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Canonical quantization of a quantum-chromodynamic effective Lagrangian

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We discuss the canonical quantizations of several slightly different effective "gluon" field terms which have been used in connection with a model designed to mock up the axial anomaly in quantum chromodynamics. A recent claim that one form of the model is inconsistent with the canonical quantization procedure is disputed.

I. INTRODUCTION

First consider an effective quantum-chromodynamic (QCD) Lagrangian of the following type[1]:

$$\mathcal{L} = \frac{c}{2}(\partial_\mu K_\mu)^2 - (\partial_\mu K_\mu)\Phi + \cdots . \tag{1.1}$$

Here K_μ is a well-known pseudovector combination of gauge fields, c is a *positive* constant, and Φ is a Hermitian combination of matter fields given by

$$\Phi = \frac{-i}{12}(\ln\det M - \ln\det M^\dagger). \tag{1.2}$$

In writing (1.2) we have specialized to a world of three flavors and adopted the normalization of K_μ given in Ref. 2 rather than Ref. 1. M is the matrix[1] of spin-zero meson fields. The three dots in Eq. (1.1) stand for "matter" terms which will not concern us in the present note. An equivalent Lagrangian was proposed in Ref. 3. One interesting feature is the elimination, by the equation of motion for K_μ, of the quantity $\partial_\mu K_\mu$ in terms of the matter field Φ (which is approximately proportional to the η' meson). Since $\partial_\mu K_\mu$ gets eliminated it is natural to try to work with a scalar field $G = \partial_\mu K_\mu$ and write instead[1-3]

$$\mathcal{L} = \frac{c}{2}G^2 - G\Phi + \cdots . \tag{1.3}$$

In fact one might want to work only with the effective Lagrangian which is obtained after all "gluon" degrees of freedom are eliminated. A derivation of this Lagrangian obtained without consideration of the intermediate stage has also been given.[4]

In addition, a more elegant description of the gluon field can be given[3,5] in terms of the so-called "topological" gauge field:

$$\mathcal{L} = \frac{-3c}{4} F_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} - \frac{i}{4}\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\Phi + \cdots . \tag{1.4}$$

The "field strengths" $F_{\mu\nu\rho\sigma}$ in (1.4) will be defined later.

Finally, it has been suggested[6] that the theory be formulated with the first term of (1.1) modified as follows:

$$\mathcal{L} = \frac{c}{2}(\partial_\mu K_\nu)(\partial_\mu K_\nu) - (\partial_\mu K_\mu)\Phi + \cdots . \tag{1.5}$$

In this case the equation of motion for K_μ will not eliminate it, but certain matrix elements of matter and gluon fields will coincide.

From the point of view of the kinds of practical (tree or semiclassical) calculations which have been made with the effective Lagrangian, it does not seem to matter much which of the four descriptions of the gluon field is used or whether it is eliminated initially. A more relevant question might be the one of the importance of possible terms like G^4 (which is expected to be suppressed by an order of $1/N_c^2$ compared to G^2). Nevertheless, for the purpose of making a deeper analysis of the structure of QCD at low energies it seems to be interesting to ask about the consistency of these Lagrangians as *quantum* theories. In particular we have in mind the question of the consistency of the quantized Hamiltonian formalisms obtained from the Lagrangians above. In doing this one is *assuming* that the fields K_μ, G, or $A_{\mu\nu\rho}$ (to be discussed later) are to be treated as *canonical* fields rather than as combinations of the underlying Yang Mills fields.

In the preceding paper[7] it is claimed that the quantization of (1.1) is inconsistent while the quantization of (1.5) is consistent. Here we attempt to show that the situation is the other way around: (1.1) can be consistently quantized while (1.5), strictly speaking, presents severe problems when one goes to the Hamiltonian formalism. We will also show that the quantization of (1.3) is (fairly trivially) consistent and in a direct way that the theory (1.4) is *identical* to (1.1). The latter result is known but we emphasize that it leads to an alternate method of quantization.

The "inconsistency" of (1.1) claimed in Ref. 7 is that since the momentum Π_4 canonically conjugate to the field K_4 vanishes by the equation of motion, we cannot satisfy the fundamental commutation relation

$$[K_4(\vec{x}, t), \Pi_4(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}). \tag{1.6}$$

However, this is not really a difficulty. In fact it occurs in ordinary Maxwell electrodynamics where $\Pi_4 = 0$. The solution is well known and requires us to set up the formalism using the theory of constrained dynamics, as expounded by Dirac,[8] for example.

II. QUANTIZATION OF EQ. (1.1)

We will work with the variables $K_\mu(x)$ and appropriate momenta $\Pi_\nu(x)$ satisfying

$$[K_\mu(\vec{x}, t), \Pi_\nu(\vec{y}, t)] = i\delta_{\mu\nu}\delta^3(\vec{x} - \vec{y}). \tag{2.1}$$

Notice that we are using the metric with "imaginary" fourth component so $g_{\mu\nu} = \delta_{\mu\nu}$. Our goal is to find a Hamiltonian formalism which is consistent with (2.1) and which reproduces the Lagrange equation of motion

$$\frac{\partial}{\partial x_\rho}(c\partial_\mu K_\mu - \Phi) = 0. \tag{2.2}$$

Equation (2.2) has as its simplest solution the matter-gluon duality relation $c\partial_\mu K_\mu = \Phi$. (Here we are neglecting the effect of the parameter θ; it can be reinstated by replacing $\Phi \to \Phi + \theta/6$.) We start by calculating the *candidates* for the canonical momenta:

$$-i\frac{\partial \mathcal{L}}{\partial(\partial_4 K_\mu)} = -i\delta_{4\mu}(c\partial_\rho K_\rho - \Phi). \tag{2.3}$$

The three space components of (2.3) vanish so we must adopt the three *constraints* or supplementary conditions,

$$\Pi_i(x)|\,\rangle = 0, \tag{2.4}$$

where $|\,\rangle$ is an allowed state of the system. Note that for compactness we are skipping the step of first working with Poisson brackets and then replacing them by commutators, etc. The fourth component of (2.3) shows that Π_4 is a true dynamical momentum given by

$$\Pi_4 = -i(c\nabla\cdot\vec{K} + c\partial_4 K_4 - \Phi). \tag{2.5}$$

Eliminating $\partial_4 K_4$ in favor of Π_4 by (2.5) we construct the Hamiltonian as

$$H = \int d^3x(\Pi_4 \dot{K}_4 - \mathcal{L}) = \int d^3x\left(\frac{1}{2c}(\Phi + i\Pi_4)^2 - i\Pi_4\nabla\cdot\vec{K}\right). \tag{2.6}$$

Notice that the $\Phi^2/2c$ term is essentially the η' mass term so it requires c to be positive. In order that the constraint (2.4) be maintained as the system evolves in time it is necessary that

$$[\Pi_i(x), H]|\rangle = 0, \tag{2.7}$$

which gives the additional constraint

$$\frac{\partial}{\partial x_i}\Pi_4(x)|\rangle = 0. \tag{2.8}$$

Equation (2.8) is maintained in time so there are no further constraints. Equations (2.4) and (2.8) commute with each other so they are *first class*, in the language of Dirac.[8] Since they are four in number they effectively eliminate all gluon degrees of freedom from the theory. Actually the theory contains four arbitrary functions $\vec{\psi}$, ψ_4 which enter as the following additional terms in the "total" Hamiltonian:

$$\int d^3x[\vec{\psi}\cdot\vec{\Pi} + (\nabla\psi_4)\cdot(\nabla\Pi_4)]. \tag{2.6a}$$

However, let us choose a gauge in which the second of these extra terms vanishes. Then the Hamilton equation for Π_4 gives

$$\partial_4\Pi_4 = -[\Pi_4, H] = 0, \tag{2.9}$$

while the Hamilton equation for K_4 gives

$$\partial_4 K_4 = -[K_4, H] = -\nabla\cdot\vec{K} + \frac{1}{c}(\Phi + i\Pi_4). \tag{2.10}$$

Taking (2.8), (2.9), and (2.10) together we see that we have once more the Lagrange equation of motion (2.2) (on allowed states). Thus the Hamiltonian treatment and hence quantization of Eq. (1.1) appear to be consistent. Integrating the second term of (2.6) by parts and using (2.8) shows that (acting on allowed states) the Hamiltonian (2.6) plus (2.6a) is manifestly positive definite. Finally it should be remarked that in Dirac's formalism, constraints are to be used only *after* evaluating all relevant commutations. Thus we should set $\Pi_4 = 0$ (or more generally $\Pi_4 = \text{const}$) only at the end of the calculation. With the gauge choice discussed above we then have for an allowed state $|\rangle$:

$$\langle|H|\rangle = \frac{1}{2c}\left\langle\left|\int d^3x\,\Phi^2\right|\right\rangle.$$

This shows how even though the starting Lagrangian has enough gauge freedom to eliminate all four gauge fields, a physical effect (η' mass term) remains. If one uses a manifestly covariant procedure to fix the choice of gauge (see Refs. 5 or 9, for example) the description of the same phenomenon appears to be considerably more complicated.

III. DISCUSSION OF EQ. (1.3)

The structure of this theory is much simpler than the preceding case. It turns out that the field G is eliminated by a *second-class* constraint so it is unnecessary to impose supplementary conditions on the states. Essentially one can then eliminate G directly by algebraic substitution. Let us demonstrate the existence of the two second-class constraints at the classical level. Introduce a canonical momentum Π obeying the Poisson bracket relation

$$[G(\vec{x},t),\Pi(\vec{y},t)]_{PB}=\delta^3(\vec{x}-\vec{y}). \tag{3.1}$$

The quantity $-i\partial\mathcal{L}/\partial(\partial_4 G)$ vanishes so we have the constraint

$$Z_1(x)\equiv\Pi(x)\approx 0. \tag{3.2}$$

The meaning of the "curly equal sign" is given in Ref. 8. The Hamiltonian is simply

$$H=-\int d^3x\,\mathcal{L}=\int d^3x\left(-\frac{c}{2}G^2+\Phi G\right), \tag{3.3}$$

so the requirement that (3.2) be maintained in time gives the additional constraint

$$Z_2\equiv -cG+\Phi\approx 0. \tag{3.4}$$

Equation (3.4) is our desired matter-gluon duality relation. Using (3.1) we find

$$[Z_1(\vec{x},t),Z_2(\vec{y},t)]_{PB}=c\delta^3(\vec{x}-\vec{y}), \tag{3.5}$$

so Z_1 and Z_2 are second class as stated.

IV. QUANTIZATION OF EQ. (1.4)

The field variable in (1.4) is a three index completely antisymmetric object $A_{\nu\rho\sigma}$. Because of the antisymmetry there are only four independent components which may be chosen to be

$$A_{123}, A_{412}, A_{413}, A_{423}. \tag{4.1}$$

Alternatively the four independent components may be taken as $K_\mu=i\epsilon_{\mu\nu\rho\sigma}A_{\nu\rho\sigma}$; explicitly

$$K_i=3i\epsilon_{ijk}A_{4jk},\qquad K_4=-i\epsilon_{ijk}A_{ijk}. \tag{4.2}$$

The field strength tensor appearing in (1.4) is a completely antisymmetric four index object $F_{\mu\nu\rho\sigma}$ defined by

$$F_{\mu\nu\rho\sigma}=\partial_\mu A_{\nu\rho\sigma}-\partial_\sigma A_{\mu\nu\rho}+\partial_\rho A_{\sigma\mu\nu}-\partial_\nu A_{\rho\sigma\mu}. \tag{4.3}$$

Because of the antisymmetry, F has only one independent component and may be written as

$$F_{\mu\nu\rho\sigma}\equiv F\epsilon_{\mu\nu\rho\sigma}. \tag{4.4}$$

The pseudoscalar quantity F is found by using (4.3) and (4.2):

$$F=\frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\partial_\mu A_{\nu\rho\sigma}=-\frac{i}{6}\partial_\mu K_\mu. \tag{4.5}$$

Thus we calculate the quantities

$$\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}=24F=-4i\partial_\mu K_\mu,\qquad F_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}=24F^2=-\tfrac{2}{3}(\partial_\mu K_\mu)^2. \tag{4.6}$$

By substituting the expressions for $\partial_\mu K_\mu$ and $(\partial_\mu K_\mu)^2$ given by (4.6) into (1.1) we arrive at the Lagrangian (1.4):

$$\mathcal{L}=-\frac{3c}{4}F_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}-\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\Phi+\cdots. \tag{1.4}$$

In other words (1.4) is just a rewriting of (1.1) with the identification of (4.2). It looks nicer[3,5] since the kinetic term displays the "right sign" and it may have a deep geometrical meaning. Since we have quantized (1.1) in Sec. II, the identification (4.2) furnishes us with a simple method to quantize the Lagrangian (1.4) written in terms of the topological gauge field. An alternative method of quantization based on the Faddeev-Popov technique is discussed in Ref. 5. Notice that our choice of normalization of $A_{\mu\nu\rho}$ given in (4.2) was made arbitrarily.

V. QUANTIZATION OF (1.5)

The canonical momenta resulting from (1.5) are

$$\Pi_\mu=-i\frac{\partial\mathcal{L}}{\partial(\partial_4 K_\mu)}=-i(c\partial_4 K_\mu-\Phi\delta_{4\mu}). \tag{5.1}$$

All four components of Π_μ involve time derivatives and are true dynamical momenta. Thus there are no constraints in this theory. This means that the theory based on (1.5) contains four real gluon degrees of freedom. The Hamiltonian is found in the usual way to be

$$H=\int d^3x\left\{\frac{1}{2c}(\Phi+i\Pi_4)^2-\frac{1}{2c}\vec{\Pi}^2-\frac{c}{2}(\partial_i K_j)^2-\frac{c}{2}(\partial_i K_4)^2+\Phi\nabla\cdot K\right\}. \tag{5.2}$$

The first term of (5.2) is the same as the first term of (2.6); this again requires c to be positive. Then we see that the second and third terms of (5.2) are each negative definite. Since there are no constraints which could somehow lead to special cancellations, the quantum theory built from the Hamiltonian formulation of (1.5) seems to be inconsistent as it stands. Finally one might consider[7] adding to (1.5) the additional term $(d/2)(\partial_\mu K_\mu)^2$ with $c\neq -d$. The above difficulty still persists however. Again the theory has no constraints [unless $c=0$ and then we are back to

(1.1)] and the Hamiltonian is

$$H=\int d^3x\left\{\frac{(\Phi+i\vec{\Pi}_1)^2}{2(c+d)}-\frac{\vec{\Pi}^2}{2c}-\frac{c}{2}(\partial_i K_j)^2-\frac{c}{2}(\partial_i K_4)^2-\frac{cd(\nabla\cdot\vec{K})^2}{2(c+d)}+\frac{1}{c+d}(\nabla\cdot\vec{K})(c\Phi-id\Pi_4)\right\}. \quad (5.3)$$

The second term of (5.3) requires $c<0$ for consistency while the fourth term requires $c>0$.

The basic difficulty with the theory based on (1.5) is that there are no constraints. Thus one is stuck with undesirable leftover terms. These include unphysical (ghost) excitations. The latter seem to be just the Kogut-Susskind ghosts[9] so it may be reasonable to argue that the effective theory should be augmented by statements that they should not be observable and should not couple to physical particles. However, these statements do not follow from the effective Lagrangian itself.

VI. DISCUSSION

One can take several points of view about what requirements to put on a low-energy effective Lagrangian for QCD. The most straightforward seems to be that all the fields (both of "matter" and "gluon" type) which remain in the Hamiltonian should correspond to real physical states. This is, after all, what one usually means by "effective." Both the Lagrangians (1.1) with a "kinetic" term $\frac{1}{2}c(\partial_\mu K_\mu)^2$ and (1.3) with a "kinetic" term $\frac{1}{2}cG^2$ obey this requirement. The quantization of these Lagrangians according to Dirac's method[8] via the Hamiltonian route was shown here to be consistent and to lead to the effective elimination of the gluon degrees of freedom. On the other hand, the Lagrangian (1.5) with a "kinetic" term $\frac{1}{2}c(\partial_\mu K_\nu)^2$ does not obey this requirement.

The long term goal, of course, is to extend the effective Lagrangian to give a description of aspects of QCD (like confinement, glueball states, etc.) other than its chiral properties. In this connection the choice of a kinetic term may point in interesting directions. The Lagrangian (1.1) was noted to be identical to the one in (1.4) which is written in terms of the topological gauge fields, $A_{\nu\rho\sigma}$. This may provide[5] a promising fundamental approach. The Lagrangian (1.3) has the advantage of being the simplest. A possible technical reason for preferring it when making an extension to also mock up the trace anomaly was noted in Ref. 2. Finally, the kinetic term itself may require modification when *other* gluon fields are present.[2,10]

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