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Petite unification of quarks and leptons

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A general discussion of a quark-lepton unification characterized by the gauge group $G_S \otimes G_W$ with two coupling constants g_S and g_W and by the unification mass scale $M = 10^{5\pm1}$ GeV is presented. The choice of G_W is quite restricted by the measured value of $\sin^2\theta_W$. The minimal model of such a unification turns out to be $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_{L'} \otimes SU(2)_{R'}$, where the first three factors constitute the well-known Pati-Salam group. The presence of $SU(2)_{L'} \otimes SU(2)_{R'}$ is required by the measured value of $\sin^2\theta_W$ and it implies the existence of mirror fermions whose masses may range from 20−30 GeV to a few TeV. The lightest mirror fermion might be relatively long lived when compared to an ordinary sequential heavy fermion. The model accommodating all known quark and lepton generations gives the correct $\sin^2\theta_W \approx 0.22$ and at the same time can be made consistent with the experimental bounds on rare transitions induced by leptoquark exchanges.

I. INTRODUCTION

If it is true that at presently attainable energies the strong and electroweak forces are described by an $SU(3) \otimes SU(2) \otimes U(1)$ gauge theory, then it is natural to expect further synthesis at higher energies. At present, the most attractive candidates for such a synthesis are the grand-unification schemes,[1−5] in particular SU(5).[2] In those schemes, strong and electroweak interactions become unified at a very high mass scale of $\sim 10^{15}$ GeV, with quite possibly relatively little new phenomena populating the energy region of $\sim 10^3$ to $\sim 10^{15}$ GeV.[6] There are good reasons, such as the approximate agreement of the measured value of $\sin^2\theta_W$ with the theoretical expectations[2,7] and the economical assignment of known fermions to SU(5) representations, to take the grand-unification idea very seriously. However, there are also well-known difficulties, in particular the large number of arbitrary parameters and especially the lack of a credible scenario of spontaneous symmetry breaking.

Given this situation, it may be of importance to carefully examine less ambitious alternatives. Our purpose here is to examine a limited class of such alternatives. We assume that at some distance scale, not too many orders of magnitude less than the Compton wavelength of the intermediate bosons $W^\pm$ and Z^0, the $SU(3) \otimes SU(2) \otimes U(1)$ gauge theory, characterized by three coupling constants, becomes embedded in a gauge theory $G_S \otimes G_W$, characterized by only two coupling constants g_S and g_W. That is, we assume the strong group G_S and weak group G_W each are either simple or pseudosimple, i.e., a direct product of simple groups with identical coupling strengths. We call such a possiblity *petite unification.* Any subsequent unification of the strong force with the weak at still shorter distances we shall leave unconsidered.

It turns out that it is not easy to find realistic models of such petite-unification schemes, at least in a reasonable economical fashion. We shall argue that the best candidate theory is based upon a Pati-Salam SU(4) strong group, where lepton number plays the role of a fourth color. The weak group G_W can be $[SU(2)]^4$ or $[SU(4)]^2$ [or, if one does not mind extravagance, SU(8)].[8]

It turns out that the choice of G_W is quite restricted. The charge operator is evidently a linear combination of generators of G_S and G_W, with coefficients of order unity. This implies that the electromagnetic potential A_μ is a linear combination of strong gauge fields with coefficient of order e/g_S, and of weak gauge fields with coefficient e/g_W. It follows that in the limit of $g_S >> g_W$, the electromagnetic gauge field resides almost completely in G_W. Because of this, a representation of $G_S \otimes G_W$ which is singlet under G_S will have the same relationship to the $SU(2) \otimes U(1)$ electroweak generators as would be the case were A_μ completely contained within G_W. In particular, the relation of Georgi, Quinn, and Weinberg,[2]

$$\sin^2\theta_W = \frac{\sum T_3{}^2}{\sum Q^2}, \tag{1.1}$$

where the sum goes over the members of a representation of G_W (for instance the adjoint representation), survives almost intact. Specifically we find

$$\sin^2\theta_W = \left.\frac{\sum T_3{}^2}{\sum Q^2}\right|_{\text{Adj of } G_W} + O\left[\frac{\alpha}{\alpha_S}\right]$$

$$+ \text{renormalization-group corrections}\,. \tag{1.2}$$

For the standard Weinberg-Salam SU(2)

$$\left.\frac{\sum T_3{}^2}{\sum Q^2}\right|_{\text{adj}} = 1 \tag{1.3}$$

and we therefore see that a considerable enlargement of G_W is required. For example, for the group $[\text{SU}(2)]^n$, one obtains

$$\left.\frac{\sum T_3{}^2}{\sum Q^2}\right|_{\text{adj}} \le \frac{1}{n}\,. \tag{1.4}$$

A detailed analysis of options for G_W is given in Secs. III and IV; the smallest acceptable G_W does turn out to be $[\text{SU}(2)]^4$. Coupling-constant renormalizations must also be considered; this is done in Sec. VI. The qualitative behavior is unaffected by these modifications, especially if one chooses the petite-unification mass scale not to be inordinately large.

What about G_S? Can it be $\text{SU}(3)_c$? If so, then the photon would be contained entirely within G_W, and the Georgi-Quinn-Weinberg formula, Eq. (1.1), would apply to *all* representations of G_W—including fermions. For example, all left-handed color-triplet fermions would necessarily form a representation of G_W. The known ones (e.g., u_L, d_L) satisfy

$$\left.\frac{\sum T_3{}^2}{\sum Q^2}\right|_{\text{color}\underline{3},L} = \frac{9}{10}, \tag{1.5}$$

implying at the least a large number of additional quarks and/or new quarks of charge > 1. We shall not consider $\text{SU}(3)_c$ further, but shall choose instead SU(4), built in the manner of Pati and Salam.[1] This provides what appears to be the most efficient choice of a strong group, and is the simplest case for which the electroweak U(1) generator is a linear combination of both G_S and G_W generators.

Hereafter in this introduction, we restrict our attention to $\text{SU}(4)\otimes[\text{SU}(2)]^4$ and inquire as to the particle content of such a "minimal" model of petite unification. The first two of the four electroweak SU(2) groups may be identified with the $\text{SU}(2)_L\otimes\text{SU}(2)_R$ of a conventional left-right-symmetric model.[9] The final $\text{SU}(2)_{L'}\otimes\text{SU}(2)_{R'}$ pair can be associated with similar "mirror" degrees of freedom which do not couple to the conventional $W^\pm$ (or its heavier right-handed counterpart). More specifically, for each generation, we introduce a set of two-component Weyl fermions Ψ_i^α ($\alpha=1,2,3,4$; $i=1,2$) transforming as $(4;\,2,1;\,1,1)_L + (\bar{4};\,1,2;\,1,1)_L$; e.g.,

$$\Psi = \begin{bmatrix} u_1 & d_1 \\ u_2 & d_2 \\ u_3 & d_3 \\ \nu_e & e^- \end{bmatrix}. \tag{1.6}$$

The mirror fields,[10] whose masses may range from 20–30 GeV to a few TeV, then transform as $(4;\,1,1;\,2,1)_L + (\bar{4};\,1,1;\,1,2)_L$:

$$\Psi' = \begin{bmatrix} U_1 & D_1 \\ U_2 & D_2 \\ U_3 & D_3 \\ N & E^- \end{bmatrix}. \tag{1.7}$$

The existence of these mirror fermions is required by the permutation symmetry among the SU(2)'s assumed *ab initio*.

From the form of Ψ we see that leptons provide the fourth color degree of freedom. It is easy to arrange spontaneous symmetry breakdown of SU(4) to SU(3). For example, a real adjoint Higgs representation $\underline{15}$ provides the six leptoquark bosons with mass, and does not generate any baryon-number violations. It also generates no fermion mass. The leptoquark mass scale must be quite large (> 100 TeV) in order not to produce unacceptable neutral-current interactions. The phenomenology is discussed in Sec. VII.

The 15th generator of SU(4) is proportional to $B-L$.[11] It combines with the sum of the four electroweak T_3 generators to become the electromagnetic field, in accordance with the generalized Gell-Mann–Nishijima relation

$$Q=\frac{1}{2}(B-L)+T_{3L}+T_{3R}+T_{3L'}+T_{3R'} \ . \quad (1.8)$$

The mass generation of the twelve electroweak gauge bosons can be accomplished by four Higgs multiplets transforming, respectively, as (4; 2,1; 1,1), (4; 1,2; 1,1), ...(4; 1,1; 1,2). Other SU(4)-singlet representations of the $(2,\bar{2})$ type may also be introduced to provide the fermion mass. However, the overall scheme for fermion-mass generation must be quite complicated in order to account for acceptable ν masses, full SU(4) breaking in the mass matrices within a generation, Cabibbo mixing, and the fermion-mass hierarchy among generations. While the scheme itself does suggest a variety of avenues to explore, the study of the fermion-mass generation is beyond the scope of this paper.

The paper is organized as follows: In Sec. II, we present in more detail the general structure of petite unification, we list our main assumptions, and we derive Eq. (1.2). In Sec. III, we study the restrictions on the choice of G_W coming from the observed value of $\sin^2\theta_W$. In Sec. IV we enlarge this consideration to include fermion representations, and show that, upon assuming SU(4) for the strong group, many possible candidates for G_W are eliminated, and that $SU(4)\otimes[SU(2)]^4$ seems to be the simplest candidate for a petite-unification model. In Sec. V we present the fermion, gauge-boson, and Higgs-boson content of the $SU(4)\otimes[SU(2)]^4$ case, and we outline spontaneous symmetry breakdown of this gauge group down to $SU(3)_c\otimes U(1)_{EM}$. In Sec. VI, we study, via a renormalization-group analysis, the predicted value of $\sin^2\theta_W$. For $SU(4)\otimes[SU(2)]^4$, the value agrees with experiment for a wide range of parameters. Section VII is devoted to phenomenological implications. The most important of these are a consequence of the lepton-quark unification at a relatively low mass scale, and are rare flavor-changing decays and neutral-current processes induced by leptoquark exchange. We also briefly discuss the phenomenology of the lightest of the mirror fermions whose masses could be as low as 20–30 GeV. Section VIII consists of concluding remarks.

II. BASIC STRUCTURE

A. Basic assumptions

We shall first list our main assumptions without yet specifying the fermion and gauge-boson content of the theory.

(A) We choose as the unifying gauge group

$$G=G_S(g_S)\otimes G_W(g_W) \ , \quad (2.1)$$

where G_S and G_W stand for the strong group and the weak group, respectively, and g_S and g_W denote the corresponding couplings. We assume G_S and G_W each are either simple or pseudosimple, i.e., a direct product of simple groups with identical couplings.

(B) We assume that G is broken down to $SU(3)_c\otimes U(1)_{EM}$ according to the following symmetry-breaking pattern:

$$G\underset{M}{\rightarrow}G_1\underset{\tilde{M}}{\rightarrow}G_2\underset{M_W}{\rightarrow}SU(3)_c\otimes U(1)_{EM} \ . \quad (2.2)$$

Here

$$G_1=SU(3)_c(g_3)\otimes\tilde{G}_S(\tilde{g}_S)\otimes G_W(g_W) \quad (2.3)$$

with $SU(3)_c\otimes\tilde{G}_S\subset G_S$, and

$$G_2=SU(3)_c(g_3)\otimes SU(2)_L(g_2)\otimes U(1)(g') \ , \quad (2.4)$$

where G_2 represents the "standard" model, i.e., quantum chromodynamics (QCD) for the strong interactions and the standard $SU(2)\otimes U(1)$ model[12] for the electroweak interactions. Furthermore, the scales at which the symmetry breakings occur satisfy

$$M_W<\tilde{M}\leq M \quad (2.5)$$

with M_W being of the order of the W-boson mass. It should be remarked that in principle the breakdown of the group G could directly occur down to the group G_2. In order to have our discussion more general we shall allow for an intermediate stage at which the group G_1 is an unbroken gauge group. Such a hierarchy also seems to be a requirement forced by data on $\sin^2\theta_W$ and rare decay processes—at least for the most economical models.

Furthermore, in accordance with our early petite-unification idea we require the following.

(C) M and $\tilde{M}$ are only a few orders of magnitude larger than M_W.

(D) The weak hypercharge U(1) group of Eq. (2.4) merges into both $\tilde{G}_S$ and G_W at the mass scale $\tilde{M}$.

Requirement (D) allows us to put quarks and leptons into identical representations of the weak group G_W and consequently make the quarks and leptons to be indistinguishable when the strong interactions are turned off. Notice that if U(1) were totally embedded into G_W such a unification would not be possible.[13] Quarks and leptons would then have to form entirely different representations under G_W, in order to account for the disparity in quark and lepton charges. Also, the total embedding of U(1) into $\widetilde{G}_S$ is unacceptable, since this would lead to a large value of $\sin^2\theta_W$. When U(1) is totally embedded into $\widetilde{G}_S$, $\sin^2\theta_W^0$ of Eq. (2.20) is equal unity. Even for M, $\widetilde{M} \cong 10^{15}$ GeV the resulting $\sin^2\theta_W(M_W^2)$ is larger than 0.4.

As far as SU(2) is concerned, it is embedded into the group G_W but the following two cases can be considered.

(E1) "Unlocked standard model," in which the generators of $SU(2)_L$ are the unbroken generators of G_W,

(E2) "Locked standard model," in which the generators of $SU(2)_L$ are the unbroken combinations of generators belonging to several disjoint SU(2) subgroups of G_W.

Finally, without loss of generality, we assume the following.

(F) $SU(3)_c$ and $\widetilde{G}_S$ are unbroken subgroups of G_S so that their generators are unbroken generators of G_S.

B. Diagonal generators and normalization conditions

In view of the derivation of Eq. (1.2) for $\sin^2\theta_W$, which we present in Sec. II C, we shall now introduce some necessary notations and list certain normalization conditions which relate various couplings in (2.1)–(2.4) at the mass scales M, $\widetilde{M}$, and M_W. For the purpose of Sec. II C only the diagonal generators need to be considered.

The electric-charge generator Q of $U(1)_{EM}$ is given as usual by

$$Q = T_{3L} + T_0 \ , \tag{2.6}$$

where T_{3L} and T_0 are diagonal generators of $SU(2)_L$ and U(1), respectively. T_{3L} and T_0 can be generally written as follows:

$$T_{3L} = \sum_\alpha C'_{\alpha W} T^0_{\alpha W} \tag{2.7}$$

and

$$T_0 = \sum_\alpha C_{\alpha W} T^0_{\alpha W} + \sum_i C_{iS} \widetilde{T}^0_{iS} \ , \tag{2.8}$$

where $T^0_{\alpha W}$ and $\widetilde{T}^0_{iS}$ are the diagonal generators of G_W and $\widetilde{G}_S$, respectively. The sets $C_{\alpha W}$ and $C'_{\alpha W}$ are orthogonal to each other. The generators $T^0_{\alpha W}$ in Eq. (2.7) are the generators of the disjoint SU(2) subgroups of G_W (see the case E2). In the case of the unlocked standard model (E1) Eq. (2.7) reads as

$$T_{3L} = T^0_{3W} \ , \tag{2.9}$$

where T^0_{3W} is a diagonal generator of one of SU(2) subgroups of G_W.

We shall normalize $T^0_{\alpha W}$ as follows:

$$\mathrm{Tr}(T^0_{\alpha W} T^0_{\beta W}) = \lambda \delta_{\alpha\beta} \ , \tag{2.10}$$

where λ depends on the representation.

Corresponding to Eq. (2.6) we have the following known relation:

$$\frac{1}{e^2(M_W{}^2)} = \frac{1}{g_2{}^2(M_W{}^2)} + \frac{1}{g'(M_W{}^2)^2} \ . \tag{2.11}$$

It is not difficult to derive analogous relations (normalization conditions) corresponding to Eqs. (2.7) and (2.8). In this respect the formalism developed by Weinberg[14] is particularly useful. We obtain

$$\frac{1}{[g_2(\widetilde{M}^2)]^2} = \frac{\sum_\alpha C'_{\alpha W}{}^2}{[g_W(\widetilde{M}^2)]^2} \ , \tag{2.12}$$

$$\frac{1}{[g'(\widetilde{M}^2)]^2} = \frac{\sum_\alpha C_{\alpha W}{}^2}{[g_W(\widetilde{M}^2)]^2} + \frac{\sum_i C_{iS}{}^2}{[\widetilde{g}_S(\widetilde{M}^2)]^2} \ . \tag{2.13}$$

Furthermore, the assumption (F) implies

$$g_3(M^2) = \widetilde{g}_S(M^2) = g_S(M^2) \ . \tag{2.14}$$

We are now in a position to derive Eq. (1.2).

C. Basic equation for $\sin^2\theta_W$

We first define $\sin^2\theta_W(M_W{}^2)$ by

$$\sin^2\theta_W(M_W{}^2) = \frac{e^2(M_W{}^2)}{g_2{}^2(M_W{}^2)} \ . \tag{2.15}$$

In order to derive Eq. (1.2) we use the evolution equations for various effective coupling constants. In the one-loop approximation to the relevant renormalization-group β functions, neglecting fermion-mass effects, and upon using the normalization conditions (2.11)–(2.14), these evolution equations read as follows:

$$\frac{1}{[g'(M_W{}^2)]^2} = \frac{C_W{}^2}{g_W{}^2(\widetilde{M}^2)} + \frac{C_S{}^2}{\widetilde{g}_S{}^2(\widetilde{M}^2)} + 2b_1 \ln\frac{\widetilde{M}}{M_W} \ , \tag{2.16}$$

$$\frac{1}{[g_2(M_W^2)]^2}=\frac{C_W'^2}{g_W^2(\widetilde{M}^2)}+2b_2\ln\frac{\widetilde{M}}{M_W}\ , \quad (2.17)$$

$$\frac{1}{g_3^2(M_W^2)}=\frac{1}{g_S^2(M^2)}+2b_3\ln\frac{M}{M_W}\ , \quad (2.18)$$

and

$$\frac{1}{\widetilde{g}_S^2(\widetilde{M}^2)}=\frac{1}{g_S^2(M^2)}+2\widetilde{b}\ln\frac{M}{\widetilde{M}}\ . \quad (2.19)$$

In order to simplify notations we have introduced $C_S^2\equiv\sum C_{is}^2$ and similarly for C_W and C_W'. The parameters b_i and $\widetilde{b}$ are the coefficients of g_i^3 in the relevant renormalization-group β_i functions. Explicit expressions for these coefficients are given in Sec. VI. There we shall also briefly discuss the fermion-mass effects.

Combining Eqs. (2.15)–(2.19) we find

$$\sin^2\theta_W(M_W^2)=\sin^2\theta_W^0\left[1-C_S^2\frac{\alpha(M_W^2)}{\alpha_S(M_W^2)}-8\pi\alpha(M_W^2)\left[K\ln\frac{\widetilde{M}}{M_W}+K'\ln\frac{M}{\widetilde{M}}\right]\right]\ , \quad (2.20)$$

where

$$\sin^2\theta_W^0=\frac{C_W'^2}{C_W^2+C_W'^2}\ , \quad (2.21)$$

$$K'=C_S^2(\widetilde{b}-b_3)\ , \quad (2.22)$$

$$K=b_1-\frac{C_W^2}{C_W'^2}b_2-C_S^2b_3\ , \quad (2.23)$$

and

$$\alpha(M_W^2)\equiv\frac{e^2(M_W^2)}{4\pi}\ ,\qquad \alpha_S(M_W^2)\equiv\frac{g_3^2(M_W^2)}{4\pi}\ . \quad (2.24)$$

We note next that for any representation of G_W which is singlet under G_S (for instance for the adjoint representation of G_W) the second sum in Eq. (2.8) is inoperative. Equations (2.6)–(2.9) lead then to

$$[(C_W)^2+(C_W')^2]^{-1}=\left[\frac{\mathrm{Tr}T_3^2}{\mathrm{Tr}Q^2}\right]_{\mathrm{adj}}\ , \quad (2.25)$$

where T_3 belongs to any subgroup of G_W. Thus finally our basic formula for $\sin^2\theta_W$ is given by (2.20) with

$$\sin^2\theta_W^0=\left[\frac{\mathrm{Tr}T_3^2}{\mathrm{Tr}Q^2}\right]_{\mathrm{adj}}C_W'^2\ . \quad (2.26)$$

For the unlocked standard model (E1) $C_W'^2=1$ and Eqs. (2.20) and (2.26) give the formula (1.2). In the case of the locked standard model (E2) $C_W'^2\neq1$ and Eq. (1.2) is modified by an overall factor in addition to a change in the parameter K.

D. Strategy

In this section we have stated our assumptions and we derived a general formula for $\sin^2\theta_W(M_W^2)$. In the next few sections we shall look for acceptable groups G_S and G_W. A necessary condition for our scheme is that it should give a value of $\sin^2\theta_W$ consistent with the experimentally measured value[15]

$$(\sin^2\theta_W)_{\mathrm{exp}}=0.23\pm0.015\ . \quad (2.27)$$

Recent calculations of Ref. 16 show that $(\sin^2\theta_W)_{\mathrm{exp}}$ of Eq. (2.27) which is measured in low-energy experiments is related to $\sin^2\theta_W(M_W^2)$ by

$$\sin^2\theta_W(M_W^2)=0.95\,(\sin^2\theta_W)_{\mathrm{exp}}\approx0.22\pm0.014\ . \quad (2.28)$$

Equation (2.20) together with $K,K'>0$ (see Sec. VI) tells us that

$$\sin^2\theta_W(M_W^2)=R\,\sin^2\theta_W^0 \quad (2.29)$$

with $R<1$. Furthermore, for $M\approx10^{5\pm1}$ GeV and $\widetilde{M}\approx300$ GeV–few TeV,

$$R\approx\begin{cases}0.95, & C_S^2=\frac{1}{6}\ ,\\ 0.85, & C_S^2=\frac{2}{3}\ ,\\ 0.65, & C_S^2=\frac{8}{3}\ ,\end{cases} \quad (2.30)$$

where the numerical values of C_S^2 are the only ones encountered in our study (see Sec. IV). Consequently only the gauge groups which give

$$0.23<\sin^2\theta_W^0<0.30 \quad (2.31a)$$

for $C_S^2=\frac{1}{6},\ \frac{2}{3}$, and

$$0.30 < \sin^2\theta_W^0 < 0.40 \tag{2.31b}$$

for $C_S{}^2 = \frac{8}{3}$ have a chance to satisfy Eq. (2.28). Therefore, our first task will be to find gauge groups G_W which have $\sin^2\theta_W^0$ consistent with Eq. (2.31).

III. $\sin^2\theta_W^0$ AND G_W

The aim of this section is to derive a formula for $\sin^2\theta_W^0$ for $G_W = [\mathrm{SU}(N)]^k$ and subsequently find which pairs (N,k) satisfy Eq. (2.31). We begin with the unlocked standard model.

A. Unlocked standard model

In this case we have

$$C_W'{}^2 = 1 , \tag{3.1}$$

and consequently

$$\sin^2\theta_W^0 = \left[\frac{\mathrm{Tr}T_{3L}{}^2}{\mathrm{Tr}Q^2}\right]_{\mathrm{adj}} = \frac{1}{1+C_W{}^2} , \tag{3.2}$$

where T_{3L} is the diagonal generator of $\mathrm{SU}(2)_L$. This equation determines $C_W{}^2$ once $\sin^2\theta_W^0$ is known.

Now in the case of $G_W = \tilde{G} \otimes \tilde{G} \cdots \tilde{G}$, where there are p identical factors of $\tilde{G}$, the "charge" generator for the adjoint representation of G_W can be written as

$$Q_{\mathrm{adj}} = \sum_{\sigma=1}^{p} Q_W(\sigma)_{\mathrm{adj}} . \tag{3.3}$$

Here $Q_W(\sigma)$ corresponds to the σth factor $\tilde{G}$ and is given in an obvious notation as follows:

$$Q_W(\sigma)_{\mathrm{adj}} = \sum_i C_{iW}(\sigma) T_{iW}^0(\sigma) , \tag{3.4}$$

where i runs over all diagonal generators in the σth factor $\tilde{G}$.

Using $\mathrm{Tr}[Q_W(\sigma)Q_W(\sigma')] = \mathrm{Tr}[Q_W{}^2(\sigma)]\delta_{\sigma\sigma'}$, we obtain

$$\mathrm{Tr}(Q^2)\,|_{\mathrm{adj}} = \sum_{\sigma=1}^{p} \mathrm{Tr}[Q_W{}^2(\sigma)]_{\mathrm{adj}} . \tag{3.5}$$

Notice that $\mathrm{Tr}[Q_W{}^2(\sigma)]_{\mathrm{adj}}$ can take on different values for different σ.

Next $(\mathrm{Tr}T_{3L}{}^2)_{\mathrm{adj}}$ is obtained by considering any of the factors $\tilde{G}$ to which the standard $\mathrm{SU}(2)_L$ belongs. We find $[\mathrm{Tr}(T_{3L}{}^2) = \frac{1}{2}$ for the fundamental representation]

$$(\mathrm{Tr}T_{3L}{}^2)_{\mathrm{adj}} = \sum_{b,c} f_{3bc} f_{3bc} = C_2(\tilde{G}) , \tag{3.6}$$

where $C_2(\tilde{G})$ is the eigenvalue of the quadratic Casimir operator for the adjoint representation of the group $\tilde{G}$. Combining (3.2), (3.5), and (3.6) we finally obtain

$$\sin^2\theta_W^0 = \frac{C_2(\tilde{G})}{\sum_{\sigma=1}^{p} \mathrm{Tr}[Q_W{}^2(\sigma)]_{\mathrm{adj}}} . \tag{3.7}$$

B. $G_W = \mathrm{SU}(N) \otimes \cdots \mathrm{SU}(N)$

1. Basic formula

We shall now evaluate (3.7) for $G_W = [\mathrm{SU}(N)]^k$. We immediately obtain

$$C_2(\mathrm{SU}(N)) = N . \tag{3.8}$$

The calculation of $\mathrm{Tr}(Q_W{}^2)_{\mathrm{adj}}$ is slightly more involved. We first notice that since the quarks and leptons are in separate (but identical) representations of G_W the gauge bosons of G_W have integer electric charges. Allowing for arbitrary integer charges of the gauge bosons we can write generally, for *each* $\mathrm{SU}(N)$,

$$\mathrm{Tr}(Q_W{}^2)\,|_{\mathrm{adj}} = \sum_{i=1}^{\alpha} i^2 n_i , \tag{3.9}$$

where α is the maximal gauge-boson charge involved and n_i is the number of gauge bosons with $|Q| = i$.

n_i can be calculated straightforwardly as follows. We first recall that the adjoint representation can be constructed from the product of the fundamental representation N and its conjugate $\bar{N}$. Therefore, n_i can be found by considering the "charge distribution" in the fundamental representation of $\tilde{G} = \mathrm{SU}(N)$.

Denote by r_j $(0 \le j \le \alpha)$ the number of elements in the fundamental representation with the charge $\tilde{Q}_W - j$, i.e.,

$$[\underbrace{\tilde{Q}_W, \ldots \tilde{Q}_W}_{r_0}, \underbrace{\tilde{Q}_W - 1, \ldots \tilde{Q}_W - 1}_{r_1}, \ldots, \underbrace{\tilde{Q}_W - \alpha, \ldots \tilde{Q}_W - \alpha}_{r_\alpha}] , \tag{3.10}$$

where $\tilde{Q}_W$ is an eigenvalue of Q_W. The gauge-boson charges are

$$|Q|=i=|(\tilde{Q}_W-j)-(\tilde{Q}_W-k)|=|k-j| .$$

Then it is easy to show that

$$n_i=2\sum_{k-j=i} r_j r_k,\quad 1\le i\le\alpha , \tag{3.11}$$

with

$$\sum_{j=0}^{\alpha} r_j=N , \tag{3.12}$$

where the factor 2 in Eq. (3.11) comes from the fact that both the positively and negatively charged gauge bosons contribute to $\mathrm{Tr}(Q_W{}^2)$. In Eq. (3.12) the summation is over all pairs r_j,r_k which satisfy $k-j=i$.

In order to illustrate the formula (3.11) and its derivation let us consider a few examples.

(a) $\alpha=1$. In this case the fundamental representation consists of r_0 fermions with charge $\tilde{Q}_W$ and r_1 fermions with charge $\tilde{Q}_W-1$. Therefore, the charges of the gauge bosons in this case are 0, ±1 and the number of charged gauge bosons (n_1) is obtained by counting how many different gauge bosons connect the r_0 fermion with charge $\tilde{Q}_W$ to the r_1 fermions with charge $\tilde{Q}_W-1$. One obtains

$$n_1=2r_0r_1,\quad r_0+r_1=N . \tag{3.13}$$

(b) $\alpha=2$. In this case the charges of the gauge bosons are 0, ±1, ±2. Proceeding as in the previous case we obtain

$$n_1=2(r_0r_1+r_1r_2) ,\qquad n_2=2r_0r_2 , \tag{3.14}$$

with

$$r_0+r_1+r_2=N . \tag{3.15}$$

It is now clear how to obtain the formula (3.11) for arbitrary α [for a single SU(N) group].

In summary $\sin^2\theta_W^0$ for $[\mathrm{SU}(N)]^k$ groups is given (in the case of the *unlocked* standard model) by Eqs. (3.7)–(3.9), (3.11), and (3.12). Note that for a given group $[\mathrm{SU}(N)]^k$ there is a set of values of $\sin^2\theta_W^0$, each value corresponding to particular charge distributions either in the adjoint or in the fundamental representations.

2. *Implications for G_W*

Having explicit formulas for $\sin^2\theta_W^0$ at hand we can easily find which gauge groups G_W give "acceptable" values of $\sin^2\theta_W^0$ [see Eq. (2.31)].

(i) We first find that gauge bosons with charges ±3 and higher are not allowed in the "unlocked" version (E1) of our scheme. Indeed if $\alpha=3$, one can derive an upper bound for $\sin^2\theta_W^0$ which is obtained for any $N\ge4$, $k=1$, $r_0=r_3=1$, and either $r_1=1$ or $r_2=1$, and which reads as follows:

$$\sin^2\theta_W^0\le\frac{1}{12-(8/N)}\le\frac{1}{10} . \tag{3.16}$$

(ii) The maximal allowed number of doubly charged gauge bosons is *two* with $Q=\pm2$. The only value of $\sin^2\theta_W^0$ consistent with Eq. (2.31) is obtained for $k=1$ and a unique charge distribution in the fundamental representation

$$(\tilde{Q}_W,\tilde{Q}_W-1,\ldots,\tilde{Q}_W-1,\tilde{Q}_W-2) . \tag{3.17}$$

For any single SU(N) with $N\ge3$ (3.17) leads to $\mathrm{Tr}(Q_W{}^2)=4N$ which implies

$$\sin^2\theta_W^0=0.25 . \tag{3.18}$$

The cases with four or more doubly charged gauge bosons or $k>1$ are excluded since they lead to $\sin^2\theta_W^0<0.20$.

(iii) If $\alpha=1$ one can derive the following bounds for $\sin^2\theta_W^0$ corresponding to $k=1$ and any N:

$$\frac{2}{N}\le\sin^2\theta_W^0\le\frac{1}{2[1-(1/N)]},\quad N\text{ even} \tag{3.19}$$

and

$$\frac{2N}{(N^2-1)}\le\sin^2\theta_W^0\le\frac{1}{2[1-(1/N)]},\quad N\text{ odd} . \tag{3.20}$$

Notice that for $N=2$, i.e., SU(2), the upper bound in (3.19) and (3.20) is 1 and becomes smaller with increasing N.

The upper bounds correspond to the charge distribution in the fundamental representation characterized by $r_0=1$, $r_1=N-1$ (or $r_0\leftrightarrow r_1$). The lower bound for even N corresponds to the symmetric charge distribution $r_0=r_1=N/2$. For odd N the lower bound corresponds to $r_0=(N-1)/2$ and $r_1=(N+1)/2$ (or $r_0\leftrightarrow r_1$).

It follows immediately from the above bounds that if $\alpha=1$ and $k=1$, i.e., G_W is a *simple* gauge group, only groups SU(N) with $N\ge7$ have values of $\sin^2\theta_W^0$ consistent with Eq. (2.31). Furthermore, combining Eqs. (3.19) and (3.20) with the general formula (3.7) we find that the maximal allowed value of k in the product $G_W=[\mathrm{SU}(N)]^k$ is $k=4$.

TABLE I. The weak groups $G_W = \mathrm{SU}(N)^k$ with $N \leq 8$ which give $\sin^2\theta_W^0$ consistent with Eq. (2.31). r_0 is defined in Eqs. (3.10) and (4.9). The last two columns give the values of the weak charges $\tilde{Q}_W^i$ in the case of representations (4.7) and (4.8), respectively.

G_W	r_0	$\sin^2\theta_W^0$	$(f,1)+(1,\bar{f})$ $\tilde{Q}_W^i$	$(f,\bar{f})$ $\tilde{Q}_W^i$
$[\mathrm{SU}(2)]^3$	1	0.333	$\pm\frac{1}{2}$	$0,\pm 1$
$[\mathrm{SU}(2)]^4$	1	0.250	$\pm\frac{1}{2}$	$0,\pm 1$
$[\mathrm{SU}(3)]^2$	1	0.375	$\frac{2}{3},-\frac{1}{3}$	$0,\pm 1$
$[\mathrm{SU}(3)]^3$	1	0.250	$\frac{2}{3},-\frac{1}{3}$	$0,\pm 1$
$[\mathrm{SU}(4)]^2$	2	0.250	$\pm\frac{1}{2}$	$0,\pm 1$
$\mathrm{SU}(4)_1\otimes\mathrm{SU}(4)_2$	1,2	0.286		$\frac{1}{4},\frac{5}{4},-\frac{3}{4}$
$[\mathrm{SU}(5)]^2$	1	0.313	$\frac{4}{5},-\frac{1}{5}$	$0,\pm 1$
$\mathrm{SU}(5)_1\otimes\mathrm{SU}(5)_2$	1,2	0.250		$\frac{4}{5},-\frac{1}{5}$
$[\mathrm{SU}(6)]^2$	1	0.300	$\frac{5}{6},-\frac{1}{6}$	$0,\pm 1$
SU(7)	3	0.292	$\frac{4}{7},-\frac{3}{7}$	
$[\mathrm{SU}(7)]^2$	1	0.292	$\frac{6}{7},-\frac{1}{7}$	$0,\pm 1$
SU(8)	3	0.267	$\frac{5}{8},-\frac{3}{8}$	
SU(8)	4	0.250	$\pm\frac{1}{2}$	

In addition if $k=4$ only the group $[\mathrm{SU}(2)]^4$ gives $\sin^2\theta_W^0$ consistent with Eq. (2.31). All other groups with $k=4$ and $N\geq 3$ have $\sin^2\theta_W^0<0.20$ and are of no interest to us. Finally in Table I we list all the groups $G_W=[\mathrm{SU}(N)]^k$ with $N\leq 8$ which gave $\sin^2\theta_W^0$ consistent with Eq. (2.31). The relevant charge distributions in the fundamental representations are also listed there.

C. Locked standard model

In this case Eq. (2.26) applies with

$$\sum_\alpha [C'_{\alpha W}]^2 \equiv C'^2_W = m \ , \tag{3.21}$$

where m is the number of disjoint SU(2) subgroups of G_W, whose diagonal generators enter Eq. (2.7). Therefore, for a fixed value of $[(\mathrm{Tr}T_{3L}^2)/(\mathrm{Tr}Q^2)]_{\mathrm{adj}}$, the value of $\sin^2\theta_W^0$ is m times larger in the locked case than in the unlocked case. For $\alpha=1$ (singly charged and neutral gauge bosons) this implies that the groups G_W which give $\sin^2\theta_W^0$ consistent with Eq. (2.31) must now be large. For instance, if $m=2$ the smallest acceptable weak groups are $[\mathrm{SU}(2)]^8$, and $[\mathrm{SU}(4)]^4$ which give $\sin^2\theta_W^0=0.25$. Therefore, the locked standard model is not economical and we shall not consider it any further. It should be, however, remarked that smaller weak groups consistent with (2.31) can be obtained in the locked case at the cost of introducing doubly ($\alpha=2$) and triply ($\alpha=3$) charged gauge bosons. For instance, if $m=2$ and $\alpha=2$ any $[\mathrm{SU}(N)]^2$ ($N\geq 3$) with the charge distribution (3.17) will give $\sin^2\theta_W^0=0.25$. Also by choosing $r_0=r_1=1$ and $r_2=N-2$ some of the simple groups SU(N) with $N\geq 6$ and doubly charged gauge bosons satisfy Eq. (2.31). Similar comments apply to the case $\alpha=3$ if $m\geq 3$ is chosen. Perhaps one interesting and economical case would be $\mathrm{SU}(3)\otimes\mathrm{SU}(3)$ (Ref. 17) with two doubly charged bosons since $\sin^2\theta_W^0=\frac{1}{4}$, but as we shall show in Secs. IV and VI this case turns out to be also unacceptable.

D. Summary

The study of this section leaves us with the following candidates for the weak group G_W.

(a) Unlocked standard model. (i) Groups listed in Table I; and (ii) SU(N) groups with $N\geq 3$, fundamental representations of Eq. (3.17), and two doubly charged gauge bosons. The corresponding $\sin^2\theta_W^0=0.25$.

(b) Locked standard model. $[\mathrm{SU}(N)]^2$ groups with $N \geq 3$, $m=2$ [see (3.21)] and two doubly charged gauge bosons. The corresponding $\sin^2\theta_W^0=0.25$. As discussed in Sec. III C other groups in the "locked" version of our scheme are not economical and will not be considered further.

IV. CONSTRAINTS FROM THE CHOICE OF G_S

To proceed further we have to choose G_S. As discussed in the Introduction the minimal strong gauge group turns out to be SU(4). We shall now show that if (i) the strong gauge group G_S is chosen to be SU(4), (ii) each standard quark $\mathrm{SU}(3)_c$ triplet is put with a lepton in the fundamental representation of G_S (lepton number being the fourth color),[1] and (iii) electric charges of quarks and leptons are restricted to

$$Q_q=\frac{d}{3}+n,\quad Q_l=n',\qquad n,n' \text{ integer},\quad d=1,2\,, \tag{4.1}$$

then many of the groups G_W listed in the Table I can be eliminated.

In order to show this we first write the electric charges of quarks and leptons in terms of their weak charges (Q_W) and the strong charges (Q_S),

$$Q=C_S T_{15}+Q_W \equiv Q_S+Q_W\,, \tag{4.2}$$

where T_{15} is the diagonal generator of SU(4) which commutes with $\mathrm{SU}(3)_c$ generators. Using the normalization of Eq. (2.10) we have ($\lambda=\frac{1}{2}$)

$$T_{15}=\frac{1}{2\sqrt{6}}\begin{bmatrix}1 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & -3\end{bmatrix}. \tag{4.3}$$

The coefficient C_S will be determined later on. Now the content of the fundamental representation of SU(4) is [see (1.6)]

$$\begin{bmatrix}q_1\\ q_2\\ q_3\\ l\end{bmatrix}, \tag{4.4}$$

where q_1,q_2,q_3 is the standard color quark triplet and l stands for a lepton. From Eq. (4.3) it follows that

$$(\tilde{Q}_S)_l=-3(\tilde{Q}_S)_q\,, \tag{4.5}$$

where $\tilde{Q}_S$ is *an* eigenvalue of Q_S. Taking next into account that the weak charges Q_W of q_1,q_2,q_3 and l in the representation (4.4) are the same and using (4.1) we obtain

$$\tilde{Q}_W^i=\tfrac{1}{4}(d+3n'+n'),\quad d=1,2\,, \tag{4.6}$$

where $\tilde{Q}_W^i$ is *an* eigenvalue of Q_W corresponding to the ith fermion.

We conclude therefore that if $G_S=\mathrm{SU}(4)$ and the quark and lepton charges are given by Eq. (4.1) then the G_W is restricted to groups and representations for which $\tilde{Q}_W^i$ are multiples of $\frac{1}{4}$. Thus it is enough to calculate $\tilde{Q}_W^i$ in order to decide whether a given group G_W and its representations can lead [in the case of $G_S=\mathrm{SU}(4)$] to acceptable charges of quarks and leptons. This is what we shall do now. To this end we have to specify the fermion representations.

We shall consider two classes of representations which we symbolically denote as follows[18]:

$$\text{(i)}\ (f,1,1,\ldots,1),\ (1,\bar{f},1,1,\ldots) \tag{4.7}$$

and

$$\text{(ii)}\ (f,\bar{f},1,1,\ldots,1)\,. \tag{4.8}$$

Here only the transformation properties under G_W have been shown. Each entry in (4.7) and (4.8) corresponds to the group $\tilde{G}$ in the product $G_W=\tilde{G}\otimes\tilde{G}\otimes\tilde{G}\cdots\otimes\tilde{G}$. In class (i) quarks and leptons transform nontrivially under one of the groups $\tilde{G}$ and are singlets under the rest. In the class (ii), which includes only semisimple groups, fermions transform as $(f,\bar{f})$ under any pair $\tilde{G}\otimes\tilde{G}\subset G_W$ and are singlets under the rest.

We begin by discussing the case of singly charged gauge bosons (the case $\alpha=1$ of Sec. III). In this case each fundamental representation of the groups $\tilde{G}$ has the charge distribution

$$[\underbrace{\tilde{Q}_W,\tilde{Q}_W,\ldots,\tilde{Q}_W}_{r_0},\underbrace{\tilde{Q}_W-1,\ldots,\tilde{Q}_W-1}_{r_1}] \tag{4.9}$$

with $r_0+r_1=N$. This is also the charge distribution for the class (i). In the case of class (ii) we have to consider the matrix

$$\begin{matrix} & \overbrace{\qquad\qquad\qquad}^{r_0} & \overbrace{\qquad\qquad\qquad}^{r_1} \\ r_0'\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. & \begin{matrix}\tilde{Q}_W & \cdots & \tilde{Q}_W \\ & \vdots & \\ \tilde{Q}_W & \cdots & \tilde{Q}_W \end{matrix} & \begin{matrix}\tilde{Q}_W-1 & \cdots & \tilde{Q}_W-1 \\ \\ \tilde{Q}_W-1 & \cdots & \tilde{Q}_W-1 \end{matrix} \\ r_1'\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. & \begin{matrix}\tilde{Q}_W+1 & \cdots & \tilde{Q}_W+1 \\ & \vdots & \\ \tilde{Q}_W+1 & \cdots & \tilde{Q}_W+1 \end{matrix} & \begin{matrix}\tilde{Q}_W & \cdots & Q_W \\ \\ \tilde{Q}_W & \cdots & \tilde{Q}_W \end{matrix} \end{matrix}, \tag{4.10}$$

where the rows refer to f and the columns to $\bar{f}$. Furthermore,

$$r_0+r_1=r_0'+r_1'=N . \tag{4.11}$$

It is now a trivial matter to calculate $\tilde{Q}_W$ by using the tracelessness condition for the charge operator Q_W. For the class (i) we obtain

$$\tilde{Q}_W=1-\frac{r_0}{N} . \tag{4.12}$$

The other eigenvalue is $\tilde{Q}_W-1$. Using Eq. (4.12) for the groups listed in Table I we obtain the relevant values for $\tilde{Q}_W^i$ which are shown in the third column of this table.

For the class (ii) $\tilde{Q}_W$ is found from condition that the sum of charges is zero,

$$r_0'r_0\tilde{Q}_W+r_0r_1'(\tilde{Q}_W+1)+r_0'r_1(\tilde{Q}_W-1) +r_1'r_1\tilde{Q}_W=0 . \tag{4.13}$$

Eliminating, by use of Eq. (4.11), r_1 and r_1' in favor of N we obtain

$$\tilde{Q}_W=\frac{r_0'-r_0}{N} . \tag{4.14}$$

Two other eigenvalues are $\tilde{Q}_W\pm 1$.

Using these equations for the groups of interest, we obtain the last column of Table I.

For the case of doubly charged bosons in the unlocked standard model only the class (i) applies. The charge distribution in the fundamental representation is given in Eq. (3.17). Using the tracelessness condition we obtain

$$\tilde{Q}_W=1 . \tag{4.15}$$

The other two eigenvalues are 0 and -1. The same eigenvalues are obtained for the "locked" cases mentioned at the end of Sec. III C.

Before discussing the implications of these results let us calculate the coefficient C_S of Eq. (4.2). We first obtain

$$C_S=2\,\mathrm{Tr}(QT_{15}) . \tag{4.16}$$

Choosing next in accordance with Eq. (4.4) the following form for Q,

$$Q=\begin{bmatrix} Q_q^i & & & 0 \\ & Q_q^i & & \\ & & Q_q^i & \\ 0 & & & Q_l^i \end{bmatrix}, \tag{4.17}$$

where

$$Q_{q,l}^i=(\tilde{Q}_S)_{q,l}+Q_W^i , \tag{4.18}$$

and using Eqs. (4.3) and (4.5), we obtain

$$\begin{aligned} C_S{}^2 &= \tfrac{1}{6}(3Q_q^i-3Q_l^i)^2 \\ &= \tfrac{8}{3}(\tilde{Q}_W^i-Q_l^i)^2 \\ &= \tfrac{8}{3}(\tilde{Q}_S)_l{}^2 , \end{aligned} \tag{4.19}$$

where $C_S{}^2$ is independent of i. In Eq. (4.19) the relation

$$\tilde{Q}_W^i=\tfrac{1}{4}(3Q_q^i+Q_l^i) \tag{4.20}$$

has been used.

TABLE II. The values of lepton (Q_l^i) and quark (Q_q^i) electric charges corresponding to the weak charges $\tilde{Q}_W^i$ discussed in the text. The values for $C_S{}^2$ have been obtained from Eq. (4.19).

$\tilde{Q}_W^i$	Q_l^i	Q_q^i	$C_S{}^2$
$\frac{1}{2}$	0	$\frac{2}{3}$	
$-\frac{1}{2}$	-1	$-\frac{1}{3}$	
$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{2}{3}$
$-\frac{1}{2}$	0	$-\frac{2}{3}$	
1	0	$\frac{4}{3}$	
0	-1	$\frac{1}{3}$	
-1	-2	$-\frac{2}{3}$	$\frac{8}{3}$
1	2	$\frac{2}{3}$	
0	1	$-\frac{1}{3}$	
-1	0	$-\frac{4}{3}$	
$\frac{5}{4}$	1	$\frac{4}{3}$	
$\frac{1}{4}$	0	$\frac{1}{3}$	$\frac{1}{6}$
$-\frac{3}{4}$	-1	$-\frac{2}{3}$	

In order to simplify the discussion of the groups in the Table I and of the assignment of the known quarks and leptons into various representations we have listed in Table II possible weak and strong multiplets, the corresponding weak and electric charges and the values of the parameter ${C_S}^2$. Only cases which satisfy Eq. (4.6) and for which $|Q_l| \leq 2$ and $|Q_q| \leq \frac{4}{3}$ have been shown there.

Having the Tables I and II at hand we can now enumerate good and bad features of the groups listed in Sec. III D. We shall only discuss the groups which give Q_W^i which are multiples of $\frac{1}{4}$ [see Eq. (4.6)]. In particular we do not consider the group $[SU(3)]^3$ any further.

(a) Certainly the most attractive ones are the groups $[SU(2)]^4$ and $[SU(4)]^2$. They do not require other electric charges than the known 0, ± 1 for leptons and antileptons and $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$ for quarks and antiquarks, if the representations of the class (i), i.e., $(f,1,1,1,\ldots)+(1,\bar{f},1,1,\ldots)$ are used. Because for these representations the parameter ${C_S}^2=\frac{2}{3}$ is small, the resulting value for $\sin^2\theta_W$ is in very good agreement with the experimental data (see Sec. VI for details). Also, the group SU(8) has the same features as the two groups above. But SU(8) is a very large group and for that reason perhaps less attractive. The representations of class (ii), i.e., $(f,\bar{f},1,\ldots,1)$ are not acceptable for the groups $[SU(2)]^4$ and $[SU(4)]^2$. They lead to a large value of ${C_S}^2=\frac{8}{3}$ and the resulting $\sin^2\theta_W({M_W}^2)$ is at least three standard deviations below its experimental value (see Sec. VI). Furthermore, in the case of these representations leptons with charges ± 2 and quarks with charges $\pm\frac{4}{3}$ are required.

(b) The groups $[SU(5)]^2$, $[SU(6)]^2$, and $[SU(7)]^2$ with $\tilde{Q}_W^i=0,\pm 1$ lead after renormalization to $\sin^2\theta_W<0.20$ if $M>10^5$ GeV. For $M<10^4$ GeV $\sin^2\theta\approx 0.22$ can be obtained but such a low value of M is inconsistent with our analysis of rare decays (see Sec. VII). Furthermore, these groups are large and require leptons with charges ± 2 and quarks with charges $\pm\frac{4}{3}$.

(c) Among the groups in Table I which have $\tilde{Q}_W^i=0,\pm 1$ only $[SU(2)]^3$ and $[SU(3)]^2$ give for $M=10^{5-6}$ GeV acceptable values of $\sin^2\theta_W$. These are $0.20-0.21$ for $[SU(2)]^3$ and $0.21-0.23$ for $[SU(3)]^2$. However, these groups require leptons with charges ± 2 and quarks with charges $\pm\frac{4}{3}$.

(d) The groups with doubly charged gauge bosons which have $\sin^2\theta_W^0=0.25$ are excluded because they have a large parameter ${C_S}^2=\frac{8}{3}$ and therefore lead to $\sin^2\theta_W<0.18$ after renormalization effects are taken into account. Furthermore, these groups also require ± 2 and $\pm\frac{4}{3}$ fermion charges. The same remarks apply to the "locked" cases mentioned at the end of Sec. III C.

(e) In the case of the group $SU(4)_1\otimes SU(4)_2$ [$r_0\neq r_0'$, Eq. (4.10)] the parameter ${C_S}^2$ is very small (${C_S}^2=\frac{1}{6}$) and the resulting $\sin^2\theta_W({M_W}^2)$ turns out after renormalization to be larger than ~ 0.27 for $M<10^8$ GeV. Smaller values of $\sin^2\theta_W$ can be obtained at the cost of increasing substantially the scale M which is against our philosophy. Furthermore, for the group in question the quarks with charges $\pm\frac{4}{3}$ are required.

In summary, the considerations of Secs. III and IV leave us with only two economical candidates for the group $G=G_S\otimes G_W$. These are

$$\text{(i) } G=SU(4)\otimes[SU(2)]^4 \tag{4.21}$$

and

$$\text{(ii) } G=SU(4)\otimes[SU(4)]^2\ . \tag{4.22}$$

Furthermore, as we shall show explicitly in Sec. VI only the fundamental representations of Eq. (4.7) are consistent with the experimentally measured values of $\sin^2\theta_W$.

V. MINIMAL PETITE-UNIFICATION MODEL

In the previous sections we have found that the minimal petit-unification gauge group consistent with the measured value of $\sin^2\theta_W$ (see also Sec. VI) and which did not require other than the conventional values of quark and lepton charges was the group $SU(4)_S\otimes[SU(2)]^4$. We shall discuss it here in some detail.[19]

More explicitly we take

$$G=SU(4)_S\otimes SU(2)_L\otimes SU(2)_R \otimes SU(2)_{L'}\otimes SU(2)_{R'}\ , \tag{5.1}$$

where $SU(2)_L\otimes SU(2)_R$ may be identified with the $SU(2)_L\otimes SU(2)_R$ part of the well-known left-right-symmetric model[9] and $SU(2)_{L'}\otimes SU(2)_{R'}$ constitutes a "mirror" left-right-symmetric counterpart. The group G is broken down to $SU(3)_c\otimes U(1)_{EM}$ in three steps as follows:

$$G\underset{M}{\rightarrow} SU(3)_c\otimes U(1)_S\otimes[SU(2)]^4$$
$$\underset{\tilde{M}}{\rightarrow} SU(3)_c\otimes SU(2)_L\otimes U(1)$$
$$\underset{M_W}{\rightarrow} SU(3)_c\otimes U(1)_{EM}\ . \tag{5.2}$$

We shall first present the fermion and gauge-boson content of the model. Subsequently we shall discuss the Higgs system necessary for the breakdown (5.2) to occur.

A. Fermions

Fermions transform under G as follows.

(i) Light fermions $\psi_i^\alpha(n)$ ($\alpha=1,\ldots 4$; $i=1,2$) are grouped in $n=1,2,\ldots$ generations and transform under G according to

$$\psi_i^\alpha(n)=(4,2,1,1,1)_L \ , \tag{5.3}$$

$$\psi_i^{*\alpha}(n)=(\bar{4},1,2,1,1)_L \ ; \tag{5.4}$$

where Weyl fields are used.

(ii) Heavy "mirror" fermions $\psi_i'^{\,\alpha}(n)$ are grouped in mirror generations which transform under G according to

$$\psi_i'^{\,\alpha}(n)=(4;1,1;2,1)_L \ , \tag{5.5}$$

$$\psi_i'^{\,\alpha *}(n)=(\bar{4};1,1;1,2)_L \ . \tag{5.6}$$

Examples of the first light generation and of its heavy mirror counterpart are given in Eqs. (1.6) and (1.7), respectively.

Recall that the existence of the mirror group $SU(2)_{L'}\otimes SU(2)_{R'}$ and its fermions is required in our scheme by the measured value of $\sin^2\theta_W$. The assignment (5.3) accommodates known quarks and leptons ($n=1,2,3$) with conventional charges. The mirror fermions have no ordinary $SU(2)_L$ weak interactions. Of course both the light and mirror fermions have ordinary electromagnetic and neutral-current interactions.

The mirror fermions have to be heavy enough to escape detection. On the other hand, they have to be lighter than $\widetilde{M}$. This is due to the assumed permutation symmetry among SU(2) groups which requires the equality $g_L(Q^2)=g_R(Q^2)=g_{L'}(Q^2)=g_{R'}(Q^2)=g_W(Q^2)$ for scales $Q^2\geq\widetilde{M}^2$, where $[SU(2)]^4$ is a good symmetry. If some of the mirror fermions had the masses larger than $\widetilde{M}$ they would not contribute to the relevant renormalization-group functions which govern the evolution of the $g_{L'}$ and $g_{R'}$ couplings for $Q^2>\widetilde{M}^2$. Consequently the equality $g_L=g_R=g_{L'}=g_{R'}$ would no longer be true for scales larger than $\widetilde{M}$ even if it was true at $\widetilde{M}$. Thus we expect the masses of the mirror fermions to populate the energy range from M_W to $\widetilde{M}$ but it is not excluded that the masses of the lightest of the mirror fermions could be as low as 20 − 30 GeV. Some phenomenology of these light mirror fermions is discussed in Sec. VII C.

B. Gauge bosons

1. Gluons and leptoquarks

The $SU(3)_c$ content of the adjoint representation $\underline{15}$ of $SU(4)_S$ is as follows:

$$\underline{15}=\underline{8}+\underline{3}+\underline{\bar{3}}+\underline{1} \ , \tag{5.7}$$

and the corresponding gauge fields $A_{\mu S}^i$ are represented by

$$\{A_{\mu S}^i(i=1,\ldots 15)\}=\begin{array}{c|ccc|c} & SU(3)_c & & U(1)_S & \\ \hline & & & G_\mu^{+,1} & \\ & A_{\mu S}^i & & G_\mu^{+,2} & SU(3)_c \\ & (i=1,\ldots,8) & & G_\mu^{+,3} & \\ & G_\mu^{-,1}\ \ G_\mu^{-,2} & G_\mu^{-,3} & \widetilde{A}_{\mu S} & U(1)_S \end{array} \tag{5.8}$$

Here the octet $A_{\mu S}^i$ stands for the gluons,

$$G_\mu^{\pm,1}=\frac{1}{\sqrt{2}}(A_{\mu S}^9\mp iA_{\mu S}^{10}) \ , \tag{5.9a}$$

$$G_\mu^{\pm,2}=\frac{1}{\sqrt{2}}(A_{\mu S}^{11}\mp iA_{\mu S}^{12}) \ , \tag{5.9b}$$

$$G_\mu^{\pm,3}=\frac{1}{\sqrt{2}}(\mp iA_{\mu S}^{14}) \ , \tag{5.9c}$$

and $\widetilde{A}_{\mu S}=A_{\mu S}^{15}$ is the neutral gauge boson which corresponds to the generator T_{15} of $SU(4)_S$ and equivalently to the generator of $U(1)_S$. The leptoquark gauge bosons $G_\mu^{\pm,i}$ carry charges $\pm\frac{2}{3}$ and connect quarks to leptons. They are responsible for the rare transitions, the phenomenology of which is presented in Sec. VII A. Under the breaking of $SU(4)_S$ down to $SU(3)_c\otimes U(1)_S$ the leptoquarks $G_i^{\pm}$ gain masses of order M whereas the gluons and the gauge boson $\widetilde{A}_{\mu S}$ remain massless.

2. *Electroweak gauge bosons*

The model has twelve massive electroweak gauge bosons in addition to the massless photon. These include (i) six charged gauge bosons $W_R^{\pm}$, $W_{L'}^{\pm}$, $W_{R'}^{\pm}$ and three neutral gauge bosons Z_1, Z_2, Z_3 all with masses of order $\widetilde{M}$, and (ii) the standard $W^{\pm}$ and Z^0 gauge bosons with the conventional masses of order M_W.

It should be remarked that the field B_μ of the standard model is expressed in terms of the fields $\widetilde{A}_{\mu S}$ and $(A_{\mu W}^3)_{R,L',R'}$, which couple to the diagonal generators of $U(1)_S \otimes SU(2)_R \otimes SU(2)_{L'} \otimes SU(2)_{R'}$, respectively, as follows:

$$B_\mu = \widetilde{A}_{\mu S}\cos\theta_S + \frac{\sin\theta_S}{\sqrt{3}}(A_{\mu R}^3 + A_{\mu L'}^3 + A_{\mu R'}^3)\ , \tag{5.10}$$

where the mixing angle θ_S is defined by

$$\tan\theta_S = \frac{\widetilde{g}_S}{g_W}\frac{3}{\sqrt{2}}\ . \tag{5.11}$$

Recall that $\widetilde{g}_S$ is the $U(1)_S$ coupling constant. Furthermore, we have

$$Z_{1\mu} = \widetilde{A}_{\mu S}\sin\theta_S - \frac{\cos\theta_S}{\sqrt{3}}(A_{\mu R}^3 + A_{\mu L'}^3 + A_{\mu R'}^3)\ , \tag{5.12}$$

$$Z_{2\mu} = \frac{1}{\sqrt{2}}(A_{\mu R}^3 - A_{\mu R'}^3)\ , \tag{5.13}$$

$$Z_{3\mu} = \frac{1}{\sqrt{6}}(A_{\mu R}^3 - 2A_{\mu L'}^3 + A_{\mu R'}^3)\ . \tag{5.14}$$

Whereas the gauge bosons $Z_2, Z_3, W_R^{\pm}, W_{L'}^{\pm}, W_{R'}^{\pm}$ have a common mass which we denote by $\widetilde{M}$, the mass of Z_1 is given by

$$M(Z_1) = \frac{\widetilde{M}}{\cos\theta_S}\ . \tag{5.15}$$

Equation (5.15) is the analog of the standard-model relation $M_W = M_Z\cos\theta_W$. As discussed in Sec. VII B, $\widetilde{M}$ must be larger than 300 GeV in order for the model to be consistent with the experimental data. Finally, notice from Eq. (2.13) that the hypercharge $U(1)_Y$ coupling constant g' is defined in terms of $\widetilde{g}_S$ and g_W by

$$g' = \frac{g_W\widetilde{g}_S\sqrt{3}}{(9\widetilde{g}_S{}^2 + 2g_W{}^2)^{1/2}} = g_W\sin\theta_S/\sqrt{3}\ , \tag{5.16}$$

where Eq. (5.11) has been used. Equation (5.16) is analogous to the well-known relation $e = g\sin\theta_W$.

C. Higgs bosons and symmetry breaking

The breakdown (5.2) can be accomplished by the scalar multiplets Φ and ϕ_i $(i = 1, \ldots, 4)$ which transform under $SU(4)_S \otimes [SU(2)]^4$ as

$$\Phi = (15; 1,1,1,1) \tag{5.17}$$

and

$$\phi_i = [4; 2 \in SU(2)_i]\ , \tag{5.18}$$

where $2 \in SU(2)_i$ means doublet under $SU(2)_i$ and singlet under the remaining SU(2)'s, i.e., each ϕ_i is a 4×2 matrix. The charge structure of ϕ_i and Φ is the same as that of the fermions and the $SU(4)_S$ gauge bosons, respectively. In particular we have

$$\phi_i = \begin{bmatrix} \vec{\phi}^{\mu} & \vec{\phi}^{d} \\ \phi^0 & \phi^- \end{bmatrix}_i\ , \tag{5.19}$$

where $\vec{\phi}$ denotes a color triplet.

The study of the symmetry breakdown (5.2) by the multiplets Φ and ϕ_i is essentially a hybrid of the analyses of Li[20] with that of Buccella *et al.*[21] Li made a general analysis for the case $SU(n) \otimes SU(m)$ with a scalar multiplet transforming as (n,m). On the other hand, Buccella *et al.* analyzed breakdown of $SU(n)$ by an adjoint and a fundamental scalar multiplet. We shall not present the details of our analysis, which is lengthy. We only remark that the desired asymmetric vacuum is characterized by the following scalar vacuum expectation values:

$$\langle\Phi\rangle = \frac{v_s}{2\sqrt{3}}\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -3 \end{pmatrix} \tag{5.20}$$

and

$$\langle\phi_i\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ v_i/\sqrt{2} & 0 \end{bmatrix}\ , \tag{5.21}$$

and that it is essential for the avoidance of massless physical Higgs particles that the cross couplings between Φ and ϕ_i are present in the Higgs potential. In particular the colored, charged Higgs particles $\vec{\phi}^u$ and $\vec{\phi}^d$ receive masses of order v_s through these cross couplings.

The physical Higgs content of the model is as follows:

(i) Color triplet: eight with charges $\pm\frac{2}{3}$, eight with charges $\pm\frac{1}{3}$, mass of the order v_s.

(ii) Color octet: one neutral color octet with mass of the order v_s.

(ii) Color singlet: one neutral singlet with mass $\sim v_s$ and four neutral singlets with mass $\sim v_i$, one of which is the Weinberg-Salam Higgs boson.

Finally, we should also remark that we may need additional Higgs multiplets such as $(1,2,\bar{2},1,1)$, $(1,1,1,2,\bar{2})$, etc., to provide the fermion masses. One might also expect that $(15,2,\bar{2},1,1)$, etc., Higgs fields may have to be introduced. However, if new $SU(4)\otimes SU(2)^4$ singlet fermions (e.g., "generation" fermions) with a large intrinsic mass are added to our scheme, there are several potential mechanisms for radiatively generated fermion masses available. In such a case it appears that no (15,2,2,1,1), etc., Higgs multiplets are required. We leave the study of fermion-mass generation for the future.

VI. RENORMALIZATION-GROUP ANALYSIS

In this section we shall present the results for $\sin^2\theta_W(M_W^2)$ for $G=SU(4)_S\otimes[SU(2)]^4$ and $G=SU(4)_S\otimes[SU(4)]^2$. In these cases the group $\widetilde{G}_S$ of Sec. II is just $\widetilde{G}_S=U(1)_S$. The basic formula for $\sin^2\theta_W$ is given in Eq. (2.20). The various parameters which enter there are for the cases in question as follows:

$$\sin^2\theta_W^0=\tfrac{1}{4},\quad C_W{}^2=3,\quad C_W'^2=1 \tag{6.1}$$

and if only gauge-boson and light-fermion contributions to the relevant β functions are taken into account:

$$b_1=\frac{10}{3}\frac{n_f}{48\pi^2},\quad b_2=\frac{2n_f-22}{48\pi^2},$$
$$b_3=\frac{2n_f-33}{48\pi^2},\quad \widetilde{b}=\frac{2n_f}{48\pi^2}, \tag{6.2}$$

where n_f is the number of light flavors (the effect of the mirror fermions is discussed at the end of this section).[22] From the analysis of Sec. IV we also have

$$C_S{}^2=\tfrac{2}{3}\text{ and }\tfrac{8}{3} \tag{6.3}$$

for representations (4.7) and (4.8), respectively.

Combining Eqs. (2.23) and (6.1)—(6.3) we obtain

$$K=\frac{1}{48\pi^2}(88-4n_f)=\frac{1}{48\pi^2}\times\begin{cases}64, & n_f=6\\ 56, & n_f=8\end{cases} \tag{6.4}$$

for $C_S{}^2=\frac{2}{3}$, and

$$K=\frac{1}{48\pi^2}(154-8n_f)=\frac{1}{48\pi^2}\times\begin{cases}106, & n_f=6\\ 90, & n_f=8\end{cases} \tag{6.5}$$

for $C_S{}^2=\frac{8}{3}$. Furthermore,

$$K'=C_S{}^2\frac{33}{48\pi^2} \tag{6.6}$$

independently of the number of flavors.

In order to complete the analysis we still need the values for $\alpha(M_W{}^2)$ and $\alpha_S(M_W{}^2)$. For $\alpha(M_W{}^2)$ we take

$$\alpha(M_W{}^2)=\tfrac{1}{128} \tag{6.7}$$

as obtained in Ref. 23. The fact that $\alpha(M_W{}^2)\neq\frac{1}{137}$ is due to QED renormalization effects. For $\alpha_S(M_W{}^2)$ we have used the standard QCD expression

$$\alpha_S(M_W{}^2)=\frac{12\pi}{(33-2n_f)\ln(M_W{}^2/\Lambda^2)} \tag{6.8}$$

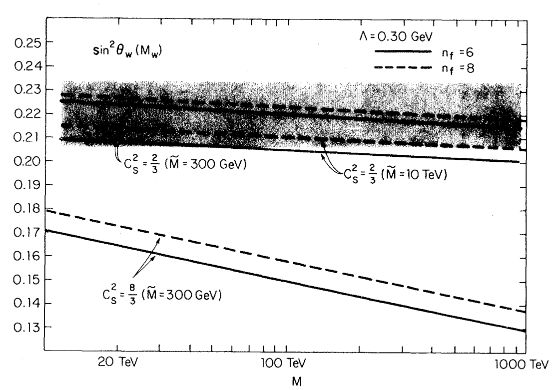

FIG. 1. $\sin^2\theta_W(M_W{}^2)$ as given by Eq. (2.20) as a function of M for $\widetilde{M}=300$ GeV, $n_f=6$ and 8, and $C_S{}^2=\frac{2}{3}$ and $\frac{8}{3}$. For the case $C_S{}^2=\frac{2}{3}$ also the curve corresponding to $\widetilde{M}=10$ TeV is shown. In all cases $\Lambda=0.3$ GeV has been used. As discussed in the text the inclusion of mirror fermions in the evolution of $\sin^2\theta_W$ changes the above curves by at most 1%.

with a typical value for the scale parameter Λ equal to 0.3 GeV. We have checked that varying Λ in the range from 0.1 to 0.5 GeV changes our results for $\sin^2\theta_W$ by at most one and five percent for $C_S{}^2=\frac{2}{3}$ and $C_S{}^2=\frac{8}{3}$, respectively.

Choosing finally $M_W=80$ GeV we have calculated $\sin^2\theta_W(M_W{}^2)$ as a function of M for $n_f=6$ and $n_f=8$, the values of $C_S{}^2$ given in (6.3), and $\tilde{M}=300$ GeV. The results are presented in Fig. 1, where also the experimental range for $\sin^2\theta_W(M_W{}^2)$ [Eq. (2.28)] is shown. We note the following features.

(i) The case $C_S{}^2=\frac{8}{3}$ is ruled out. The corresponding $\sin^2\theta_W$ is by at least three standard deviations below the experimental data. Slightly higher values of $\sin^2\theta_W$ could be obtained by decreasing M below 10 TeV but such low values of M are excluded on the basis of our analysis of rare decays (see Sec. VII).

(ii) The case $C_S{}^2=\frac{2}{3}$ is in agreement with the experimentally measured value for $\sin^2\theta_W$ for both $n_f=6$ and $n_f=8$ and for the whole range $10\leq M\leq 1000$ TeV considered.

Increasing $\tilde{M}$ from 300 GeV to 1 TeV would *decrease* the predicted values of $\sin^2\theta_W$ as shown in Fig. 1 by at most 0.005 and hence would not change our conclusions. We also find that the maximal value of $\tilde{M}$ consistent with the experimental data for $\sin^2\theta_W$ is roughly 10 TeV.

These results are essentially unchanged when the contributions of the mirror fermions to the relevant β functions are taken into account. As discussed in Sec. V we expect the masses of these fermions to populate the energy range from M_W to $\tilde{M}$. Therefore, the contributions of the mirror fermions to the b_i coefficients of Eq. (6.2) are in the energy range from M_W to $\tilde{M}$ suppressed by mass effects as compared to the corresponding light-fermion contributions. In the approximation of neglecting these mass effects we find

$$(\Delta K)_{\rm mirror}=\pm\frac{2n_F}{48\pi^2}\,, \tag{6.9}$$

where + and − correspond to $C_S{}^2=\frac{2}{3}$ and $C_S{}^2=\frac{8}{3}$, respectively. Here n_F is the number of mirror flavors which is equal to n_f. Furthermore, K' of Eq. (6.6) remains unchanged. Combining Eq. (6.9) with (6.4)−(6.6) we find that the inclusion of the mirror fermions in our renormalization-group analysis lowers (increases) the value of $\sin^2\theta_W(M_W{}^2)$ in the case of $C_S{}^2=\frac{2}{3}$ ($C_S{}^2=\frac{8}{3}$) by *at most* 0.002.

VII. PHENOMENOLOGY

A. Rare decays of K mesons

Rare decays of K mesons serve as a strong constraint on our ideas of petite unification. Specifically, we concentrate on the $K\rightarrow\mu e,\pi\mu e$ modes whose experimental upper bounds on branching ratios are given by[24]

$$B(K_L\rightarrow\mu e)<2\times10^{-9}\,,$$
$$B(K^{\pm}\rightarrow e^{+}\mu^{\pm}\pi^{\mp})<7\times10^{-9}\,. \tag{7.1}$$

Such decays have been considered in Ref. 25 in the context of generation-changing horizontal gauge groups. In our case, the elementary processes $q+l\rightarrow G^{\pm}\rightarrow l'+q'$ become, after a Fierz-Michel rearrangement, $q+\bar{q}'\rightarrow\bar{l}+l'$. Here $G^{\pm}$ are the massive SU(4)/[SU(3)$_c\otimes$U(1)$_S$] leptoquark gauge bosons of Eq. (5.9).

According to Sec. V, we have the following SU(4) light-fermion representation for each generation:

$$\psi=\begin{bmatrix} u_1 & d_1\\ u_2 & d_2\\ u_3 & d_3\\ \nu & e^-\end{bmatrix}\updownarrow \mathrm{SU}(4)\,, \qquad \leftarrow \mathrm{SU}(2)_L\rightarrow \tag{7.2}$$

where $Q(u)=\frac{2}{3}$, $Q(d)=-\frac{1}{3}$, $Q(e^-)=-1$, and $Q(\nu)=0$. To simplify the discussion, we make some sort of "kinship" hypothesis whereby we have (d_i,e^-), (s_i,μ^-), ... In principle we could have generation mixing but this could only complicate our estimates of $K\rightarrow\mu e,\pi\mu e$. Specifically, the Lagrangian describing the interaction of light fermions with the SU(4)/[SU(3)$_c\otimes$U(1)$_S$] gauge bosons is given by

$$\mathscr{L}=(g_S/\sqrt{2})\left[\sum_{a=1}^{N}\sum_{i=1}^{3}(\bar{u}_i^a\gamma_\mu\nu^a+\bar{d}_i^a\gamma_\mu e^a)G^{\mu}_{+,i}+\text{H.c.}\right], \tag{7.3}$$

where $G^{\mu}_{+,i}$ are defined by Eq. (5.9), while a and i are "generation" and color indices, respectively.

For $q^2<<m_G{}^2$, we have the following effective Lagrangian:

$$\mathscr{L}_{\text{eff}}=\sqrt{2}G_S\sum_{\substack{a'=1\\a=1}}^{N}\sum_{i=1}^{3}(\bar{u}_i^a\gamma_\mu\nu^a+\bar{d}_i^a\gamma_\mu e^a)\times(\bar{\nu}^{a'}\gamma^\mu u_i^{a'}+\bar{e}^{a'}\gamma^\mu d_i^{a'})\,, \tag{7.4}$$

where G_S is defined by

$$g_S^2/2m_G^2=\sqrt{2}G_S\,. \tag{7.5}$$

We are particularly interested in that part of Eq. (7.4) which describes $d+\mu\to e+s$, namely

$$\mathscr{L}_{\text{eff}}^{d\mu\to es}=\sqrt{2}G_S\sum_{i=1}^{3}(\bar{d}_i\gamma_\mu e\bar{\mu}\gamma^\mu s_i+\text{H.c.})\,. \tag{7.6}$$

A Fierz-Michel rearrangement of Eq. (7.6) gives

$$\mathscr{L}_{\text{eff}}^{ds\to\mu e}=\sqrt{2}G_S\sum_{i=1}^{3}(d_is_i\bar{\mu}e-\bar{d}_i\gamma_5 s_i\bar{\mu}\gamma_5 e-\tfrac{1}{2}\bar{d}_i\gamma_\mu s_i\bar{\mu}\gamma^\mu e-\tfrac{1}{2}\bar{d}_i\gamma_\mu\gamma_5 s_i\bar{\mu}\gamma^\mu\gamma_5 e+\text{H.c.})\,. \tag{7.7}$$

Equation (7.7) is the basic formula which we now use to describe $K\to\mu e,\pi\mu e$.

1. $K_L\to\mu e$

For simplicity, we ignore the effect of *CP* violation here and use

$$K_{L,S}=(K^0\pm\bar{K}^0)/\sqrt{2}\,. \tag{7.8}$$

With our "kinship" hypothesis, we have the following transitions $K^0\to\mu^+e^-$ and $\bar{K}^0\to\mu^-e^+$. In reality we are looking at $K_{L,S}\to\mu^+e^-+\mu^-e^+$ with a 50% probability in each mode.

From Eq. (7.7), we learn that there are two contributions to the amplitudes $T(K_{L,S}\to\mu^+e^-,\mu^-e^+)$, one coming from the axial-vector-current part and the other from the pseudoscalar density part. To see the relative importance of these two contributions, we need to evaluate $\langle 0|(\bar{d}\gamma_\mu\gamma_5 s+\text{H.c.})|{}^{(}\bar{K}^{)0}\rangle$ and $\langle 0|(\bar{d}\gamma_5 s+\text{H.c.})|{}^{(}\bar{K}^{)0}\rangle$. Using PCAC (partial conservation of axial-vector current), we obtain

$$\langle 0|\bar{s}\gamma_\mu\gamma_5 d\,|K^0\rangle=\langle 0|\bar{d}\gamma_\mu\gamma_5 s\,|\bar{K}^0\rangle=if_Kp_\mu^K\,. \tag{7.9}$$

With the use of the equation of motion and PCAC, we also obtain

$$\langle 0|\bar{s}\gamma_5 d\,|K^0\rangle=\langle 0|\bar{d}\gamma_5 s\,|\bar{K}^0\rangle=-if_K\left[\frac{m_K^2}{m_s+m_d}\right], \tag{7.10}$$

where m_K is the kaon mass and m_s and m_d are the current-algebra masses of the s and d quarks, respectively. Since

$$p_\mu^K(\bar{\mu}\gamma_\mu\gamma_e e)=-(m_\mu+m_e)\bar{\mu}\gamma_5 e\simeq -m_\mu\bar{\mu}\gamma_5 e\,, \tag{7.11}$$

using Eq. (7.7) in conjunction with Eqs. (7.9), (7.10), and (7.11), we see that the contribution of the pseudoscalar density of the decay amplitude is larger by a factor $[2m_K^2/m_\mu(m_s+m_d)]$ than the contribution coming from the axial-vector current. With $m_s\simeq 200$ MeV, $m_d\simeq 10$ MeV, $[m_K^2/m_\mu(m_s+m_d)]\simeq 23$. The axial-vector-current contribution will be neglected from here on.

From Eqs. (7.7), (7.9), and (7.10), it is straightforward to obtain the following amplitudes:

$$A(K_{L,S}\to\mu^\pm e^\mp)=iG_Sm_K^2[f_K/(m_s+m_d)]\begin{Bmatrix}\bar{\mu}\gamma_5 e\\ \pm\bar{e}\gamma_5\mu\end{Bmatrix}, \tag{7.12}$$

where the + and − signs in the curly brackets correspond to K_L and K_S, respectively. We have to compare $K_{L,S}\to\mu^\pm e^\mp$ with $K_{L,S}\to\mu^+\mu^-$. It is generally accepted that the dominant contribution to $K_{L,S}\to\mu^+\mu^-$ comes from the two-photon intermediate states, i.e., $K_{L,S}\to\gamma\gamma\to\mu^+\mu^-$, giving the following estimated amplitudes:

$$A(K_{L,S}\to\mu\bar{\mu})\simeq iG_Fm_K^2\sin\theta_C\frac{\alpha^2}{\pi}(m_\mu/m_K)\begin{Bmatrix}\bar{\mu}\gamma_5\mu\\ \bar{\mu}\mu\end{Bmatrix}. \tag{7.13}$$

This estimation correctly predicts $B(K_L\to\mu\bar{\mu})\simeq 10^{-8}$.

Since K_S is so short-lived, we will concentrate on the comparison of $K_L\to\mu^\pm e^\mp$ with $K_L\to\mu\bar{\mu}$. Experimentally, it is known that[26]

$$B(K_L\to\mu\bar{\mu})=(9.1\pm 1.8)\times 10^{-9}\,, \tag{7.14}$$

giving

$$\frac{\Gamma(K_L\to\mu^+e^-+\mu^-e^+)}{\Gamma(K_L\to\mu\bar{\mu})}<0.2\,, \tag{7.15}$$

where $\Gamma(K_L\to\mu^+e^-+\mu^-e^+)=\Gamma(K_L\to\mu^+e^-)+\Gamma(K_L\to\mu^-e^+)$ and where $B(K_L\to\mu e)$ is given by Eq. (7.1). Using Eqs. (7.12), (7.14), and (7.15) we obtain

$$\left[\frac{G_S}{G_F\alpha^2}\right]^2\frac{2\pi^2}{\sin^2\theta_C}\left[\frac{f_K}{m_\mu}\right]^2\left[\frac{m_K}{m_s+m_d}\right]^2<0.2\ . \tag{7.16}$$

With $f_K\simeq 1.3f_\pi$ [the factor 1.3 is due to flavor SU(3) breaking] and $G_S=g_S{}^2/2\sqrt{2}m_G{}^2$, the bound (7.16) is translated into

$$\frac{\alpha_S{}^2(m_G)}{m_G{}^4}\leq 10^{-24}\ \text{GeV}^{-4}\ , \tag{7.17}$$

where $\alpha_S(m_G)=g_S{}^2(m_G)/4\pi$.

Now, according to our "minimal" petit-unification scheme,

$$\alpha_S(m_G)=\alpha_3(m_G)\ , \tag{7.18}$$

where $\alpha_3(m_G)$ is the $SU(3)_c$ coupling evaluated at the mass scale m_G.

$\alpha_3(m_G)$ can be estimated by using the formula

$$\frac{1}{\alpha_3(m_G{}^2)}=\frac{1}{\alpha_3(M_W{}^2)}-8\pi b_3\ln\frac{\widetilde{M}}{M_W}-8\pi\widetilde{b}_3\ln\frac{M_G}{\widetilde{M}}\ , \tag{7.19}$$

where

$$b_3=\frac{2n_f-33}{48\pi^2}\ , \tag{7.20}$$

$$\widetilde{b}_3=\frac{2(n_f+n_F)-33}{48\pi^2}\ , \tag{7.21}$$

and n_F is the number of mirror flavors which is equal to n_f, the number of light flavors. As discussed in Secs. V and VI the masses of mirror fermions are expected to populate the energy range from M_W to $\widetilde{M}$. Therefore, the contributions of the mirror fermions to the coefficient b_3 are in the energy range from M_W to $\widetilde{M}$ suppressed by mass effects as compared to the corresponding light-fermion contributions. In order to simplify our analysis in writing Eq. (7.19) we have neglected the contributions of mirror fermions in the evolution of α_3 from M_W to $\widetilde{M}$. On the other hand, we have included the mirror-fermion contributions to the parameter $\widetilde{b}_3$ which characterizes the evolution of α_3 from $\widetilde{M}$ to m_G. Using $\widetilde{M}=1$ TeV and Eq. (6.8) for $\alpha_S(M_W{}^2)$ with $\Lambda=0.3$ GeV and $n_f=6$ we find from (7.17) and (7.19)

$$m_G>300\ \text{TeV}\ . \tag{7.22}$$

For $n_f=8$ the bound (7.22) is changed to 350 TeV. These estimates are consistent with those of Ref. 27.

Whereas the rare decays of K mesons give the *lower* bound (7.22) on $m_G\simeq M$, the renormalization-group analysis of $\sin^2\theta_W$ of Sec. VI gives an *upper* bound on m_G. This upper bound is shown in Fig. 2 as a function of $\widetilde{M}$. It has been obtained from the requirement $\sin^2\theta(M_W{}^2)>0.206$ [see (2.28)]. Combining this upper bound with the lower bound on m_G (horizontal line in Fig. 2) as given in (7.22) and with the lower bound on $\widetilde{M}$ (vertical line in Fig. 2) as obtained from neutral-current phenomenology we observe that only a certain range of $m_G=M$ values is allowed. We can also see that as $\widetilde{M}$ increases the allowed range of m_G values decreases. In any case it is important to notice that our model can be made *simultaneously* consistent with the value of $\sin^2\theta_W$, the rates for the rare K-meson decays and with the low-energy neutral-current phenomenology.

Is there any interest on a *narrow* range for m_G? The answer is yes. The reason is the following: $\Gamma(K_L\to\mu e)$ behaves like $m_G{}^{-4}$ and any increase in m_G by a factor of 10 will bring down the rate by four orders of magnitude. For example, if $300<m_G<3000$ TeV, then $2.0\times10^{-13}<B(K_L\to\mu e)<2.0\times10^{-9}$. Experimentally it would be hard to reach the sensitivity of $B(K_L\to\mu e)\sim10^{-13}$ or less.

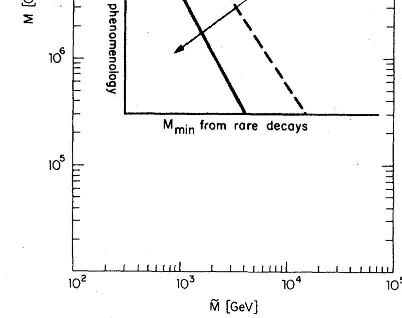

FIG. 2. The allowed region (shaded triangle) in the M-$\widetilde{M}$ plane as obtained from the measured $\sin^2\theta_W$, $K_L\to\mu e$, and electroweak phenomenology.

Looking at Fig. 2, we can see that the most experimentally interesting allowed range for m_G is $300\text{ TeV} < m_G < 1000\text{ TeV}$. This is the range for which $\tilde{M} \sim 1-10$ TeV, for n_f (light)~ 6 and 8, respectively. In this range of m_G, $2.0\times 10^{-11} < B(K_L \to \mu e) < 2.0\times 10^{-9}$. It would be interesting if future experiments on K-meson rare decays can reach the above sensitivity.

2. $K \to \pi\mu e$

Owing to theoretical and experimental uncertainties in the calculation and observations of the "normal" rare decay $K \to \pi\mu\mu$, it is hard to use the mode $K \to \pi\mu e$ to set a useful bound of m_G. We will therefore rely only on the previous decay mode, $K_L \to \mu^{\pm} e^{\mp}$, which has less theoretical uncertainties.

B. Corrections to normal charged- and neutral-current processes

Let us recall that in the minimal petite-unification model considered here the light fermions have $SU(2)_L$ and $SU(2)_R$ weak interactions. There have been numerous studies on the effect of right-handed currents to neutral-current interactions.[28] We will not repeat the analysis here but only summarize the results. It has been shown that in a typical $SU(2)_L \otimes SU(2)_R \otimes U(1)$ model, the mass of the heavier of the Z bosons must be larger than ~ 300 GeV. Similar lower limits are found for W_R. In conclusion, we have to have $m_{W_R}, m_{Z_R} > 300$ GeV. This bound sets a lower limit for $\tilde{M}$ discussed earlier.

We conclude this section by mentioning that the contribution to neutral-current interactions coming from the effective Lagrangian (7.4) which is *parity-conserving,* is negligible in view of the large leptoquark-gauge-boson mass. Furthermore, since it is parity conserving, its effects would be completely overwhelmed by the electromagnetic interactions in atomic physics and SLAC polarized $e_{L,R}$D experiments.

C. Remarks on mirror fermions

The value of $\sin^2\theta_W$ forces us to introduce mirror fermions whose weak interactions are described by $SU(2)_{L'} \otimes SU(2)_{R'}$. These fermions do not interact directly with the electroweak $W_L^{\pm}$ bosons. As we have explained earlier, their masses could range anywhere between 20 GeV to a few TeV. We have seen earlier that these mirror fermions are the exact duplicates of the ordinary ones as far as SU(4) interactions and electric charges are concerned. How then can one distinguish experimentally these genuine "new" fermions from the ordinary heavy sequential fermions?

Let us consider the *lightest* among the charge mirror fermions which we assume to be the mirror electron (positron) $E^{\pm}$. If they exist and are light enough, they could be produced in a reaction such as $e^+e^- \to E^+E^-$. Since E^+ and E^- do not couple directly to $W_L^{\pm}$, they presumably live longer than one would expect on the basis of an ordinary weak decay. The only question is whether or not there is any substantial mixing between $W_L^{\pm}$ and $W_{L',R'}^{\pm}$ which could be induced by a Higgs representation which transforms as (2,2) under $SU(2)_L \otimes SU(2)_{L',R'}$. Such mixing is expected to be small in order to keep $M_{W_L^{\pm}}$ small compared to $M_{W_{L',R'}^{\pm}}$. More serious is the question of how small or how big is the possible Yukawa coupling of the (2,2) Higgs to ordinary and mirror fermions. On one hand, such a Yukawa coupling may naturally be expected to be small if the (2,2) Higgs is dynamical, i.e., it arises through higher-order corrections of the simplest Higgs system. On the other hand, the masses of all these mirror fermions are large, suggesting a relatively large Yukawa coupling. Irrespective of these considerations, it is clearly of interest to see if the next heavy leptons (if any) have any unexpected long lifetime. Similar considerations apply to mirror quarks.

VIII. SUMMARY

In this paper we have studied a possibility of a quark-lepton unification characterized by the gauge group $G_S \otimes G_W$ with two coupling constants g_S and g_W and by the unification mass scale $M = 10^{5\pm1}$ GeV. We call such a possibility *petite unification.* In this scheme the $SU(3)_{\text{color}}$ and the standard $SU(2)_L$ group are embedded into the strong group G_S and the weak group G_W, respectively. The generator of the standard $U(1)_Y$ group is a linear combination of the generators of G_S and G_W. The latter property allows us to put quarks and leptons into identical representations of the weak group G_W and consequently make the quarks and leptons to be indistinguishable when the strong interactions are turned off. The simplest candidate for G_S turns out to be the $SU(4)_{PS}$ of Pati and Salam, in which the fundamental representation consists of a standard quark $SU(3)_c$ triplet and a lepton (lepton number being the fourth color). The choice of G_W, the type of the fermion representations under G_W, and the charges of weak gauge bosons are quite restricted by the measured value of $\sin^2\theta_W$.

This restriction becomes even stronger if we want at the same time to satisfy the experimental bounds on reactions induced by leptoquark exchanges and right-handed gauge-boson exchanges.

In particular we have found the following.

(i) Weak gauge bosons with electric charges $|Q| \geq 2$ are not allowed in our scheme (unlocked standard model) since they would lead to a too small value of $\sin^2\theta_W$.

(ii) If G_W is a pseudosimple group (i.e., a direct product of simple groups $\tilde{G}_W$ with identical coupling strength) then certain fermion representations are favored. These are the representations in which quarks and leptons transform nontrivially under one of the groups $\tilde{G}_W$ and are singlets under the rest.

We have analyzed the general case of $G_W = [SU(N)]^k$ and have found that the most economical and at the same time realistic models are $G_W = [SU(2)]^4$ and $G_W = [SU(4)]^2$.

We have presented in some detail the minimal petit-unification model $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_{L'} \otimes SU(2)_{R'}$, where the first three factors constitute the well-known Pati-Salam group.[1] The presence of $SU(2)_{L'} \otimes SU(2)_{R'}$ is required by the measured value of $\sin^2\theta_W$. For $M = 10^{5\pm1}$ GeV, as assumed in our paper, the Pati-Salam group by itself would give $\sin^2\theta_W(M_W{}^2) \approx 0.45$ which is inconsistent with experiment.

Our minimal petite-unification group $G = SU(4)_{PS} \otimes [SU(2)]^4$ is broken down to $SU(3)_c \otimes U(1)_{EM}$ in three steps as follows:

$$G \underset{M}{\rightarrow} SU(3)_c \otimes U(1)_S \otimes [SU(2)]^4$$

$$\underset{\tilde{M}}{\rightarrow} SU(3)_c \otimes SU(2)_L \otimes U(1)$$

$$\underset{M_W}{\rightarrow} SU(3)_c \otimes U(1)_{EM} ,$$

where the mass scales M, $\tilde{M}$, and M_W characterize the masses of leptoquarks (M), the weak gauge bosons of $SU(2)_R \otimes SU(2)_{L'} \otimes SU(2)_{R'}(\tilde{M})$, and the standard weak gauge bosons (M_W).

The following properties of the minimal petit-unification model should be emphasized.

(i) It accomodates all known quark and lepton generations, which are assumed to transform nontrivially under $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R$ and are singlets under the rest.

(ii) It implies the existence of mirror fermions whose masses may range from 20 − 30 GeV to a few TeV. The mirror fermions carry the standard electric charges ($+\frac{2}{3}$, $-\frac{1}{3}$ for quarks and 0, −1 for leptons) and transforming nontrivially under $SU(4)_{PS} \otimes SU(2)_{L'} \otimes SU(2)_{R'}$ are singlets under $SU(2)_L \otimes SU(2)_R$. As discussed in Sec. VII C the lightest mirror fermions might be relatively "long" lived, as opposed to an ordinary sequential heavy fermion with the same mass.

(iii) It gives the correct value of $\sin^2\theta_W(M_W) \approx 0.22$ and at the same time can be made consistent with the experimental bounds on rare decays induced by leptoquark exchanges such as $K \rightarrow \mu e$ and $K \rightarrow \pi\mu e$.

(iv) In our model as it stands the proton is stable. However, by complicating the Higgs system it is in principle possible to generate induced Yukawa couplings between quarks and leptons which in higher orders could lead to proton decay.[29]

The study of fermion-mass generation in our scheme is left for the future. Similar comments apply to a possible embedding of our petite-unification model into a grand-unification gauge group or to exploration of synthesis at a higher mass scale based on composite structures for quarks, leptons, and/or gauge quanta. Much work has to be done to explore the ideas presented in this paper. The class of models presented here imply a lot of new physics in the $10^3 - 10^6$ GeV regime without assuming what happens in the $10^6 - 10^{15}$ GeV range. For this reason it may well be that experimental hints for the relevance of the petite-unification models will be sooner visible than in the case of their grand sisters and brothers such as SU(5), SO(10), and E_6.

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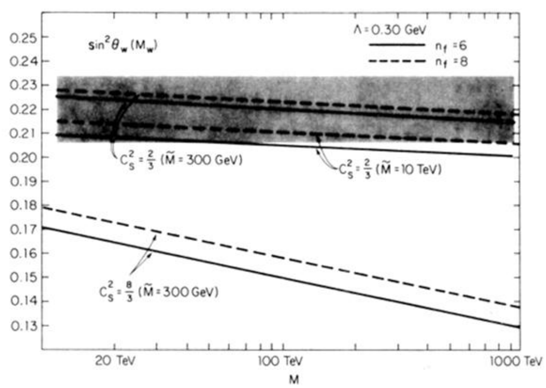

FIG. 1. $\sin^2\theta_W(M_W{}^2)$ as given by Eq. (2.20) as a function of M for $\tilde{M} = 300$ GeV, $n_f = 6$ and 8, and $C_S{}^2 = \frac{2}{3}$ and $\frac{8}{3}$. For the case $C_S{}^2 = \frac{2}{3}$ also the curve corresponding to $\tilde{M} = 10$ TeV is shown. In all cases $\Lambda = 0.3$ GeV has been used. As discussed in the text the inclusion of mirror fermions in the evolution of $\sin^2\theta_W$ changes the above curves by at most 1%.