begin{pmatrix}p\\ n\end{pmatrix}_L,\quad \psi_{2,L}=\begin{pmatrix}\psi_2^1\\ \psi_2^2\end{pmatrix}_L=\begin{pmatrix}\nu\\ e\end{pmatrix}_L. \tag{4.1}$$

The right-handed fermions (p_R, n_R, e_R) are taken as weak-isospin singlets and do not couple to the SU(2) gauge fields.

All currents that are coupled to gauge fields are free of anomalies, and the model is renormalizable. However, the baryon (ψ_1) and lepton (ψ_2) currents

$$J^5_\mu=i\bar\psi_1\gamma_\mu\gamma_5\psi_1+i\bar\psi_2\gamma_\mu\gamma_5\psi_2 \tag{4.2}$$

have an anomaly[2,5]

$$\partial_\mu J^5_\mu=\frac{-i}{8\pi^2}(g^2F^a_{\mu\nu}\tilde F^a_{\mu\nu}-g'^2F'_{\mu\nu}\tilde F'_{\mu\nu}), \tag{4.3}$$

where $F_{\mu\nu}$, $F'_{\mu\nu}$ are the four SU(2)×U(1) field strengths and $\tilde F_{\mu\nu}$, $\tilde F'_{\mu\nu}$ are their duals. Since the topological quantum number for an instanton is

$$n=\frac{g^2}{32\pi^2}\int F^a_{\mu\nu}\tilde F^a_{\mu\nu}d^4x, \tag{4.4}$$

a unit-winding-number instanton can be associated with a violation of axial-vector charge

$$\Delta Q^5=4, \tag{4.5}$$

or, equivalently, baryon and lepton number

$$\Delta B=\Delta L=2, \tag{4.6}$$

in this model.

The instanton-generated effective Lagrangian describing the baryon- and lepton-number-violating process is, according to (3.22),

$$\mathcal{L}_{\rm eff}=\mathcal{G}\left(\tfrac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\sigma}-\tfrac{1}{6}\delta_{\alpha\sigma}\delta_{\beta\gamma}\right)\times(\bar\psi_1^\alpha\Lambda_+\psi_1^\beta\bar\psi_2^\gamma\Lambda_+\psi_2^\sigma-\bar\psi_1^\alpha\Lambda_+\psi_2^\beta\bar\psi_2^\gamma\Lambda_+\psi_1^\sigma)+\text{H.c.}, \tag{4.7}$$

where

$$\mathcal{G}=8\pi^2\left(\frac{8\pi^2}{g^2(\mu)}\right)^4\exp[-\alpha(1)+3\alpha(\tfrac{1}{2})]\times\mu^{35/6}(2\pi^2F^2)^{-47/12}\Gamma(\tfrac{47}{12})\exp[-8\pi^2/g^2(\mu)] \tag{4.8}$$

and[4]

$$\alpha(\tfrac{1}{2})=0.145\,873,\quad \alpha(1)=0.443\,307. \tag{4.9}$$

The Higgs vacuum expectation value is

$$F^2=\frac{1}{2\sqrt2 G}, \tag{4.10}$$

where G is the Fermi coupling constant[10]

$$G=(1.166\,32\pm0.000\,04)\times10^{-5}\ \text{GeV}^{-2}. \tag{4.11}$$

The effective coupling constant (4.8) has the value

$$\mathcal{G}=3.71^{+3.27}_{-2.27}\times10^{-83\mp4}\ \text{GeV}^{-2}, \tag{4.12}$$

using the SU(2) coupling constant

$$g^2(\mu)=g^2(100\ \text{GeV})=0.412^{-0.018}_{+0.019}, \tag{4.13}$$

and neglecting threshold corrections. (By way of comparison, Newton's gravitational constant is of order $G_N\sim10^{-38}$ GeV^{-2}.) The errors are due to the experimental uncertainty in the Weinberg parameter[10]

$$\sin^2\theta_W=0.228\pm0.010. \tag{4.14}$$

In (4.12) and all subsequent equations, the upper and lower errors on the preexponentials [of interest where $O(1)$ factors are important] and exponents correspond to the upper and lower errors, respectively, on $\sin^2\theta_W$ in Eq. (4.14). For example, for $\sin^2\theta_W=0.238$, we have $g^2(100\ \text{GeV})=0.394$ and $\mathcal{G}=6.98\times10^{-87}$ GeV^{-2}. We have taken the renormalization point as $\mu=100$ GeV since this is the relevant renormalization point for the instanton scale size integration in Eq. (3.18). The Higgs mechanism cuts off the ρ integral somewhere above 100 GeV, while below 100 GeV the $W^\pm$, Z, and Higgs fields decouple.

The value for $g^2(100\ \text{GeV})$ was obtained by running the SU(2) coupling constant

$$\frac{g^2(Q)}{4\pi}=\frac{\alpha_{\rm EM}}{\sin^2\theta_W} \tag{4.15}$$

from $Q^2=\mu_0{}^2=1$ GeV^2 to $Q^2=\mu^2=10^4$ GeV^2 using the two-loop renormalization- group equation[11]

$$\frac{4\pi}{g^2(Q)}=\frac{4\pi}{g^2(\mu_0)}-\frac{1}{3\pi}N_f\ln\left(\frac{Q}{\mu_0}\right)+\frac{3}{8\pi}\frac{N^2-1}{N}\ln\ln\left(\frac{Q}{\mu_0}\right). \tag{4.16}$$

For our simple two-doublet electroweak theory, we have $N_f=2$ and SU(N) = SU(2). The present experimental value for the Weinberg parameter given in (4.14) is determined in experiments with typical momentum transfers of the order $Q^2\sim1$ GeV^2. Consequently, we have taken the starting value $g^2(\mu_0)$ in (4.16) at $\mu_0=1$ GeV with $\alpha_{\rm EM}=\frac{1}{137}$ and $\sin^2\theta_W$ given by (4.14).

Let us now turn to the phenomenology of Eq. (4.7). From the $1+\gamma_5$ structure in (4.7), we see that a unit-winding-number instanton can cause antibaryons and antileptons to annihilate and form baryons and leptons. The Hermitian conjugate in (4.7) describes the fermion to antifermion reactions mediated by anti-instantons. Because the interaction (4.7) conserves electric charge but not fermion number, the field $\frac{1}{2}(1+\gamma_5)\psi_i^\alpha$ represents

TABLE I. Differential and total cross sections for the 12 reactions described by the Lagrangian (4.22). Here t is the invariant momentum transfer squared, s is the center-of-momentum energy squared, and $\Sigma = \mathcal{G}^2/72\pi$.

Reaction	$d\sigma/dt$	σ_{total}
$e+\nu\to\bar{p}+\bar{n}$	Σ	Σs
$n+\nu\to\bar{n}+\bar{\nu}$	$\Sigma\left(1+\frac{t}{s}+\frac{t^2}{4s^2}\right)$	$\frac{7}{12}\Sigma s$
$p+n\to\bar{e}+\bar{\nu}$	$\frac{1}{2}\Sigma$	$\frac{1}{2}\Sigma s$
$p+\nu\to\bar{n}+\bar{e}$	$\Sigma\frac{t^2}{s^2}$	$\frac{1}{3}\Sigma s$
$n+\nu\to\bar{p}+\bar{e}$	$\Sigma\frac{t^2}{s^2}$	$\frac{1}{3}\Sigma s$
$p+e\to\bar{p}+\bar{e}$	$\frac{1}{2}\Sigma\left(1+\frac{t}{s}+\frac{t^2}{4s^2}\right)$	$\frac{7}{24}\Sigma s$
$\nu+\nu\to\bar{n}+\bar{n}$	$\frac{1}{4}\Sigma$	$\frac{1}{4}\Sigma s$
$p+e\to\bar{n}+\bar{\nu}$	$\frac{1}{2}\Sigma\frac{t^2}{s^2}$	$\frac{1}{6}\Sigma s$
$n+e\to\bar{p}+\bar{\nu}$	$\frac{1}{2}\Sigma\frac{t^2}{s^2}$	$\frac{1}{6}\Sigma s$
$p+p\to\bar{e}+\bar{e}$	$\frac{1}{16}\Sigma$	$\frac{1}{16}\Sigma s$
$n+n\to\bar{\nu}+\bar{\nu}$	$\frac{1}{16}\Sigma$	$\frac{1}{16}\Sigma s$
$e+e\to\bar{p}+\bar{p}$	$\frac{1}{16}\Sigma$	$\frac{1}{16}\Sigma s$

the charge-conjugate field $(\psi^c)^\alpha_{i,R}$. Hence

$$\bar{\psi}_1^1\Lambda_+=\bar{\psi}_{p_L},\quad \bar{\psi}_1^2\Lambda_+=\bar{\psi}_{n_L},\qquad \bar{\psi}_2^1\Lambda_+=\bar{\psi}_{\nu_L},\quad \bar{\psi}_2^2\Lambda_+=\bar{\psi}_{e_L} \tag{4.17}$$

and

$$\Lambda_+\psi_1^1=\psi^c_{n_L},\quad \Lambda_+\psi_1^2=\psi^c_{p_L},\qquad \Lambda_+\psi_2^1=\psi^c_{e_L},\quad \Lambda_+\psi_2^2=\psi^c_{\nu_L}\,. \tag{4.18}$$

In order to study the particle-to-antiparticle reactions (which are of inherent interest in our part of the universe), we expand the Hermitian conjugate in (4.7) explicitly and obtain

$$\left(\tfrac{1}{3}\,\delta_{\alpha\beta}\delta_{\gamma\sigma}-\tfrac{1}{6}\,\delta_{\alpha\sigma}\delta_{\beta\gamma}\right)\left(\bar{\psi}_2^{\sigma}\Lambda_-\psi_2^{\gamma}\bar{\psi}_1^{\beta}\Lambda_-\psi_1^{\alpha}-\bar{\psi}_1^{\sigma}\Lambda_-\psi_2^{\gamma}\bar{\psi}_2^{\beta}\Lambda_-\psi_1^{\alpha}\right)$$

$$=\tfrac{1}{6}\,(2\bar{\psi}_e^c\Lambda_-\psi_\nu\bar{\psi}_p^c\Lambda_-\psi_n+\bar{\psi}_p^c\Lambda_-\psi_\nu\bar{\psi}_e^c\Lambda_-\psi_n \tag{4.19a}$$

$$+2\bar{\psi}_\nu^c\Lambda_-\psi_e\bar{\psi}_n^c\Lambda_-\psi_p+\bar{\psi}_n^c\Lambda_-\psi_e\bar{\psi}_\nu^c\Lambda_-\psi_p \tag{4.19b}$$

$$+\bar{\psi}_e^c\Lambda_-\psi_\nu\bar{\psi}_n^c\Lambda_-\psi_p-\bar{\psi}_n^c\Lambda_-\psi_\nu\bar{\psi}_e^c\Lambda_-\psi_p \tag{4.19c}$$

$$+\bar{\psi}_\nu^c\Lambda_-\psi_e\bar{\psi}_p^c\Lambda_-\psi_n-\bar{\psi}_p^c\Lambda_-\psi_e\bar{\psi}_\nu^c\Lambda_-\psi_n \tag{4.19d}$$

$$-2\bar{\psi}_p^c\Lambda_-\psi_e\bar{\psi}_e^c\Lambda_-\psi_p-\bar{\psi}_e^c\Lambda_-\psi_e\bar{\psi}_p^c\Lambda_-\psi_p \tag{4.19e}$$

$$-2\bar{\psi}_n^c\Lambda_-\psi_\nu\bar{\psi}_\nu^c\Lambda_-\psi_n-\bar{\psi}_\nu^c\Lambda_-\psi_\nu\bar{\psi}_n^c\Lambda_-\psi_n)\,. \tag{4.19f}$$

This expression can be simplified by recalling that[12]

$$\psi^c=C\bar{\psi}^T,\quad \bar{\psi}=(C^{-1}\psi^c)^T,\qquad \psi=C\bar{\psi}^{cT},\quad \bar{\psi}^c=(C^{-1}\psi)^T, \tag{4.20}$$

where in the standard representation

$$C=i\gamma^2\gamma^0=-C^{-1}=-C^\dagger=-\ C^T\,. \tag{4.21}$$

Using (4.20) and (4.21), one sees that the second terms in (4.19e) and (4.19f) vanish, while (4.19a)–(4.19d) can be consolidated into three terms. The final result is

$$\left(\tfrac{1}{3}\,\delta_{\alpha\beta}\delta_{\gamma\sigma}-\tfrac{1}{6}\,\delta_{\alpha\sigma}\delta_{\beta\gamma}\right)\left(\bar{\psi}_2^{\sigma}\Lambda_-\psi_2^{\gamma}\bar{\psi}_1^{\beta}\Lambda_-\psi_1^{\alpha}-\bar{\psi}_1^{\sigma}\Lambda_-\psi_2^{\gamma}\bar{\psi}_2^{\beta}\Lambda_-\psi_1^{\alpha}\right)$$

$$=\tfrac{1}{3}\,(\bar{\psi}_p^c\Lambda_-\psi_n\bar{\psi}_e^c\Lambda_-\psi_\nu-\bar{\psi}_p^c\Lambda_-\psi_\nu\bar{\psi}_n^c\Lambda_-\psi_e+\bar{\psi}_p^c\Lambda_-\psi_e\bar{\psi}_n^c\Lambda_-\psi_\nu$$

$$+\bar{\psi}_p^c\Lambda_-\psi_e\bar{\psi}_p^c\Lambda_-\psi_e+\bar{\psi}_n^c\Lambda_-\psi_\nu\bar{\psi}_n^c\Lambda_-\psi_\nu)\,. \tag{4.22}$$

The terms in (4.22) give rise to the free decay $n\to\bar{p}+\bar{e}+\bar{\nu}$ plus the 12 two-body reactions listed in the first column of Table I.

The neutron lifetime and the cross sections for these reactions are straightforwardly calculated using the charge-conjugate spinor identities (B4) in Appendix B and the familar Dirac algebra. The initial states are taken as unpolarized, and final spin states are summed over. A brief outline of the calculation for $p+n\to\bar{e}+\bar{\nu}$ and $n\to\bar{p}+\bar{e}+\bar{\nu}$ is given in Appendix B. We find the neutron decay width to be

$$\Gamma(n\to\bar{p}+\bar{e}+\bar{\nu})=\frac{\mathcal{G}^2 m_n m_p}{36\pi^3}\,(1.678\,58\times10^{-10}\ \text{GeV}^3)\,. \tag{4.23}$$

Using the calculated value of $\mathcal{G}$ from (4.12), the neutron lifetime is

$$\tau=1.14^{+2.09}_{-0.381}\times10^{146\pm7}\,\text{yr}\,. \tag{4.24}$$

The invariant differential cross sections and total cross sections for the remaining 12 reactions are summarized in Table I. For notational convenience, we denote

$$\Sigma=\frac{\mathcal{G}^2}{72\pi}\,. \tag{4.25}$$

Using the value for $\mathcal{G}$ in (4.12), this quantity is

$$\Sigma=2.37^{-1.531}_{+1.200}\times10^{-195\mp7}\ \text{GeV}^{-2}\,\text{cm}^2\,. \tag{4.26}$$

The total cross sections are all of order $s\times10^{-195\mp7}\,\text{cm}^2$, where s is the center-of-momentum energy squared in GeV^2.

As is usual with four-fermion point interactions, the total cross sections rise linearly with the center-of-momentum energy squared s. The effective Lagrangian (4.7) is probably not physically valid beyond $\sqrt{s}=10^{19}$ GeV, where gravity becomes important. In the next two sections we explore the effect of temperature on these rates.

V. FINITE-TEMPERATURE EFFECTIVE LAGRANGIAN

The effect of temperature on the chiral-U(1)-breaking amplitude (3.5) is determined by the temperature dependence of the instanton density (3.6). Gross, Pisarski, and Yaffe[6] have recently made an extensive study of finite-temperature SU(N) gauge theories and imbedded self-dual SU(2) instantons. They find that at high temperature thermal excitations produce a plasma of quarks and gluons which screens all color-electric flux, and quarks become unconfined. This is evident from the correlation function of the timelike component of the gauge field,

$$\langle A_0(\bar{x})A_0(\bar{y})\rangle \sim \exp(-m_{el}|\bar{x}-\bar{y}|)\,. \tag{5.1}$$

The one-loop electric screening length is

$$\begin{aligned} m_{el}^{-1} &= \Pi_{00}^{-1/2}(\omega=0,\bar{k}=0) \\ &= [\tfrac{1}{3}g^2T^2(N+\tfrac{1}{2}N_f)]^{-1/2}\,, \end{aligned} \tag{5.2}$$

for an SU(N) gauge theory with N_f fermion fundamental representation multiplets. Here $\Pi_{\mu\nu}(k)$ is the gluon self-energy.

At finite temperature the boundary condition of fields vanishing at temporal infinity is replaced by periodic boundary conditions in Euclidean time. A periodic instanton can be constructed from the multi-instanton solution which describes an infinite string of instantons located at $\bar{x}=0$ (arbitrary) and $\tau=n\beta$, $n\in Z$, with identical sizes and gauge orientations. The 't Hooft aligned instanton solution is

$$A_\mu = \frac{1}{g}\Pi\,\bar{\eta}^a_{\mu\nu}(\tau^a/2i)\partial_\nu\Pi^{-1}\,, \tag{5.3}$$

where

$$\Pi(x) = 1+\sum_{n=1}^{K}\rho_n^2/(x-z_n)^2 \tag{5.4}$$

describes K instantons with positions z_n and sizes ρ_n. For $\rho_n=\rho$ and $z_n=n\beta\hat{\tau}$ $(n\in Z)$, this becomes the single periodic instanton[13]

$$\Pi(\bar{x},\tau) = 1+\frac{\pi\rho^2}{\beta r}\frac{\sinh 2\pi r/\beta}{\cosh 2\pi r/\beta - \cos 2\pi\tau/\beta}\,, \tag{5.5}$$

where $r=|\bar{x}|$. Here the instanton solution is expressed in the so-called "singular" gauge where A_μ has a pure gauge singularity at $r=\tau=0$ (which can be removed by a periodic gauge transformation).

For distances $|r^2+\tau^2|^{1/2}\ll\beta$, the periodic instanton is approximately

$$A^a_\mu \approx \frac{2}{g}\frac{\rho'^2}{r^2+\tau^2}\frac{\bar{\eta}^a_{\mu\nu}x_\nu}{r^2+\tau^2+\rho'^2}\,, \tag{5.6}$$

where $\rho'^2=\rho^2/(1+\frac{1}{3}\lambda^2)$ and $\lambda=\pi\rho/\beta$. This is like a zero-temperature instanton with a renormalized size ρ'. When $\rho'\ll x\ll\beta$, the solution is like a four-dimensional self-dual dipole. For distances $r\gg\beta$, the periodic instanton is approximately

$$A^a_0 \sim \frac{-x^a}{gr^2(1+r/\lambda\rho)}\,, \quad A^a_i \sim \frac{\epsilon^{aij}x^j}{gr^2(1+r/\lambda\rho)}\,. \tag{5.7}$$

If $r\gg\lambda\rho$, this describes a three-dimensional dipole field. If $\beta\ll r\ll\lambda\rho$, the solution is like a dyon with unit electric and magnetic charges.

Periodic Euclidean solutions are interpretable as "finite-temperature most probable escape paths" (FTMPEP's) in the WKB sense.[7] They are the tunneling paths with the maximum barrier-crossing probability at a given temperature. For example, in the familiar one-dimensional double-well potential of Fig. 3, the finite-temperature barrier-crossing probability is given by the product of the Boltzmann factor and tunneling factor:

$$\exp\left(\int_0^\beta L_{eff}\,d\tau\right) = \exp\left\{-\beta E - 2\int_{-a}^{+a}\{2m[V(q)-E]\}^{1/2}dq\right\}, \tag{5.8}$$

where

$$L_{eff} = -\tfrac{1}{2}m\left(\frac{dq}{d\tau}\right)^2 - V(q)\,. \tag{5.9}$$

The competing effects of the WKB tunneling factor and the Boltzmann factor result in there being a path with some optimal energy maximizing (5.8).

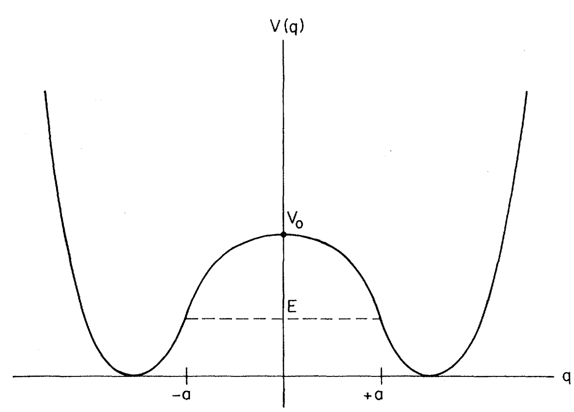

FIG. 3. The potential for the symmetric double-well anharmonic oscillator. The dashed tunneling path with optimal energy E and turning points $\pm a$ represents the periodic instanton for a given temperature T. Above a critical temperature T_c the instanton becomes static at the energy $E=V_0$.

This thermally activated path is a periodic instanton and results in an increase of the barrier-crossing probability. However, above a critical temperature

$$T_c = -\frac{1}{2\pi}\frac{1}{m}\frac{d^2V(0)}{dq^2}, \tag{5.10}$$

the instanton becomes static.[14] Physically this means that the most probable crossing path is the one with energy equal to the barrier height V_0. Above T_c the crossing probability is simply $e^{-\beta V_0}$.

The periodic instanton (5.5) is a self-dual solution, and consequently it automatically satisfies the field equations and has zero energy. This solution becomes a pure gauge at spatial infinity. Under these conditions it is a local minimum of the Euclidean action with one unit of topological charge in the physical strip $0 \leq \tau \leq \beta$.[13] The periodic instanton (5.5) then apparently connects zero-energy Minkowski states, which makes it physically understandable why its classical tunneling amplitude is

$$W = \exp(-8\pi^2/g^2), \tag{5.11}$$

independent of temperature. By relaxing the pure gauge boundary condition, it might be possible to find finite-energy tunneling paths like those in the double-well potential, but this is speculative at present.

Combined quantum and thermal fluctuations alter this classical thermodynamic picture. Generalizing 't Hooft's one-loop calculation of the fluctuation about an instanton to finite temperature, Gross *et al.*[6] have reevaluated the fluctuation determinants in an SU(N) gauge theory using periodic temporal boundary conditions.

Using the fact that $\Pi^{-1}\partial^2\Pi = 0$ for any 't Hooft solution (5.3), the zero-mode normalization integrals can be changed to surface integrals at spatial infinity. These integrals may be evaluated by using the asymptotic form (5.7) and turn out to be completely temperature independent. Furthermore, the determinants can be written as

$$\ln\det(-D^2/-\partial^2)\big|_T = \ln\det(-D^2/-\partial^2)\big|_{T=0} + \delta(\lambda), \tag{5.12}$$

where

$$\delta(\lambda) = \int_0^T dT' \frac{\partial}{\partial T'} \operatorname{tr}\ln(-D^2/\partial^2)\big|_{T'}. \tag{5.13}$$

All finite-temperature determinants are thus given by the zero-temperature determinant multiplied by a temperature-dependent correction $\exp\delta(\lambda)$.

For the SU(2) gauge theory described by (3.1), in the absence of fermions, the result of Gross *et al.*[6] for the finite-temperature instanton density is

$$n(\rho, T) = n(\rho, 0)\exp\left[-\delta_1(\lambda) - \sum_I N^s(I)\,\delta_I(\lambda)\right]. \tag{5.14}$$

Here

$$\lambda = \pi\rho T, \tag{5.15}$$

$$\delta_{1/2}(\lambda) = \tfrac{1}{3}\eta\lambda^2 + A(\lambda) \tag{5.16}$$

($\eta = +1$ for periodic fields and $-\frac{1}{2}$ for antiperiodic fields),

$$\delta_1(\lambda) = \tfrac{4}{3}\lambda^2 + 16A(\lambda), \tag{5.17}$$

$$A(\lambda) \simeq -\tfrac{1}{12}\ln(1+\lambda^2/3) + \alpha(1+\gamma\lambda^{-3/2})^{-8}, \tag{5.18}$$

$$\alpha = 0.012\,897\,64, \tag{5.19}$$

$$\gamma = 0.158\,58, \tag{5.20}$$

and $n(\rho, 0)$ is the zero-temperature density (3.7).

The instanton density decreases with temperature due to color-electric screening generated by thermal fluctuations. Comparison with (5.2) shows that the finite-temperature correction in (5.14) from the gauge (plus ghost) fields can be written as

$$\exp[-\delta_1(\lambda)] = (1+\pi^2\rho^2T^2/3)^{4/3}\exp\{-16\alpha[1+\gamma(\pi\rho T)^{-3/2}]^{-8}\} \times \exp\left(-\frac{2\pi^2}{g^2}m_{\rm el}{}^2\rho^2\right) \tag{5.21}$$

$$\underset{T\to\infty,\,\rho\text{ fixed}}{\sim} (\rho T)^{8/3}\exp\left(-\frac{2\pi^2}{g^2}m_{\rm el}{}^2\rho^2\right). \tag{5.22}$$

The exponential factor in (5.22) is the screening factor, with large instantons being suppressed compared to small ones.

The contribution of fermions to (5.14) is

$$\prod_{s=1}^{N_f}[\det(\not{D}+m_s)] = \prod_{s=1}^{N_f}\{m_s\rho\exp[-\tfrac{2}{3}\ln\mu\rho + 2\alpha(\tfrac{1}{2})] \times\exp[-\tfrac{1}{3}\lambda^2 + 2A(\lambda)]\}. \tag{5.23}$$

In the chiral limit of massless fermions, the fermion zero-mode determinant

$$\det Z(J) = \det(\psi^* J\psi) \tag{5.24}$$

replaces the $m_s\rho$ factors, just as it did in the zero-temperature case. The finite-temperature fermion zero-mode wave functions are[6]

$$\psi^\alpha = \Pi^{1/2}\tau\cdot\partial(\phi/\Pi)\not{\tau}^\dagger u^\alpha, \tag{5.25}$$

where Π is given by (5.4), $\tau_\mu \equiv (-i, \tau_i)$, $\tau_\mu^\dagger \equiv (i, \tau_i)$, and u^α is the Dirac spinor defined in (3.11). Here

$$\phi = (\Pi - 1)\frac{\cos\pi\tau/\beta}{\cosh\pi r/\beta}. \tag{5.26}$$

As Gross *et al.*[6] have noted, the periodic instanton (5.5) has various length scales such as ρ', β, and $\lambda\rho$. Different scales are relevant for different physical effects. Most of the instanton action density

$$\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}=\tfrac{1}{2}\partial^2(\partial_\mu \ln\Pi)^2 \tag{5.27}$$

is concentrated in a region of size ρ' about the center of the instanton rather than ρ. At finite temperature the field strengths do not spread out over larger regions as $\rho\to\infty$. For baryon- and lepton-number-violating effects then, we see from (5.6) that, having interacted with the finite-temperature instanton in the core region $x\sim\rho'$, the massless fermions propagate away from it into the region $x\gg\rho'$. Since the finite-temperature fermion zero modes approach the zero-temperature modes in the limit $\beta\to\infty$ with a temperature-independent normalization, the form of the effective fermion interaction (3.14) is unchanged at finite temperature.

The finite-temperature factors in (5.14) and (5.23) now multiply the zero-temperature Lagrangian (3.18) to yield the finite-temperature fermion interaction Lagrangian

$$\mathcal{L}_{eff}=\mathcal{G}(T)\left\langle\prod_{s=1}^{N_f}(\bar\psi_s\omega)(\bar\omega\psi_s)\right\rangle+\text{H.c.}, \tag{5.28}$$

where the temperature-dependent coupling constant is given by

$$\mathcal{G}(T)=ce^{-a}\mu^b\exp\left(\frac{-8\pi^2}{g^2(\mu)}\right)\int_0^\infty \rho^{-5+3N_f+b}\,d\rho\,\exp\left[-\left(4\pi^2 lF^2+\frac{4+N_f}{3}\pi^2T^2\right)\rho^2 -(16-2N_f)A(\pi\rho T)-\sum_I N^s(I)\,\delta_I(\pi\rho T)\right]. \tag{5.29}$$

VI. FINITE-TEMPERATURE CROSS SECTIONS

We can now apply the results of Sec. V to study the effect of temperature on the baryon- and lepton-number-violating processes discussed in Sec. IV.

Because the form of the effective fermion interaction (3.14) is temperature independent, the finite-temperature cross sections for the 13 reactions in Sec. IV can be obtained by simply taking $\mathcal{G}$ in (4.7) to be the temperature-dependent coupling (5.29). As is evident from (5.29), the effective coupling $\mathcal{G}$ decreases with temperature. This is a direct consequence of the decrease in the self-dual instanton density at finite temperature due to color-electric-charge screening. Hence, all of the reaction rates in Sec. IV decrease with temperature in the dilute-instanton-gas approximation.

For the SU(2)×U(1) electroweak theory of Sec. IV, the temperature-dependent coupling is explicitly

$$\mathcal{G}(T)=16\pi^2\left(\frac{8\pi^2}{g^2(\mu)}\right)^4\exp[-\alpha(1)+3\alpha(\tfrac{1}{2})]\,\mu^{35/6}\exp\left(\frac{-8\pi^2}{g^2(\mu)}\right)\int_0^\infty d\rho\,\rho^{41/6}(1+\pi^2\rho^2T^2/3)^{13/12} \times\exp\left\{-\left(2\pi^2F^2+\frac{7\pi^2T^2}{3}\right)\rho^2-13\alpha[1+\gamma(\pi\rho T)^{-3/2}]^{-8}\right\}. \tag{6.1}$$

[The constant α in the last exponent of (6.1) is given by (5.19) and is not related to $\alpha(1)$ and $\alpha(\frac{1}{2})$ in (4.9).] Introducing the dimensionless variable

$$x=f(T)\rho\equiv\left(2\pi^2F^2+\frac{7\pi^2T^2}{3}\right)^{1/2}\rho, \tag{6.2}$$

the coupling (6.1) may be written as

$$\mathcal{G}(T)=\mathcal{G}(0)\frac{2}{\Gamma(\frac{47}{12})}\left(1+\frac{7T^2}{6F^2}\right)^{-47/12}\int_0^\infty dx\,x^{41/6}\left(1+\frac{\pi^2T^2x^2}{3f^2(T)}\right)^{13/12}\exp\left\{-x^2-13\alpha\left[1+\gamma\left(\frac{\pi Tx}{f(T)}\right)^{-3/2}\right]^{-8}\right\}, \tag{6.3}$$

where $\mathcal{G}(0)$ is given by (4.8).

The above integral is easily evaluated numerically for any temperature T. In Fig. 4, the ratio $\mathcal{G}(0)/\mathcal{G}(T)$ is plotted as a function of temperature up to $T=10^{19}$ GeV (10^{32} °K), where gravity becomes important. The effective finite-temperature Lagrangian (5.29) is probably not physically valid above this temperature. Below $T=100$ GeV (10^{15} °K) the coupling is approximately temperature independent. At high temperatures ($T\gg 100$ GeV) the coupling decreases as a power of T,

$$\mathcal{G}(T\gg 100\text{ GeV})\approx 0.014\,046\,\mathcal{G}(0)(G^{1/2}T)^{-47/6}, \tag{6.4}$$

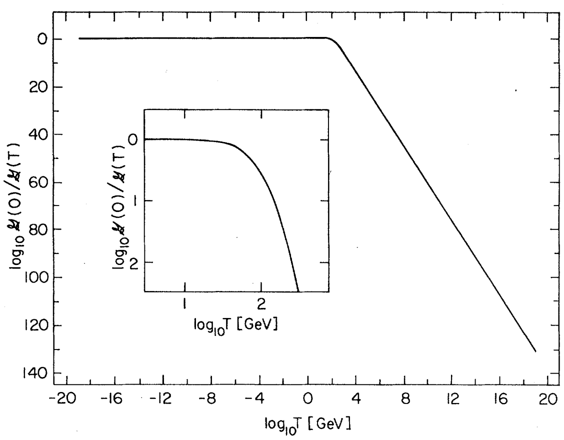

FIG. 4. The logarithm of the ratio of the four-fermion couplings $\mathcal{G}(0)/\mathcal{G}(T)$ as a function of the temperature T (in GeV). The inset figure is an enlargement of the knee region around $T = 100$ GeV (10^{15} °K).

where G is the Fermi coupling constant in (4.11).

The neutron lifetime for temperatures $T \gg 100$ GeV increases as

$$\tau(T) = 5.0687 \times 10^{3} (G^{1/2} T)^{47/3} \tau(0) . \qquad (6.5)$$

The cross sections in Table I at such temperatures decrease as

$$\sigma(T) = 1.9729 \times 10^{-4} (G^{1/2} T)^{-47/3} \sigma(0) . \qquad (6.6)$$

VII. DISCUSSION

Using 't Hooft's effective-Lagrangian method, it has been straightforward to quantitatively calculate the magnitude of various instanton-mediated baryon- and lepton-number-violating processes. To study more diverse reactions than we have, one can add more lepton doublets, and quark doublets can be used instead of our phenomenological proton-neutron doublet. One effect of additional SU(2) fermion doublets is to increase the coupling $\mathcal{G}(0)$. The magnitude of $\mathcal{G}(0)$ is governed by the factor $\exp[-8\pi^2/g^2(\mu)]$ in (4.8). For six flavors $g^2(100$ GeV) in (4.13) increases from a median value of 0.412 to 0.440 (neglecting threshold corrections). The coupling $\mathcal{G}(0)$ then increases from a median value of 10^{-83} GeV^{-2} to around 10^{-77} GeV^{-2}. Another consequence of more doublets is to cause $\mathcal{G}(T)$ to fall off faster at high temperature, specifically like

$$T^{d}, \quad d = -\frac{1}{3}\left[10 + 7N_f - \tfrac{1}{2}\sum_{I} N^{s}(I) C(I)\right] \qquad (7.1)$$

(the final term in the exponent d being due to the scalar fields). For six flavors and one Higgs isodoublet, $\mathcal{G}(T)$ falls off like $T^{-103/6}$.

The smallness of these rates is experimentally disappointing. It should be remembered that our results were obtained by considering only a single unit-winding-number instanton. As discussed by Berg and Luscher,[15] and Berg and Stehr,[16] the (zero-temperature) instanton gas is not dilute but in fact very dense. One would expect the probability for baryon- and lepton-number-violating processes to increase in such a dense gas because at any point in spacetime the number of instantons with which fermions could interact would be much greater than one. Presumably, generalizing the calculations of Berg *et al.* to finite temperature would allow one to determine the effect of temperature on these violating processes in a dense instanton gas.

As was noted in Sec. V, the currently known periodic Yang-Mills instantons become pure gauges at spatial infinity, and their classical tunneling amplitudes are temperature independent. Finite-energy tunneling paths would obey a different boundary condition and could be local minima of the Euclidean action independent of the above instantons. Their classical tunneling probability could conceivably increase with temperature. As the vacuum tunneling barrier in Fig. 1 suggests, such paths might be periodic non-self-dual solutions with noninteger winding numbers. Recently non-self-dual complex solutions with finite, complex Euclidean action have been found in an SU(2) gauge theory.[17] Periodic non-self-dual solutions may well exist too.

Finally, since all of our calculations assume thermodynamic equilibrium, nonequilibrium effects on our rates are unknown. Hence, although the smallness of these rates is not encouraging, one cannot rule out that finite-energy tunneling paths along with nonequilibrium and dense-instanton effects might increase them in, for example, heavy-ion collisions, dense stellar interiors, or the early universe. At the very least, it is gratifying to know that we can now perform some quantitative nonperturbative calculations.

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APPENDIX A: EFFECTIVE LAGRANGIAN FUNCTIONS

We wish to construct an effective Lagrangian satisfying

$$\int D\psi D\bar{\psi}\,\mathcal{L}_{\text{eff}}(\bar{\psi},\psi)e^{\bar{\psi}J\psi}=\det(\psi^*J\psi)\,. \tag{A1}$$

This can clearly be done if there is only one fermion doublet, since the effective Lagrangian must have the general form

$$\mathcal{L}_{\text{eff}}(\bar{\psi},\psi)=\int d^4y\,\bar{\psi}(y)A(y)\psi(y)\,, \tag{A2}$$

where $A(y)$ is to be determined. Substituting this into (A1) gives

$$\int D\psi D\bar{\psi}\int d^4y\,\bar{\psi}(y)A(y)\psi(y)e^{\bar{\psi}J\psi}=\psi^*J\psi\,. \tag{A3}$$

(The $\bar{\psi}$ and ψ on the left-hand side are fields, while the ψ^* and ψ on the right-hand side are zero-mode wave functions.)

When we expand the exponential in (A3) in powers of J, only the linear term will survive upon contraction with $\bar{\psi}(y)A(y)\psi(y)$:

$$\int d^4y\,\bar{\psi}(x)J(x)\psi(x)\bar{\psi}(y)A(y)\psi(y)=\psi^*J\psi \tag{A4}$$

or

$$-\text{tr}\int d^4y\,J(x)S_F(x-y)A(y)S_F(y-x)=\psi^*J\psi\,, \tag{A5}$$

where

$$S_F(x-y)=\frac{\gamma\cdot(x-y)}{2\pi^2(x-y)^4}\,. \tag{A6}$$

Using the zero-mode wave function (3.11) with the instanton at the origin, (A5) becomes

$$\begin{aligned}\text{tr}\int d^4y\,J(x)\frac{\gamma\cdot(x-y)A(y)\gamma\cdot(x-y)}{4\pi^4(x-y)^8}&=\frac{2\rho^3}{\pi^2}\text{tr}(u^*Ju)(\rho^2+x^2)^{-3}\\&=\frac{2\rho^3}{\pi^2}\text{tr}(Ju_\alpha u_\alpha^*)(\rho^2+x^2)^{-3}\\&=\frac{2\rho^3}{\pi^2}(\rho^2+x^2)^{-3}\,\text{tr}\left(J\frac{(1-\gamma_5)}{2}\right),\end{aligned} \tag{A7}$$

where we have used the identity[4]

$$\sum_\alpha u_\alpha\bar{u}_\alpha=\tfrac{1}{2}(1-\gamma_5)\,. \tag{A8}$$

For large x^2, then, we see that

$$A(y)=\frac{1+\gamma_5}{2}f(y)\,, \tag{A9}$$

where $f(y)$ satisfies

$$\int d^4y\frac{1}{4\pi^4}\frac{f(y)}{(x-y)^6}=\frac{2\rho^3}{\pi^2}\frac{1}{x^6}\,. \tag{A10}$$

This implies that $f(y)=8\pi^2\rho^3\delta^4(y)$ and

$$A(y)=8\pi^2\rho^3\frac{1+\gamma_5}{2}\delta^4(y)\,. \tag{A11}$$

Hence, for a single doublet, we find that (with the instanton at the origin, $z^\mu=0$)

$$\mathcal{L}_{\text{eff}}(\bar{\psi},\psi)=8\pi^2\rho^3\bar{\psi}(0)\frac{1+\gamma_5}{2}\psi(0)\,. \tag{A12}$$

In the two-doublet case, the zero-mode determinant is

$$\begin{aligned}\det(\psi^*J\psi)&=\begin{vmatrix}\psi_1^*J_{11}\psi_1 & \psi_1^*J_{12}\psi_2\\ \psi_2^*J_{21}\psi_1 & \psi_2^*J_{22}\psi_2\end{vmatrix}\\&=\left(\frac{2\rho^3}{\pi^2}\right)^2(\rho^2+x^2)^{-6}[\text{tr}(u^*J_{11}u)\text{tr}(u^*J_{22}u)-\text{tr}(u^*J_{12}u)\text{tr}(u^*J_{21}u)]\\&=\left(\frac{2\rho^3}{\pi^2}\right)^2(\rho^2+x^2)^{-6}\left[\text{tr}\left(J_{11}\frac{1-\gamma_5}{2}\right)\text{tr}\left(J_{22}\frac{1-\gamma_5}{2}\right)-\text{tr}\left(J_{12}\frac{1-\gamma_5}{2}\right)\text{tr}\left(J_{21}\frac{1-\gamma_5}{2}\right)\right].\end{aligned} \tag{A13}$$

Drawing upon our experience with the one doublet case, we take the effective Lagrangian to be of the general form

$$\mathcal{L}_{\text{eff}}(\bar{\psi},\psi)=(8\pi^2\rho^3)^2\left(\bar{\psi}_1^\alpha\frac{1+\gamma_5}{2}\psi_1^\beta\bar{\psi}_2^\gamma\frac{1+\gamma_5}{2}\psi_2^\sigma A_{\alpha\beta\gamma\sigma}+\bar{\psi}_1^\alpha\frac{1+\gamma_5}{2}\psi_2^\beta\bar{\psi}_2^\gamma\frac{1+\gamma_5}{2}\psi_1^\sigma B_{\alpha\beta\gamma\sigma}\right) \tag{A14}$$

for large x^2. The only terms from $e^{\bar{\psi}J\psi}$ that will survive when contracted with $\mathcal{L}_{\text{eff}}(\bar{\psi},\psi)$ are

$$\bar{\psi}_1^\delta J_{11}\psi_1^\delta\bar{\psi}_2^\lambda J_{22}\psi_2^\lambda+\bar{\psi}_1^\delta J_{12}\psi_2^\delta\bar{\psi}_2^\lambda J_{21}\psi_1^\lambda\,. \tag{A15}$$

This contraction yields, when equated to (A13),

$$(8\pi^2\rho^3)^2\{[\delta_{\alpha\beta}\delta_{\gamma\sigma}\,\mathrm{tr}(J_{11}S_F\Lambda_+S_F)\mathrm{tr}(J_{22}S_F\Lambda_+S_F)-\delta_{\alpha\sigma}\delta_{\beta\gamma}\,\mathrm{tr}(J_{12}S_F\Lambda_+S_FJ_{21}S_F\Lambda_+S_F)]A_{\alpha\beta\gamma\sigma}$$
$$+[-\delta_{\alpha\sigma}\delta_{\beta\gamma}\,\mathrm{tr}(J_{11}S_F\Lambda_+S_FJ_{22}S_F\Lambda_+S_F)+\delta_{\alpha\beta}\delta_{\gamma\sigma}\,\mathrm{tr}(J_{12}S_F\Lambda_+S_F)\mathrm{tr}(J_{21}S_F\Lambda_+S_F)]B_{\alpha\beta\gamma\sigma}\}$$
$$=\left(\frac{2\rho^3}{\pi^2}\right)^2(x^2)^{-6}[\mathrm{tr}(J_{11}\Lambda_-)\mathrm{tr}(J_{22}\Lambda_-)-\mathrm{tr}(J_{12}\Lambda_-)\mathrm{tr}(J_{21}\Lambda_-)]\,, \quad \text{(A16)}$$

where

$$\Lambda_\pm=\tfrac{1}{2}(1\pm\gamma_5)\,. \quad \text{(A17)}$$

Because of the symmetry in the isospin indices of (A16), the most general forms for $A_{\alpha\beta\gamma\sigma}$ and $B_{\alpha\beta\gamma\sigma}$ we need consider are

$$A_{\alpha\beta\gamma\sigma}=a_1\delta_{\alpha\beta}\delta_{\gamma\sigma}+a_2\delta_{\alpha\sigma}\delta_{\beta\gamma} \quad \text{(A18)}$$

and

$$B_{\alpha\beta\gamma\sigma}=b_1\delta_{\alpha\beta}\delta_{\gamma\sigma}+b_2\delta_{\alpha\sigma}\delta_{\beta\gamma}\,. \quad \text{(A19)}$$

Substituting these into (A16), one finds that

$$\delta_{\alpha\beta}\delta_{\gamma\sigma}A_{\alpha\beta\gamma\sigma}=4a_1+2a_2=1\,,\qquad \delta_{\alpha\sigma}\delta_{\beta\gamma}A_{\alpha\beta\gamma\sigma}=2a_1+4a_2=0 \quad \text{(A20)}$$

and

$$\delta_{\alpha\sigma}\delta_{\beta\gamma}B_{\alpha\beta\gamma\sigma}=2b_1+4b_2=0\,,\qquad \delta_{\alpha\beta}\delta_{\gamma\sigma}B_{\alpha\beta\gamma\sigma}=4b_1+2b_2=-1\,. \quad \text{(A21)}$$

The solutions are

$$a_1=\tfrac{1}{3},\quad a_2=-\tfrac{1}{6},\quad b_1=-\tfrac{1}{3},\quad b_2=\tfrac{1}{6}. \quad \text{(A22)}$$

The effective Lagrangian (A14) is then

$$\mathcal{L}_{\mathrm{eff}}(\overline{\psi},\psi)=\frac{(8\pi^2\rho^3)^2}{6}(2\delta_{\alpha\beta}\delta_{\gamma\sigma}-\delta_{\alpha\sigma}\delta_{\beta\gamma})\times(\overline{\psi}{}_1^\alpha\Lambda_+\psi_1^\beta\overline{\psi}{}_2^\gamma\Lambda_+\psi_2^\sigma-\overline{\psi}{}_1^\alpha\Lambda_+\psi_2^\beta\overline{\psi}{}_2^\gamma\Lambda_+\psi_1^\sigma)\,. \quad \text{(A23)}$$

This method can be extended straightforwardly to more doublets.

APPENDIX B: PROTON-NEUTRON ANNIHILATION AND FREE-NEUTRON DECAY

A. $p+n\to\overline{e}+\overline{\nu}$

The four-momenta of p, n, $\overline{e}$, and $\overline{\nu}$ will be denoted p_1, p_2, p_3, and p_4, respectively. Using the fact that

$$\overline{\psi}{}^c=\sum_{p,s}(b\overline{v}e^{-ip\cdot x}+d^\dagger\overline{u}e^{ip\cdot x})\,,\qquad \psi=\sum_{p,s}(bue^{-ip\cdot x}+d^\dagger ve^{ip\cdot x})\,, \quad \text{(B1)}$$

the Feynman amplitude for $p+n\to\overline{e}+\overline{\nu}$ arising from (4.22) is

$$M=\frac{\mathcal{G}}{3}(\overline{v}_1\Lambda_-u_2\overline{u}_3\Lambda_-v_4-\overline{v}_1\Lambda_-v_4\overline{v}_2\Lambda_-v_3+\overline{v}_1\Lambda_-v_3\overline{v}_2\Lambda_-v_4)\,. \quad \text{(B2)}$$

Owing to the presence of u and v spinors, one will get cross terms such as

$$\overline{v}_1\Lambda_-u_2\overline{u}_3\Lambda_-v_4\overline{v}_3\Lambda_+v_2\overline{v}_4\Lambda_+v_1 \quad \text{(B3)}$$

in the expression for $|M|^2$. Use of the identities[12]

$$v=C\overline{u}^T,\quad \overline{u}=(C^{-1}v)^T,\qquad u=C\overline{v}^T,\quad \overline{v}=(C^{-1}u)^T, \quad \text{(B4)}$$

[where C is given by (4.21)] will always transform these into the more familiar forms in which the usual Dirac-Algebra identities are applicable. In the case of (B3), for example, one finds

$$\overline{v}_1\Lambda_-u_2\overline{u}_3\Lambda_-v_4\overline{v}_3\Lambda_+v_2\overline{v}_4\Lambda_+v_1=-\overline{v}_1\Lambda_-u_2\overline{u}_2\Lambda_+u_3\overline{u}_3\Lambda_-v_4\overline{v}_4\Lambda_+v_1\,. \quad \text{(B5)}$$

Averaging over initial spin states and summing over all final spin states, the amplitude (B2) squared becomes

$$\langle|M|^2\rangle=\tfrac{1}{4}\sum_{\text{spins}}|M|^2=\frac{\mathcal{G}^2s^2}{9}\,, \quad \text{(B6)}$$

where $\sqrt{s}$ is the total energy in the center-of-momentum frame. The invariant differential cross section is defined as

$$\frac{d\sigma}{dt}=\frac{\pi}{p^2}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{c.m.}}=\frac{1}{64\pi p^2s}\langle|M|^2\rangle\,, \quad \text{(B7)}$$

where $p=|\vec{p}_1|_{\mathrm{c.m.}}$. In the negligible-mass limit, $p^2=s/4$, and we obtain, for $p+n\to\overline{e}+\overline{\nu}$,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{c.m.}}=\frac{\mathcal{G}^2s}{576\pi^2}\,, \quad \text{(B8)}$$

$$\frac{d\sigma}{dt}=\frac{\mathcal{G}^2}{144\pi}\,, \quad \text{(B9)}$$

$$\sigma_{\mathrm{total}}=\int_{-s}^{0}\frac{d\sigma}{dt}\,dt=\frac{\mathcal{G}^2s}{144\pi}. \quad \text{(B10)}$$

B. $n\to\overline{p}+\overline{e}+\overline{\nu}$

The four-momenta of n, $\overline{p}$, $\overline{e}$, and $\overline{\nu}$ will be denoted p_1, p_2, p_3, and p_4, respectively. The Feynman amplitude for $n\to\overline{p}+\overline{e}+\overline{\nu}$ arising from (4.22) is

$$M=\frac{g}{3}(\bar{u}_2\Lambda_-u_1\bar{u}_3\Lambda_-v_4-\bar{u}_2\Lambda_-v_4\bar{v}_1\Lambda_-v_3+\bar{u}_2\Lambda_-v_3\bar{v}_1\Lambda_-v_4). \quad \text{(B11)}$$

Averaging over the initial neutron spin states and summing over all final spin states, the amplitude (B11) squared is found to be

$$\langle |M|^2\rangle=\frac{1}{2}\sum_{\text{spins}}|M|^2=\frac{8}{9}g^2(p_1\cdot p_2)^2. \quad \text{(B12)}$$

In neutron decay, the $\bar{p}$ carries away a negligible amount of kinetic energy in the neutron rest frame. In this frame, the n and $\bar{p}$ are approximately at rest. Equation (B12) becomes

$$\langle |M|^2\rangle=\frac{8}{9}g^2m_n^2m_p^2. \quad \text{(B13)}$$

The neutron decay width is

$$\Gamma=\frac{1}{2m_n}\int\langle |M|^2\rangle d\Phi_3, \quad \text{(B14)}$$

where the three-body phase-space factor is

$$d\Phi_3=(2\pi)^4\delta^4(p_1-p_2-p_3-p_4)\frac{d^3p_2d^3p_3d^3p_4}{(2\pi)^9 8E_2E_3E_4}. \quad \text{(B15)}$$

Using the fact that the n and $\bar{p}$ are nearly at rest, and that the $\bar{\nu}$ is (essentially) massless, the integral (B14) can be evaluated exactly to yield

$$\int d\Phi_3=\frac{1}{16\pi^3m_p}\left\{\frac{E_0}{2}\left[E_0(E_0^2-m_3^2)^{1/2}-m_3^2\ln\left(\frac{E_0+(E_0^2-m_3^2)^{1/2}}{m_3}\right)\right]-\frac{1}{3}(E_0^2-m_3^2)^{3/2}\right\} \quad \text{(B16)}$$

$$=\frac{1}{16\pi^3m_p}(1.678\,58\times10^{-10}\ \text{GeV}^3), \quad \text{(B17)}$$

where E_0 is the neutron-proton rest-mass difference, and m_3 is the electron rest mass. The decay width is then

$$\Gamma=\frac{g^2m_nm_p}{36\pi^3}(1.678\,58\times10^{-10}\ \text{GeV}^3). \quad \text{(B18)}$$

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