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Electron-positron annihilation into three gluons

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We present the differential and the total cross sections for the process $e^+e^- \to ggg$ mediated by a virtual photon in the continuum. A comparison with $e^+e^- \to q\bar{q}g$ and a brief discussion of $e^+e^- \to gg\gamma$ are also given.

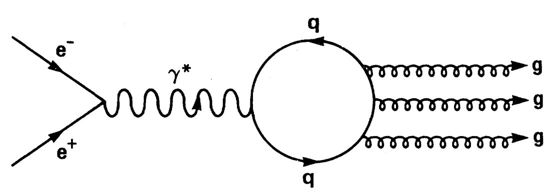
The study of three-jet final states in electron-positron collisions has flourished both in experiment and in theory over the past several years.[1,2] To lowest order in QCD the experimental results are interpreted to be electron-positron annihilation into a quark, an antiquark, and a gluon, all three materializing as jets of hadrons. The only other three-jet final state accessible in e^+e^- annihilations is a state of three gluons. We have studied the reaction $e^+e^- \to ggg$ and present the results in this paper. As a corollary, we also discuss $e^+e^- \to gg\gamma$.

We study three-gluon states as a test of higher-order QCD. The reaction proceeds via quark loops, in particular via the box diagram shown in Fig. 1(a). The triangle diagram, Fig. 1(b), vanishes essentially because of charge-conjugation symmetry. For the same reason, electron-positron annihilation into two gluons is forbidden to lowest order, $e^+e^- \to \gamma^* \to gg$, but allowed in higher order, $e^+e^- \to \gamma^*\gamma^* \to gg$.

Since $\sigma(ggg)/\sigma(q\bar{q}g) \sim \alpha_s{}^2$, we expect very few ggg events at PEP and PETRA though they are energetically accessible. We hope that the results presented here will help encourage hunting for such events, which will probably require reliable identification of gluon jets[3,4] (for example, by the fatness of the jets or their charge multiplicity). One might have to wait for a higher-energy machine like LEP, not because of some intrinsic scale (there are no thresholds to be crossed), but because of the identification problem.

Here we consider the limit of massless-quark loops only. Using the work of Costantini *et al.*[5] on photon splitting, we have found that in the limit $m \to 0$ several delicate cancellations take place to yield a finite result. These cancellations serve as a check of our calculation.

The kinematics for $e^+e^- \to ggg$ and $q\bar{q}g$ being identical, we follow the conventions of Ellis *et al.*[6] Note that θ is the angle between the positron momentum and its projection in the plane containing the three final jets. In that plane ϕ_1, ϕ_2, and ϕ_3 are the angles between jets 1, 2, and 3, respectively,



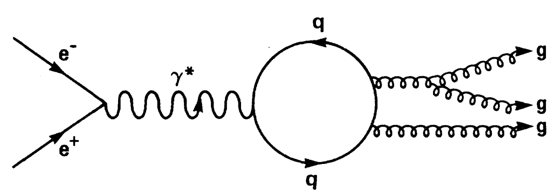


FIG. 1. Feynman diagrams for $e^+e^- \to ggg$. Permutations must be added. Only box diagrams (a) contribute. Triangle diagrams (b) vanish by charge conjugation.

and this projection axis, measured counterclockwise when seen along the direction of the positron, with $\phi_1 > \phi_2 > \phi_3$. The ϕ_i range between 0 and 2π, and θ between $-\pi/2$ and $+\pi/2$. The Euler angle χ denotes the angle between the projection axis and an arbitrary axis in the three-jet plane.

For $e^+e^- \to ggg$ we find the completely differential cross section

$$\frac{d^4\sigma(e^+e^-\to ggg)}{d\chi\, d\sin\theta\, dx_1 dx_2} = \left[\frac{\alpha}{8\pi}\right]^2 \frac{\alpha_s^3}{Q^2} C^{ggg} \left[\sum_i q_i\right]^2 \frac{d^4F}{d\chi\, d\sin\theta\, dx_1 dx_2}, \tag{1}$$

where $C^{ggg} = \frac{10}{3}$ is a color factor, and

$$\begin{aligned}\frac{d^4F}{d\chi\, d\sin\theta\, dx_1 dx_2} = \frac{2}{\pi} \Bigg\{ & A(x_1,x_2,x_3) + B(x_1,x_2,x_3)\cos^2\theta \\ & - \frac{1}{4(1-x_1)(1-x_2)(1-x_3)} [\cos\theta(\cos\phi_2 - \cos\phi_1) C(x_1,x_2,x_3) \\ & \qquad - \cos\theta(\cos\phi_3 - \cos\phi_1) C(x_1,x_3,x_2)]^2 \\ & + (1\leftrightarrow 2) + (1\leftrightarrow 3) \Bigg\} + \frac{8}{\pi}\cos^2\theta ,\end{aligned} \tag{2}$$

where $x_i = E_i/E$, the jet energy E_i divided by the beam energy $E = \frac{1}{2}\sqrt{s} = \frac{1}{2}\sqrt{Q^2}$, and $q_i =$ electric charge of quark i in units of $|e|$. The x_i satisfy $x_1 + x_2 + x_3 = 2$, and

$$\begin{aligned} A(x_1,x_2,x_3) = \{ & x_2(1-x_2)[E^1(x_1,x_2,x_3)]^2 + x_3(1-x_3)[E^1(x_1,x_3,x_2)]^2 \\ & -2(1-x_2)(1-x_3)E^1(x_1,x_2,x_3)E^1(x_1,x_3,x_2)\}/x_1(1-x_1) ,\end{aligned} \tag{3a}$$

$$B(x_1,x_2,x_3) = [E^2(x_1,x_2,x_3)]^2 , \tag{3b}$$

$$C(x_1,x_2,x_3) = x_2(1-x_2)E^1(x_1,x_2,x_3) , \tag{3c}$$

where E^1 and E^2 are helicity amplitudes occurring in the box diagram:

$$\begin{aligned} E^1(x_1,x_2,x_3) = & \frac{2(1-x_3)}{x_2} + \left[3 - \frac{1}{x_2} + \frac{2(1-x_2)}{(1-x_1)} - \frac{2(1-x_1)}{x_2^2}\right]\ln(1-x_2) \\ & + (1-x_3)\left[\frac{1}{x_3} + \frac{2}{(1-x_1)}\right]\ln(1-x_3) \\ & + \frac{1}{(1-x_1)}\left[x_2 - x_3 + \frac{2(1-x_2)(1-x_3)}{1-x_1}\right] G(x_2,x_3) ,\end{aligned} \tag{4a}$$

$$\begin{aligned} E^2(x_1,x_2,x_3) = & \left[1 - \frac{1}{x_2} + \frac{2(1-x_2)}{1-x_1}\right]\ln(1-x_2) + \left[1 - \frac{1}{x_3} + \frac{2(1-x_3)}{1-x_1)}\right]\ln(1-x_3) \\ & + \frac{1}{(1-x_1)}\left[x_1 + \frac{2(1-x_2)(1-x_3)}{1-x_1}\right] G(x_2,x_3) .\end{aligned} \tag{4b}$$

The function G is

$$G(x,y) = \ln(1-x)\ln(1-y) + \mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \pi^2/6 , \tag{5}$$

where

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\ln(1-t)\ .$$

The completely differential cross section, Eq. (1), contains too many variables to be useful. We will integrate it step by step until we get to the total cross section beginning with the variable χ. We find several simplifications which allow the result to be written in a compact form:

$$\frac{d^3\sigma(e^+e^-\to ggg)}{ds\,\sin\theta\,dx_1dx_2} = \frac{\alpha^2\alpha_s^{\,3}}{32\pi^2Q^2}C^{ggg}\left[\sum_i q_i\right]^2[\,(1+\sin^2\theta)A(x_1,x_2,x_3)+2\cos^2\theta\,B(x_1,x_2,x_3)$$
$$+(1\leftrightarrow 2)+(1\leftrightarrow 3)+8\cos^2\theta]\ . \qquad (6)$$

In Fig. 2 we plot this triply differential cross section as a function of $\sin\theta$ for $x_1=x_2=0.9$. The magnitude of the curve is different for other values of x_i, but the shape does not change much. The angular distribution $(1/\sigma_T)[d\sigma(ggg)/d\sin\theta]$ is also shown in Fig. 2. The dashed curve $1+\frac{1}{2}\cos^2\theta$ is included for comparison with $e^+e^-\to q\bar{q}g$.

Integrating over θ we obtain the energy distributions of the final jets, which are identical to the distributions obtained for the decay $Z^0\to ggg$.[7] In Fig. 3 we show these energy distributions as functions of x_1 for two values of x_2. Comparing the ggg distributions with the $q\bar{q}g$ distributions, we see that they are substantially different, particularly around the region $x_1+x_2\approx 1$, i.e., $x_3\approx 1$. In ggg we find an integrable $\ln^2(1-x_i)$ divergence as any one of the $x_i\to 1$, while $q\bar{q}g$ diverges only when $x_1\to 1$ and/or $x_2\to 1$, and is nonintegrable unless one introduces a cutoff, as discussed below. The same comments apply also to the infrared divergences $x_i\to 0$: the ggg is integrable while $q\bar{q}g$ is not.[8] The absence of the usual $1/x_i$ infrared divergence in the reaction $e^+e^-\to ggg$ can be understood because there is no "bremsstrahlung" from external legs in that case. Several cancellations occur so that Eq. (6) diverges only as $\ln^2(1-x)$ and $\ln^2(x)$ when $x\to 1$ and $x\to 0$, respectively. An important property of the G function used to prove this result is $G(x,1-x)=0$.

To obtain the total cross sections we integrate over x_1 and x_2 after introducing a cutoff parameter ϵ as was done by Ellis *et al.*[2] For $e^+e^-\to ggg$ such a cutoff is made for experimental rather than theoretical reasons, since our result is finite for $\epsilon=0$. We now perform a two-dimensional numerical integration[9] and express the result in terms of $F(\epsilon)$

$$\sigma_T(e^+e^-\to ggg) = \left[\frac{\alpha}{8\pi}\right]^2\frac{\alpha_s^{\,3}}{Q^2}C^{ggg}\left[\sum_i q_i\right]^2 F(\epsilon)\ . \qquad (7)$$

The function $F(\epsilon)$ occurs also in the decay

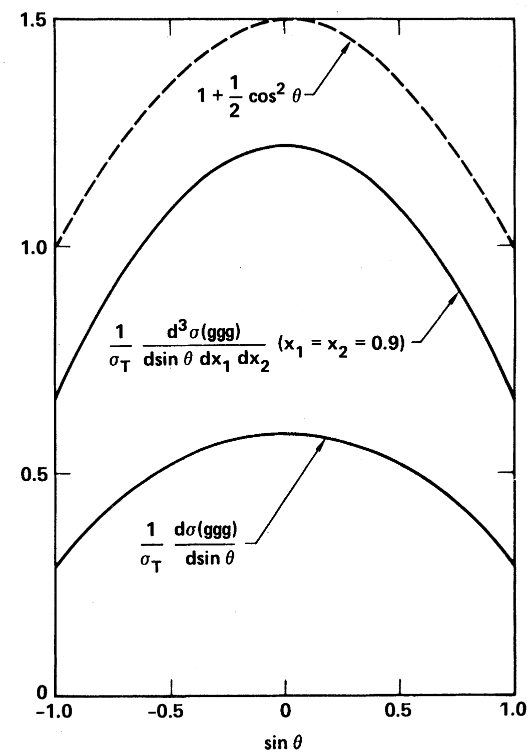


FIG. 2. The triply and singly differential angular distributions for the process $e^+e^-\to ggg$. The dashed curve is included for comparison with $e^+e^-\to q\bar{q}g$. (Note that the normalization of the dashed curve is not the same as the solid curves.)

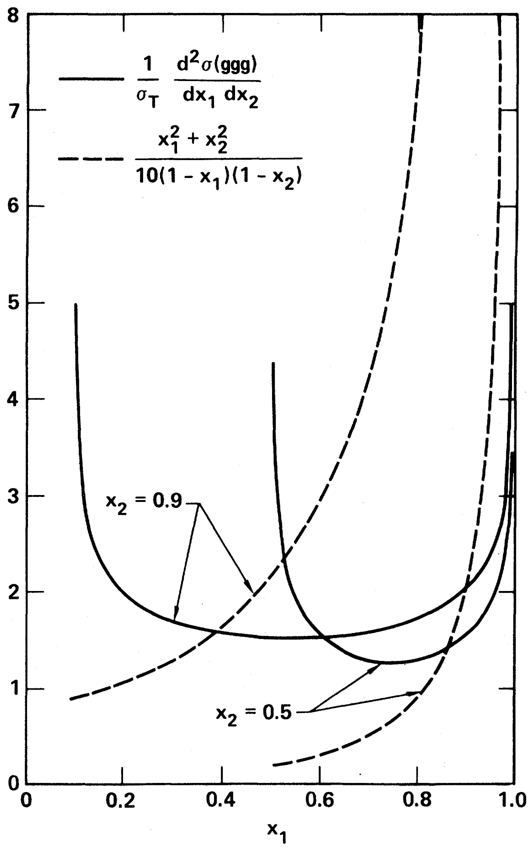


FIG. 3. The energy distribution $(1/\sigma_T)[d^2\sigma(ggg)/dx_1dx_2]$ as a function of x_1 for $x_2=0.5$ and $x_2=0.9$ (continuous curves). The dashed curves are included for comparison with $e^+e^-\to q\bar{q}g$.

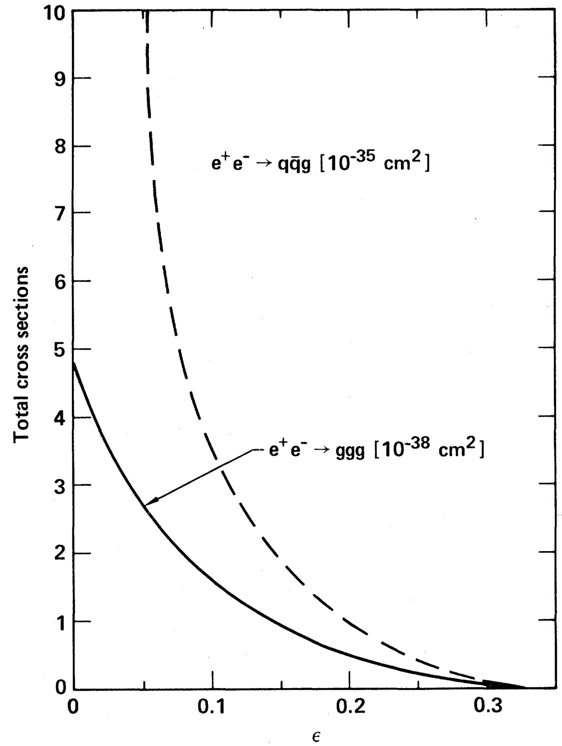


FIG. 4. The total cross sections for $e^+e^-\to ggg$ and $e^+e^-\to q\bar{q}g$ as functions of the cut-off parameter ϵ at $\sqrt{s}=40$ GeV, assuming three generations of "massless" quarks.

width $\Gamma(Z^0\to ggg)$ discussed elsewhere.[7] Numerically, we find $F(0)\simeq 80$ which, using[10] $\alpha_s=0.21$ at $Q^2=1600$ GeV^2, gives $\sigma_T(e^+e^-\to ggg)\simeq 4.8\times 10^{-38}$ cm^2. In Fig. 4 we plot the total cross sections for $e^+e^-\to ggg$ and, for comparison,[11] $e^+e^-\to q\bar{q}g$ as a function of ϵ. From this figure we expect one ggg event for every couple of thousand $q\bar{q}g$ events.

$e^+e^-\to gg\gamma$. We need only replace the factor occurring in Eqs. (1) and (7) by

$$\left[\frac{\alpha}{8\pi}\right]^2 \frac{\alpha\alpha_s^2}{Q^2} C^{gg\gamma}\left[\sum_i q_i^2\right]^2 ,$$

where the color factor $C^{gg\gamma}=8$. Hence, the branching ratio $d\sigma(gg\gamma)/d\sigma(ggg)=20\alpha/3\alpha_s \approx 23\%$.

Note that Feynman diagrams for this process with the external photon radiated off the $e^\pm$, i.e., $e^+e^-\to\gamma^*\gamma\to gg\gamma$, vanish identically.

Remarks. (i) We find that all of our helicity amplitudes are real, even though we are above the $q\bar{q}$ production threshold. The vanishing of the imaginary parts, true only in the limit $m_q/\sqrt{s}\to 0$, is connected with the absence of mass singularities,[12] but we have no physical explanation. Of course the box diagram does have imaginary parts in other channels like $\gamma\gamma\to gg$.

(ii) Substantial enhancement in three-gluon final states is expected near $q\bar{q}$ resonances, since quarkonia like J/ψ and Υ decay predominantly into three gluons. The continuum contribution that we have calculated is of course cleaner, but also very small.

(iii) Other applications of the box diagram in QED like photon splitting or Delbruck scattering have their analogs in QCD when some of the photons are replaced by gluons: e.g., photon scattering and photon conversion into one or two gluons in the color field of a target.

Note added in proof. In this paper, as well as in Ref. 7, we have assumed that the gluons are, in principle, distinguishable and, therefore, have not included the statistical factors $1/n!$ in the total cross sections and total rates (n is the number of identical particles in the final state). If, however, one considers indistinguishable *gluon jets,* the factor $1/n!$ should be included.

After this work was completed, we became aware of a related paper by V. Baĭer, E. Kurayev, and V. Fadin.[13] Our results are basically in agreement, except for the above-mentioned statistical factor. We would like to thank N. Bilić and S. Meljanac for bringing this paper to our attention.

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[9]It was necessary to calculate the dilogarithm functions very accurately because of delicate cancellations. We used the programs developed by C. Chlouber and M. A. Samuel, Comput. Phys. Commun. 15, 513 (1978).

[10]We have included the t quark in

$$\alpha_s = \frac{12\pi}{(33-2n_f)\ln(Q^2/\Lambda^2)}$$

and set $n_f = 6$, $\sum_i q_i = 1$, though presumably our approximation $m_q << \sqrt{s}$ fails for t quarks at $\sqrt{s} = 40$ GeV. $\Lambda = 0.5$ GeV.

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