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Families from spinors

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The possible utility of spinor representations of large orthogonal internal-symmetry groups is explored. The repetitive structure of families is incorporated quite naturally, but there is a difficulty with extra "conjugate" families having $V+A$ weak currents. Possible methods for removing these conjugate families from the low-energy spectrum are discussed. An SO(18) example is discussed in some detail. An occurrence of spinors as a classification of composite particles is discussed. A long appendix discusses useful techniques for practical calculations involving spinors.

I. FAMILIES AND SPINORS

Fundamental forces are unified, but not fundamental fermions. The thrust of contemporary physics has been to describe the strong, electromagnetic, and weak interactions in terms of a grand unified gauge theory based on a simple Lie group. Ambitious though these theories are, they fail to shed any light on the mystery that fermions appear to occur in identical families. Why does Nature repeat herself? In all the grand unified theories presently on the market, fermions in a given family are assigned to some representation (not necessarily irreducible) of the gauge group and the repetitive structure seen in Nature is accommodated simply by having this representation as many times as there are families.

It is reasonable to suppose that in the ultimate grand unified theories the fundamental fermions are unified into one single irreducible representation R of the gauge group . When one breaks $\mathcal{G}$ into some subgroup G the irreducible representation R should decompose into several copies of the representation R of G, thus reproducing the repetitive structure seen in Nature. (We are implicitly assuming that quarks and leptons are fundamental. Another approach to the family problem might involve the possibility that quarks and leptons are composite. See Sec. IV.)

Remarkably enough, the group-theory condition mentioned above goes a long way towards deciding the relevant group and representation. It is easy to see, for instance, that if $\mathcal{G}$ and G are both simple unitary groups, the tensor representations of $\mathcal{G}$ do not decompose into a direct sum of identical representations of G. As an illustration, consider the traceless tensor $T_p^{[uv]}$ of SU(8). When SU(8) is broken down to SU(5) in the standard way, the decomposition of this tensor representation contains $\underline{\bar{5}}+\underline{10}$ three times, but together with such unwanted representations as $\underline{24}$ and $\underline{5}$. Another serious problem with the simple unitary groups is that their representations are not in general free from anomalies. One then has two choices. Assign the fermions to several different representations whose total anomalies cancel [as in the SU(5) theory of Georgi and Glashow[1]] and thus frustrate the desire to unify the fermions in a single irreducible representation. Alternatively, one has to use a real representation of the simple unitary groups. There are reasons[2] for not favoring real representations. The observed fermions form a complex representation under $SU(3)\times SU(2)\times U(1)$, SU(5), and SO(10). If fermions are assigned to a real representation, then invariant bare-mass terms for the fermions are allowed in the Lagrangian.[3]

A survey of all Lie algebras reveals that only the spinor representations of orthogonal groups come close to having the desired decomposition property. The spinor representations of SO($2n$) are 2^{n-1}-dimensional. Most amazingly, the 2^{n+m-1}-dimensional spinor of SO($2n+2m$) decomposes into 2^m spinors of SO($2n$). We believe that this striking group-theoretic fact justifies exploration of a connection between repetitive family structure and spinor representations.

For the sake of completeness, a review of the group theory of spinor representations is given in the Appendix. Essentially, the reason that a spinor decomposes into a sum of spinors rests on two facts: (1) spinor representations exist because Clifford algebras exist, and (2) Clifford algebras may be constructed iteratively. Let us briefly explain. The Clifford algebra associated with SO($2n$) consists of $2n$ Hermitian matrices γ_i which satisfy

$$\{\gamma_i,\gamma_j\}=2\delta_{ij}\,. \tag{1}$$

If such a set of matrices exists, then clearly the matrices $\sigma_{ij} \equiv \frac{1}{2}i[\gamma_i, \gamma_j]$ generate rotations in the i-j plane. To show that Clifford algebras exist, one constructs them by iteration. Given $\gamma_i^{(n)}$, $i = 1, \ldots, 2n$ associated with SO($2n$), one defines

$$\gamma_i^{(n+1)} = \gamma_i^{(n)} \otimes \tau_3, \quad i = 1, \ldots, 2n\,, \tag{2}$$

$$\gamma_{2n+1}^{(n+1)} = 1 \otimes \tau_1\,, \tag{3}$$

$$\gamma_{2n+2}^{(n+1)} = 1 \otimes \tau_2\,. \tag{4}$$

(The iteration starts with $\gamma_i^{(0)} = 0$.) Note that the matrix

$$\gamma_{\text{FIVE}} \equiv (-i)^n (\gamma_1 \gamma_2 \cdots \gamma_{2n}) \tag{5}$$

anticommutes with γ_i. Thus, if ψ transforms under SO($2n$) as

$$\psi \to e^{i\omega_{ij}\sigma_{ij}}\psi\,, \tag{6}$$

then the "chiral" components

$$\psi_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_{\text{FIVE}})\psi \tag{7}$$

transform irreducibly. Thus, SO($2n$) has two spinor representations $S_\pm$ which are clearly 2^{n-1}-dimensional by construction. From the iterative construction it is obvious that spinors decompose into spinors under an orthogonal subgroup. For further details, see the Appendix.

Another significant feature is that the orthogonal groups, except for SO(6), are all anomaly free.[4] For SO($4N+2$) the spinor representations $S_\pm$ are complex and conjugates of each other.

The fact that the group decomposition property of spinors is highly suggestive of the observed repetitive family structure was noted independently by the present authors[5] and by Gell-Mann, Ramond, and Slansky.[6] A number of other authors have also studied the use of spinor representations to unify the fermions.[7] There is, unfortunately, a serious obstacle to constructing a reasonable theory. The group theory almost works out as desired, but not quite. The hitch is that S_+ of SO($2n+2m$) decomposes into an equal number of S_+ and S_- when restricted to SO($2n$). This incontrovertible group-theoretic fact may be seen readily from the construction in Eqs. (2)–(5):

$$\gamma_{\text{FIVE}}^{(n+1)} = \gamma_{\text{FIVE}}^{(n)} \otimes \tau_3\,. \tag{8}$$

Physically, we wish to start out with an SO($10+4N$) gauge theory with the fermions assigned to a complex spinor representation (S_+ or S_-, it does not matter which) and decompose the theory down to the standard SO(10) theory.[8] The difficulty, as just stated, is that a representation complex under SO($10+4N$) turns out to be real under SO(10). The SO(10) theory contains 2^{2N-1} $\underline{16}_+$'s and 2^{2N-1} $\underline{16}_-$'s. If our convention is such that the observed electron family with $V-A$ weak interactions belong to a $\underline{16}_+$, then the fermions in the $\underline{16}_-$'s would have $V+A$ interactions.

II. WHERE ARE THE $V+A$ FERMIONS?

Experimentalists assert that they have not observed fermions with $V+A$ weak interactions. The τ lepton evidently[9] decays by $V-A$ interactions. Establishing the same fact for the b quark will be an important experimental task.

We are thus left with several possibilities, which are not necessarily mutually exclusive. (1) The idea of obtaining repetitive family structure from spinor representations may be altogether wrong. Perhaps quarks and leptons are not elementary. (2) Perhaps $V+A$ fermions will be seen after all, in the next round of experiments. (3) The idea of using spinor representations is correct. However, the $V+A$ fermions are to be concealed somehow. In Ref. 5 the present authors suggested giving the $V-A$ fermions large masses by using the Higgs mechanism. This can be done. However, the scheme appears somewhat artificial and *ad hoc*, as is often the case when explicit Higgs fields are employed. A more ingenious scheme was suggested by Gell-Mann *et al.* in Ref. 6. They utilize the idea of heavy color[10] to conceal the unwanted $V+A$ fermions. After breaking SO($10+4n$) down to SO(10)×SO($4n$), they assume that some subgroup (call it HC) of SO($4n$) remains unbroken and is to be identified as heavy color. The point is that the two spinor representations $S_\pm$ of SO($4n$) in general decompose quite differently under the subgroup HC and, in particular, will contain a different number of HC singlets. The fundamental hypothesis of heavy color, that HC nonsinglets are confined by HC-strong forces, is then invoked to conceal the $V+A$ fermions.

In this paper, we explore the heavy-color scheme[11] outlined by Gell-Mann *et al.* While the explicit Higgs scheme suffers from a certain amount of arbitrariness, the heavy-color scheme is not without its share of arbitrary choices stemming from our ignorance of strong-interaction dynamics. Starting with a gauge symmetry SO($10+4n$) one can in general break the symmetry down stepwise into $\text{SU}(3)_{\text{color}} \times \text{U}(1)_{\text{em}}$ in many different ways. Without a detailed theory of symmetry breaking we do not know which symmetry-breaking chain is favored. It is also possible, and in view of the fairly complicated symmetry breaking required (Sec. II) even likely, that the pure gauge theory with spinor fermions will have to be supplemented with other fields (e.g., explicit Higgs fields).

Another problem of this program of fitting the fermion families into a single spinorial representation is the proliferation of fermions, which threatens to modify the renormalization-group analysis of Georgi, Quinn, and Weinberg.[12] As we will see, this actually constrains fairly severely the symmetry-breaking chain.

III. AN SO(18) THEORY

SO(14) is not large enough to contain more than two $V-A$ families. Thus, the smallest theory we may consider is SO(18). Let us take the spinor S_+ of SO(18). Upon breaking $SO(18)\to SO(10)\times SO(8)$ we have $S_+\to(16_+,8_+)+(16_-,8_-)$.

We are now faced with a well-defined group-theory problem. What is a subgroup HC of SO(8) such that under HC, the decomposition of $\underline{8}_-$ does not contain singlets while the decomposition of $\underline{8}_+$ does contain singlets?

According to the Appendix, the spinors $S_\pm$ decompose quite differently under the unitary subgroups. In particular, referring to the Appendix we see that upon breaking $SO(8)\to SU(4)$ we have

$$\underline{8}_+\to[0]+[2]+[4]=[0]+[2]+[\overline{0}]\,,$$

$$\underline{8}_-\to[1]+[3]=[1]+[\overline{1}]\,.$$

The notation $[k]$ ($[\overline{k}]$) denotes the representation of SU(n) realized by an antisymmetric tensor with k upper (lower) indices. If we identify SU(4) as HC we would then have two unconfined $V-A$ families. Suppose we go further and consider the breaking $SO(8)\to SU(4)\to SU(2)\times SU(2)$, then we have

$$8_+\to[0]+[2]+[\overline{0}]\to[0,0]+[0,2]+[1,1]+[2,0]+[0,0]\,,$$

$$8_-\to[1]+[\overline{1}]\to[0,1]+[1,0]+[0,\overline{1}]+[\overline{1},0]\,.$$

(Obviously, under $SU(n+m)\to SU(n)\times SU(m)$, $[k]\to\oplus_{j+l=k}[j,l]$ with $[j,l]\equiv[j]\otimes[l]$.) For SU(2), the representation [2] is equivalent to [0]. Thus, if we identify $SU(2)\times SU(2)$ as HC, then this theory contains four $V-A$ fermion families. All $V+A$ fermions are confined. Thus, the theory predicts one additional family beyond the τ family.

Incidentally, Gell-Mann *et al.* identified HC as Sp(4). The group Sp(4) contains $SU(2)\times SU(2)$ and is itself contained in SU(4). Under the breaking chain $SU(4)\to Sp(4)\to SU(2)\times SU(2)$ we have the decomposition

$$\underline{4}\to\underline{4}\to[1,0]+[0,1]\,,$$

$$\underline{6}\to\underline{5}+\underline{1}\to[1,1]+[0,0]+[0,0]\,.$$

Thus, if one stops the symmetry breaking at Sp(4), as Gell-Mann *et al.* do, one would have three $V-A$ families.

This discussion underscores the fact that, owing to our ignorance of symmetry breaking, the physical predictions of the scheme depend completely on the unbroken subgroup chosen. In the present state of the art, this choice is largely arbitrary. We could, for instance, break down further into the diagonal SU(2) subgroup of $SU(2)\times SU(2)$. In this case we end up with five $V-A$ families. It is amusing to note the local isomorphisms $SU(4)\simeq SO(6)$, $Sp(4)\simeq SO(5)$, $SU(2)\times SU(2)\simeq SO(4)$, and $SU(2)\simeq SO(3)$. Thus, in the present discussion, the number of $V-A$ families is "predicted" to be two, three, four, or five, respectively, according to whether SO(8) is broken down to SO(6), SO(5), SO(4), or SO(3). [These orthogonal subgroups are not embedded in SO(8) in the obvious way, however.]

The group SO(8) is beloved by group theorists because of its remarkable symmetry properties (as may be seen from its Dynkin diagram). In the present context it suffices to note that the two spinor representations $\underline{8}_\pm$ of SO(8) have the same dimension as the fundamental vector representation $\underline{8}_v$. There is an outer automorphism of the group which rotates these three eight-dimensional representations $\underline{8}_+$, $\underline{8}_-$, and $\underline{8}_v$ into each other. With the standard embedding of $SO(6)\cong SU(4)$ into SO(8) such that $\underline{8}_v\to\underline{6}+\underline{1}+\underline{1}$ the spinors decompose into spinors [see Eq. (A7)] $\underline{8}_\pm\to\underline{4}+\overline{\underline{4}}$. The "peculiar" embedding of SO(6) into SO(8) giving the decomposition we want is obtained by applying the outer automorphism so that the spinor $\underline{8}_+\to\underline{6}+\underline{1}+\underline{1}$ while $\underline{8}_-\to\underline{4}+\overline{\underline{4}}$. We could also easily understand the embedding of $SO(5)\cong Sp(4)$ into SO(8) such that $\underline{8}_+\to\underline{5}+\underline{1}+\underline{1}+\underline{1}$ and $\underline{8}_-\to\underline{4}+\overline{\underline{4}}$. It corresponds to the standard embedding of SO(5) into SO(6) such that $\underline{6}_v\to\underline{5}+\underline{1}$. According to the Appendix the spinors of SO($2n$) and SO($2n-1$) have the same dimensions. Thus, the embeddings given above correspond to the standard embedding followed by a "twist." Note also that SO(8) belongs to the sequence SO($4n$) of orthogonal groups in which the two spinors $S_\pm$ are real and so could decompose quite differently. From the group-theoretic point of view the SO(18) theory is thus particularly attractive.

Our discussion of the theory is incomplete without an analysis of the behavior of the gauge coupling constants under the renormalization group. As noted before, this imposes certain restrictions on the allowed chain of symmetry breaking. The group SO(18) may be broken down in many different ways. For example, we may have the chain $SO(18)\to SU(9)\to SU(5)\times SU(2)\to SU(3)\times SU(2)\times U(1)\times SU(2)$. Without a detailed dynamical theory of symmetry breaking we can only examine each of the many possible chains in turn and find the one which works best.

We suggest the symmetry breaking occurs through SO(18),

$$\mathfrak{M} \atop \rightarrow \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{SO}(8)\,, \tag{9}$$

$$\underset{\rightarrow}{M}\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\,.$$

We know of no dynamical reason why the symmetry should break down in this (somewhat peculiar) fashion.

The running gauge coupling constant behaves as[13]

$$\frac{1}{\alpha(\mu)}=\frac{1}{\alpha(M)}+\frac{1}{6\pi}\ln\frac{M}{\mu}[-11C_2(G)+2T(R)]\,. \tag{10}$$

Here $\alpha\equiv g^2/4\pi$. The index T and the Casimir invariant C_2 are defined for a given representation R by

$$\mathrm{tr}\lambda^a\lambda^b=T(R)\delta^{ab} \tag{11}$$

and

$$\sum_a\lambda^a\lambda^a=C_2(R)1\,. \tag{12}$$

The left-hand sides of Eqs. (11) and (12) are evaluated over the representation R. In Eq. (10) fermions are to be treated as two-component Weyl fields and $C_2(G)$ denotes C_2 (adjoint representations). An obvious identity follows,

$$C_2(R)d(R)=T(R)d(\mathrm{adjoint})\,, \tag{13}$$

where $d(R)$ = dimension of R. We normalize λ^a so that for SU(N), T (fundamental representation) $=\frac{1}{2}$.

An easy computation shows that for SU(N), T (fundamental) $=\frac{1}{2}$ and $C_2(\mathrm{SU}(N))=N$, while for SO($2n$) T (fundamental) $=1$, T (spinor) $=2^{n-4}$, and $C_2(\mathrm{SO}(N))=N-2$.

Incidentally, we notice that in the sequence of theories based on SO($4N+2$) and with all fermions assigned to a single spinor representation, SO(18) is the largest group for which one has an asymptotically free theory. The quantity in square brackets in Eq. (10) is equal to $[-176+64]$ for SO(18) and to $[-220+256]$ for SO(22). Whether or not this fact is significant we do not know.

There are two mass scales in the problem: a grand unifying mass scale $\mathfrak{M}$ at which SO(18) breaks down to SU(3)×SU(2)×U(1)×SO(8) and an intermediate mass scale M at which SO(8) breaks into SU(2)×SU(2) [see Eq. (9)]. Between M and $\mathfrak{M}$ there are sixteen fermion families while between $\mathfrak{M}$ and some "low-energy" mass scale μ (100 GeV say) there are four fermion families. Working to lowest order, ignoring threshold effect, etc., we write down a set of approximate renormalization-group equations:

$$\alpha_3(\mathfrak{M})=\alpha_2(\mathfrak{M})=\tfrac{5}{3}\alpha_1(\mathfrak{M})=\alpha_8(\mathfrak{M})=\alpha_G\,, \tag{14}$$

$$\alpha_3^{-1}(\mu)=\alpha_3^{-1}(M)+\frac{1}{6\pi}\ln\frac{M}{\mu}[-33+4F] \tag{15}$$

$$=\alpha_G^{-1}+\frac{1}{6\pi}\ln\frac{\mathfrak{M}}{M}[-33+4F]$$

$$+\frac{1}{6\pi}\ln\frac{M}{\mu}[-33+4f]\,. \tag{16}$$

We define

$$A\equiv\frac{4F}{6\pi}\ln\frac{\mathfrak{M}}{M}+\frac{4f}{6\pi}\ln\frac{M}{\mu}\,. \tag{17}$$

Here F and f denote the number of high- and low-energy fermion families, respectively. In the case under discussion, $F=16$, $f=4$. Introduce the standard notation: $\alpha_3\equiv\alpha_s,\alpha_2\equiv\alpha/\sin^2\theta$, and $\alpha_1\equiv\alpha/\cos^2\theta$. Then

$$\alpha_s^{-1}=\alpha_G^{-1}+\frac{1}{6\pi}\ln\frac{\mathfrak{M}}{\mu}[-33]+A\,, \tag{18}$$

$$\sin^2\theta\alpha^{-1}=\alpha_G^{-1}+\frac{1}{6\pi}\ln\frac{\mathfrak{M}}{\mu}[-22]+A\,, \tag{19}$$

$$\tfrac{3}{5}\cos^2\theta\alpha^{-1}=\alpha_G^{-1}+A\,. \tag{20}$$

Finally, the evolution of α_8 between M and $\mathfrak{M}$ is given by

$$\alpha_8^{-1}(M)=\alpha_G^{-1}+\frac{1}{6\pi}\ln\frac{\mathfrak{M}}{M}[-66+4F]\,. \tag{21}$$

We assume that below the mass scale M the coupling for SU(2)×SU(2) is in a strong-coupling regime and that we cannot trace its behavior by using lowest-order renormalization-group analysis. [Of course, if one were to naively treat the SU(2) ×SU(2) coupling to lowest order it could not be asymptotically free.]

It is also noteworthy that the SO(8) coupling is asymptotically free, albeit by a tiny margin. This is of course the reason why we chose this particular symmetry-breaking chain.

We notice that Eqs. (18)–(20) have exactly the same form as in the standard SU(5) analysis. This is of course due to the fact that low-lying fermions contribute equally to the renormalization of the SU(3), SU(2), and U(1) couplings. Thus, the two standard results

$$\alpha/\alpha_s=\tfrac{3}{10}(6\sin^2\theta-1)\,, \tag{22}$$

$$\sin^2\theta\alpha^{-1}-\alpha_s^{-1}=\frac{1}{6\pi}\ln\frac{\mathfrak{M}}{\mu}[11] \tag{23}$$

are derivable from Eqs. (18)–(20). The grand unified predictions for $\sin^2\theta$ and for $\mathfrak{M}$ are preserved.

What is changed from the standard SU(5) analysis is the value of α_G. Combining Eq. (21) with Eqs. (18)–(20), we may determine M by

$$\frac{1}{6\pi}\ln\frac{M}{\mu}\left(\tfrac{8}{3}[66-4f]\right)=14\left(\frac{\sin^2\theta}{\alpha}-\frac{1}{\alpha_s}\right)-\left(\frac{1}{\alpha}-\frac{8}{3\alpha_8(M)}\right). \qquad (24)$$

Since $\alpha_8(M)$ is supposed to be of order 1, its precise value is unimportant in determining M. In contrast, the precise value of α_G does depend on $\alpha_8(M)$.

For the sake of definiteness we put in some numbers. We take $\sin^2\theta \simeq \frac{1}{5}$. Then from Eqs. (22) and (23) we have $\alpha_s \simeq \frac{50}{3}\alpha$ and $\mathfrak{M}/\mu \simeq 1.8\times 10^{14}$. The intermediate energy scale M, above which we have 256 effectively light fermions, is determined from Eq. (24) to be $M/\mu \simeq 1.2\times 10^{8}$. Taking μ to be ~100 GeV, we find that

$$M \simeq 10^{10}\ \text{GeV}. \qquad (25)$$

Finally, we determine from Eq. (21)

$$\alpha_G^{-1} = \alpha_8^{-1}(M) + 1.5\,. \qquad (26)$$

Remembering that in the large-N limit the effective coupling for a gauge theory based on SU(N) or SO(N) is actually g^2N, we see that if we take $8\alpha_8(M)$ to be order 1, $18\alpha_G \sim 2$ is roughly of order 1. We should remind the reader that our renormalization-group analysis is very approximate and that the various output numbers are quite sensitive to the input value of $\sin^2\theta$.

Incidentally, if we were to identify as HC some subgroup of SO(8) other than SU(2)×SU(2), then the only quantity affected is M. The coefficient of $(1/6\pi)\ln(M/\mu)$ in Eq. (24) will be replaced by $\frac{8}{3}(66-4f)$, where f = the number of $V-A$ families at low energies.

The value 10^{10} GeV which we obtained in Eq. (25) for the scale at which SO(8) becomes strong is a very undesirable result. The reason is that, as one can see from the Appendix, the mass term for the $V+A$ fermions, transforming as $\underline{16}_-\times\underline{16}_-$, do not contain a piece which transforms as a singlet under the electroweak SU(2)×U(1) subgroup of SO(10). Thus, SU(2)×U(1) is necessarily broken at 10^{10} GeV which is not acceptable. To put it somewhat differently, there is an upper bound of the order of 10^2 GeV on the masses of $V+A$ fermions in theories of this kind.[14]

One might suggest at this point that the heavy-color framework be jettisoned and that an explicit Higgs mechanism be used to give the $V+A$ fermions masses of the order of 10^2 GeV. Unfortunately this is inconsistent with Eq. (24).

One possible view, though not a very satisfactory one, is that SO(18) may be a good classification symmetry but only a subgroup of SO(18) is gauged. For instance, perhaps only SO(10)×SO(8) is gauged and one does not seek to unify heavy color with SO(10). In that case, one can arbitrarily set the scale at which SO(8) becomes strongly coupled to be 10^2 GeV. There may even be some dynamical basis for thinking that SO(18) may be good only as a classification symmetry. One such scheme is sketched in the next section. Another possible scheme involves preons. For instance, suppose preons (as two-component Weyl fields) are assigned to $R\oplus R^*$ repeated n times, where R denotes a complex representation of some heavy-color gauge group which binds the preons into quarks and leptons. The theory then possesses an SU(n)×SU(n)×U(1) global symmetry. One might quite naturally gauge some subgroup of SU(n)×SU(n)×U(1) such as SO(10)×SO(8). There would be no reason why one necessarily has to unify the SO(10) vertical group with the SO(8) horizontal group. All these possibilities, however, negate our motivating philosophy of placing all the fundamental fermions in one single irreducible representation.

IV. SPINORS AS COMPOSITES

The notation of the Appendix, in which spinors under SO($2n$) are represented as n-component strings of + and − signs, suggests that in some sense these spinors are composites of n more elementary objects. In this section we shall make this notion a little more precise and note a concrete physical realization of spinor multiplets arising as composites.

A construction of spinors different, but almost trivially equivalent to, that supplied in the Appendix goes as follows.[15] Consider the algebra of N-fermion creation and destruction operators:

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \qquad i,j=1,\ldots,N \qquad (27a)$$

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0. \qquad (27b)$$

These have an obvious representation in the occupation-number space spanned by N-component strings of + and − signs:

$$|+-++-\cdots\rangle\,. \qquad (28)$$

Here in the ith place + represents the occupied, and − the unoccupied, state for fermion i. These strings span a 2^N-dimensional space.

Now the Hermitian bilinear operators

$$a_i a_j^\dagger + a_j a_i^\dagger\,, \qquad (29a)$$

$$a_i a_j + a_j^\dagger a_i^\dagger\,,\quad i(a_i a_j - a_j^\dagger a_i^\dagger) \qquad (29b)$$

close as an $[N^2+N(N-1)=2N^2-N]$-dimensional Lie algebra. It is not difficult to convince oneself that this Lie algebra is isomorphic to that of SO($2N$).

The fermion bilinears, and hence the algebra of SO(2N), are now represented acting on the 2^N-dimensional occupation-number space. Since the bilinears either leave the total number of occupied states unchanged or change it by an even number, this representation decomposes into two pieces—one with an even number of states occupied, the other with an odd number occupied. Each of these 2^{N-1}-dimensional representations, labeled S_+ and S_- in the Appendix, is an irreducible spin representation.

This representation of the spinors shows that in a certain sense they may be regarded as composites of fermions.

An interesting physical realization of this construction exists. Consider a magnetic monopole in a vector SO(3) gauge theory broken down by an isovector Higgs field to U(1). Suppose that there are N isodoublet fermions in the theory as well.

For $N=1$ this situation was analyzed by Jackiw and Rebbi.[16] They found that there is a bound state of the fermion in the presence of the monopole with exactly zero energy. This means that there are two degenerate states, i.e., the monopole with or without the zero-energy fermion added. Because of the charge-conjugation symmetry of the theory, and the fact that the states differ in fermion number by one, these states have fermion number $\pm\frac{1}{2}$.

With N fermions we will have zero-energy bound states for each, which we can fill independently. We then have 2^N states, which can be reached from one another by the creation and destruction operators $a_i^\dagger, a_j$ which add or remove a zero-energy fermion of type i. In fact, we have reproduced the mathematical setup of the spinor representation just reviewed above. The monopole states form degenerate multiplets organized into spinor representations of SO(2N).

These considerations still have us far from a realistic theory of fermions as composites, for several reasons. First of all the composites are spin-zero bosons. (This could conceivably be solved by binding an additional boson.) Second, there will be a long-ranged force between the composites generated by the unbroken U(1) in the underlying SO(3) gauge theory. The reader will note that this scheme is also plagued with the problem of having equal numbers of $V-A$ and $V+A$ fermions. We must overcome many difficulties before we have a physical theory along this line, but the mathematical framework certainly seems ripe for exploitation. Notice that in any scheme of this kind it is very easy to include gauge interactions for SU(N), which simply transform one type of fermion into another. The full SO(2N) seems very awkard (in terms of the original fermions) to gauge however. Perhaps this suggests that the spinor representations orthogonal groups are good for classification of fermions but that the interactions only gauge the unitary subgroup.

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APPENDIX

In this appendix we present various techniques for dealing with the spinorial representation of orthogonal groups in the hope that others working in this field may find this "handbook" useful. Certainly, there is nothing here that has not been known to the mathematicians for a long time. However, some of the explicit notations introduced below may prove to be efficient in physical calculations.

A list of the topics to be discussed below is as follows:

(1). Existence of spinorial representations.
(2). Construction of spinors.
(3). Conjugation properties.
(4). An explicit notation.
(5). Restriction to orthogonal subgroups.
(6). Internal parity.
(7). SO(N) for N odd.
(8). Embedding of SU(n) into SO(2n).
(9). Decomposition of spinors.
(10). Mass pattern of SU(2n) gauge bosons.
(11). Explicit identification of fermion states.

Existence of spinorial representations

We assume the reader is familiar with the definition of the orthogonal groups SO(N) and O(N) and with the construction of tensor representations. To discuss the spinorial representations one begins by proving the following fundamental theorem: There exist $2n$ Hermitian matrices γ_i, $i=1,\ldots,2n$, which are 2^n by 2^n and which satisfy

$$\{\gamma_i,\gamma_j\}=2\gamma_{ij}\,. \tag{A1}$$

The γ_i's are said to satisfy a Clifford algebra. The proof is by explicit iterative construction. For $n=1$, the matrices desired can be chosen to be two of the Pauli matrices

$$n=1:\ \ \gamma_1=\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$\gamma_2=\tau_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}.$$

To iterate from n to $n+1$, one constructs the γ matrices for $n+1$, denoted by $\gamma^{(N+1)}$, in terms of $\gamma^{(n)}$ by

$$\gamma_i^{(n+1)}=\begin{pmatrix}\gamma_i^{(n)} & 0\\ 0 & -\gamma_i^{(n)}\end{pmatrix} \text{ for } i=1,\ldots,2n\,, \tag{A2}$$

$$\gamma_{2n+1}^{(n+1)}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{A3}$$

$$\gamma_{2n+2}^{(n+1)}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}. \tag{A4}$$

This proves the theorem.

We note here, for future use, that if we define $\sigma_{ij}\equiv\frac{1}{2}i[\gamma_i,\gamma_j]$, the Hermitian matrices $\sigma_{ij}^{(n+1)}$ for SO($2n+2$) are then iteratively related to the matrices for SO($2n$) as follows:

$$\sigma_{ij}^{(n+1)}=\begin{pmatrix}\sigma_{ij}^{(n)} & 0\\ 0 & \sigma_{ij}^{(n)}\end{pmatrix}, \quad i,j=1,\ldots,2n$$

$$\sigma_{i,\,2n+1}^{(n+1)}=i\begin{pmatrix}0 & \gamma_i^{(n)}\\ -\gamma_i^{(n)} & 0\end{pmatrix},$$

$$\sigma_{i,\,2n+2}^{(n+1)}=\begin{pmatrix}0 & \gamma_i^{(n)}\\ \gamma_i^{(n)} & 0\end{pmatrix},$$

$$\sigma_{2n+1,\,2n+2}^{(n+1)}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Construction of spinors

The construction of spinors follows essentially the same procedure, familiar to physicists, of constructing spinors for the Lorentz group. We introduce the Hermitian matrices $\sigma_{ij}=\frac{1}{2}i[\gamma_i,\gamma_j]$ and the unitary matrix $U(R)=e^{i\omega\sigma}$, where $\omega\sigma\equiv\omega_{ij}\sigma_{ij}$, ω_{ij} antisymmetric in ij and real. (Repeated indices are always summed.) Then one can easily verify that

$$U^{-1}(R)\gamma_k U(R)=R_{kl}(\omega)\gamma_l\,, \tag{A5}$$

where R_{kl} is the $2n$-dimensional orthogonal matrix associated with rotations through angle ω_{ij} in the i-j plane in $2n$-dimensional space.

The map $R\to U(R)$ then defines a $2n$-dimensional unitary representation of SO($2n$). However, this representation is not irreducible. To see this we construct the matrix $\gamma_{\rm FIVE}$ by

$$\gamma_{\rm FIVE}=(-i)^n(\gamma_1\gamma_2\cdots\gamma_{2n}) \tag{A6}$$

which clearly anticommutes with γ_i since $2n$ is even. Thus, if γ transforms under SO($2n$) as $\psi\to U(R)\psi$ the "chiral" components $(1+\gamma_{\rm FIVE})\psi$ and $(1-\gamma_{\rm FIVE})\psi$ transform separately. In other words there are two irreducible spinor representations, which we denote by S^+ and S^-, with dimension $=2^{n-1}$. They may be referred to as the "right-handed" and the "left-handed" spinorial representations, respectively.

The key fact for our purposes, as is explained in the text, is that the dimensions of spinorial representations increase exponentially with the rank of the group. In contrast, the dimensions of tensorial representations only increase like a power.

Because of the block construction given in Eqs. (A2)–(A4), it is evident that spinorial representations of SO($2n$) decompose into direct sum spinorial representations of SO($2n'$) $\subset$ SO($2n$). The construction of $\gamma_{\rm FIVE}$ given in (A6) means that in block notation

$$\gamma_{\rm FIVE}^{(n+1)}=\begin{pmatrix}\gamma_{\rm FIVE}^{(n)} & 0\\ 0 & -\gamma_{\rm FIVE}^{(n)}\end{pmatrix}. \tag{A7}$$

As a consequence, the right-handed spinor S_+ of SO($2n+2m$) actually contains 2^{m-1} right-handed spinors and 2^{m-1} left-handed spinors of SO($2n$).

Conjugation properties

The notion of "conjugation" can be introduced. Let ψ transform as a spinor. Then the combination $\psi C\psi$ will be invariant under SO($2n$) if the matrix C is such that

$$C^{-1}\sigma_{ij}^T C=-\sigma_{ij} \quad (T=\text{transpose}). \tag{A8}$$

The conjugation matrix C could be constructed by iteration. Start with $C^{(1)}=i\tau_2$ for $n=1$ and define

$$C^{(n+1)}=\begin{pmatrix}0 & C^{(n)}\\ (-)^{n+1}C^{(n)} & 0\end{pmatrix}. \tag{A9}$$

One verifies easily that

$$C^{(n)-1}\gamma_i^T C^{(n)}=(-)^n\gamma_i\,. \tag{A10}$$

Notice the necessary factors of $(-)^n$ in Eqs. (A9) and (A10). They account for some of the peculiar properties of SO($2n$). In particular, Eq. (A9) and Eq. (A10) imply that for SO($2n$)

$$C^T=(-)^{n(n+1)/2}C \tag{A11}$$

and

$$C^{-1}\gamma_5 C=(-)^n\gamma_5\,. \tag{A12}$$

It follows from Eqs. (A7), (A8), and (A12) that

$$C^{-1}[\sigma_{ij}(1+\gamma_5)]^*C=C^{-1}\sigma_{ij}^*(1+\gamma_5)C$$
$$=-\sigma_{ij}[1+(-)^n\gamma_5]\,. \tag{A13}$$

This means that for SO($2n$) with n even, S^+ and S^- are real (i.e., self-conjugate). In contrast, for SO($2n$) with n odd, S^- is the conjugate of S^+ (and vice versa).

An explicit notation

In assigning physical particles to spinorial representations we find it useful to have a more explicit notation. Some of the properties discussed above also become obvious in this notation. Let us rewrite Eqs. (A2)–(A4) in a cross-product notation:

$$\gamma_i^{(n+1)}=\gamma_i^{(n)}\times\tau_3,\quad i=1,2,\ldots,2n \tag{A2$'$}$$

$$\gamma_{2n+1}^{(n+1)}=1\times\tau_1\,, \tag{A3$'$}$$

$$\gamma_{2n+2}^{(n+1)}=1\times\tau_2\,. \tag{A4$'$}$$

Here 1 denotes the unit matrix. This iterative construction yields for SO($2n$) the following forms for the γ matrices

$$\gamma_{2k}=1\times1\times1\times\cdots\times1\times\tau_2\times\tau_3\times\tau_3\times\cdots\times\tau_3 \tag{A14}$$

with 1 appearing $k-1$ times and τ_3 appearing $n-k$ times,

$$\gamma_{2k-1}=1\times1\times1\times\cdots\times1\times\tau_1\times\tau_3\times\tau_3\times\cdots\times\tau_3 \tag{A15}$$

with 1 appearing $k-1$ times and τ_3 appearing $n-k$ times. The matrices $\sigma_{2k-1,2k}$ are thus diagonal:

$$\sigma_{2k-1,2k}=-1\times1\times\cdots\times1\times\tau_3\times1\times1\cdots\times1 \tag{A16}$$

with 1 appearing $k-1$ times preceding τ_3 and $n-k$ times following τ_3.

We could denote the states in a spinorial representation of SO($2n$) by

$$|\epsilon_1\epsilon_2\cdots\epsilon_n\rangle\,, \tag{A17}$$

where ϵ_k can take the values ±1. Note that in this basis $\gamma_{\rm FIVE}=(-i)^n\gamma_1\cdots\gamma_{2n}$ has the form

$$\gamma_{\rm FIVE}=\tau_3\times\tau_3\times\cdots\times\tau_3 \tag{A18}$$

with τ_3 appearing n times. Thus the right-handed spinor consists of all those states $|\epsilon_1\epsilon_2\cdots\epsilon_n\rangle$ such that $\prod_{i=1}^n\epsilon_i=+1$, while for the left-handed spinor, the product $\prod_{i=1}^n\epsilon_i=-1$. By the iterative construction given in Eq. (A9) we see that the conjugation matrix C has the form

$$C=i\tau_2\times i\tau_2\times i\tau_2\times\cdots \tag{A19}$$

with n $i\tau$ matrices on the right-hand side. Thus C acting on $|\epsilon_1\epsilon_2\cdots\epsilon_n\rangle$ has the effect of flipping the sign of all the ϵ's. The conjugation properties of S^+ and S^- reached earlier thus follow immediately since obviously $\prod_{i=1}^n\epsilon_i$ changes its sign under C for n odd, and does not change its sign for n even.

Restriction to orthogonal subgroups

When we restrict a spinor of SO($2n+2m$) to the subgroup SO($2n$) we merely rewrite $|\epsilon_1\epsilon_2\cdots\epsilon_{n+m}\rangle$ as $|\epsilon_1\epsilon_2\cdots\epsilon_n;\ \epsilon_{n+1}\cdots\epsilon_{n+m}\rangle$. Since $\prod_{i=1}^{n+m}\epsilon_i=\prod_{i=1}^n\epsilon_i\prod_{i=n+1}^{n+m}\epsilon_i$ this confirms the previous observation about a definitely handed spinor of SO($2n+2m$) containing equal number of right-handed and left-handed spinors of SO($2n$). Explicitly, under SO($2n\pm2m$) $\rightarrow$ SO($2n$) $\times$ SO($2m$), spinors decompose as follows:

$$2_+^{n+m-1}\rightarrow(2_+^{n-1},2_+^{m-1})+(2_-^{n-1},2_-^{m-1})\,, \tag{A20a}$$

$$2_-^{n+m-1}\rightarrow(2_+^{n-1},2_-^{m-1})+(2_-^{n-1},2_+^{m-1})\,. \tag{A20b}$$

Internal parity

One may wish to extend SO($2n$) to O($2n$) $\simeq$ SO($2n$) $\times P$ where P denotes reflection in the $2n$-dimensional space. Since $\gamma_a\gamma_i\gamma_a=-\gamma_i$ if $a\neq i$ and γ_i if $a=i$, the reflection operator may be chosen to be γ_a. We see that under internal reflection $S_+\leftrightarrow S_-$, since γ_a anticommutes with $\gamma_{\rm FIVE}$. The irreducible spinorial representation under O($2n$) thus consists of (S_+,S_-). Consulting Eqs. (A14), (A15), and and (A17), we note the internal reflection acting on $|\epsilon_i\rangle$ flips one of the ϵ's.

SO($2n-1$)

Most of the discussion in this appendix will be focused on SO($2n$). For completeness, we will now briefly indicate how most of the remarks on SO($2n$) could be taken over for SO($2n-1$). The rotations in SO($2n$) are realized as transformations over the $2n$ matrices γ_i $i=1,\ldots,2n$ in the manner indicated by Eq. (A5). We consider SO($2n-1$) as a subgroup of SO($2n$) and simply throw out the matrix γ_{2n}. The rotations in SO($2n-1$) are then realized over the ($2n-1$) matrices γ_i $i=1,\ldots,2n-1$.

One may think that $\gamma_{\rm FIVE}$ could not be defined for SO($2n-1$). However, one could simply employ the $\gamma_{\rm FIVE}$ defined for SO($2n$) (Eq. A6) which obviously commutes with the generators σ_{ij} of SO($2n-1$). Thus, the irreducible spinorial representations of SO($2n-1$) in fact have the same dimension as the irreducible spinorial representations of SO($2n$), namely 2^{n-1}.

Embedding of SU(n) into SO($2n$)

Low-energy physics, however, is done with unitary groups, and not with orthogonal groups.

Color gluons and the electroweak bosons are supposed to "know" only about the unitary groups $SU(3)\times SU(2)\times U(1)$. Thus, for physical purposes, we must discuss how unitary groups are embedded into $SO(2n)$.

There is a natural embedding of $SU(n)$ into $SO(2n)$. The group $U(n)$ consists of transformations on n-dimensional complex vectors a, b leaving invariant the sum $\sum_{i=1}^n b_j^* a_j$. Writing $a_j = x_j + iy_j, b_j = x'_j + iy'_j$ we find that $U(n)$ leaves invariant $\sum_{j=1}^n (x'_j x_j + y'_j y_j)$ and $\sum_{j=1}^n (x'_j y_j - y'_j x_j)$ and is thus the "natural" subgroup of the $SO(2n)$ of rotations on the real $2n$-dimensional vector

$$v=\begin{pmatrix} x \\ y \end{pmatrix}. \tag{A21}$$

Let $R=e^M \in SO(2n)$, then M is antisymmetric and may be written as [in the basis of (A21)]

$$M=\begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}, \tag{A22}$$

where A, C are antisymmetric $n\times n$ matrices while B is an $n\times n$ matrix. R also belongs to $U(n)$ if M satisfies

$$JM + M^T J = 0\ , \tag{A23}$$

where

$$J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A24}$$

where A is antisymmetric and B is symmetric. Taking out the trace of B we are left with $SU(n)$ Thus, we have the decomposition $SO(2n)\to SU(n)\times U(1)$. Under this decomposition it is clear that the $2n$ vector representation of $SO(2n)$ decomposes into $n+\bar{n}$ of $SU(n)$ corresponding to $x\pm iy$.

It thus follows that the $n(2n-1)$ generators of $SO(2n)$, which transform as $(2n\times 2n)_A$ under $SO(2n)$ and as $[(n+\bar{n})\times(n+\bar{n})]_A$ under $SU(n)$ may be classified as follows:

$$\begin{aligned} &n^2-1 \text{ the adjoint of } SU(n);\\ &1 \text{ singlet under } SU(n);\\ &n(n-1)/2 \text{ of } SU(n);\\ &\overline{n(n-1)/2} \text{ of } SU(n)\ . \end{aligned} \tag{A25a}$$

As a check, the total number of generators is equal to $2n^2-n$. The singlet 1 generates the $U(1)$ in $SO(2n)\to SU(n)\times U(1)$. Thus, for example, the $\underline{45}$ generating $SO(10)$ decomposes as

$$\underline{45}\to \underline{24}+\underline{1}+\underline{10}+\overline{\underline{10}}\ .$$

Decomposition of spinors

This classification of the generators allows us to decompose the spinorial representation S^+ of $SO(2n)$. Let S^+ decompose into a direct sum of representations of $SU(n)$. The generators n^2-1 and 1 transform each of these $SU(n)$ representations into itself, while the generators $n(n-1)/2$ and $\overline{n(n-1)/2}$ transform each of these representations into another. Now the generators $n(n-1)/2$ and $\overline{(n(n-1)/2)}$ carry two upper (lower) $SU(n)$ indices, antisymmetrized. At this point let us introduce the notation $[k]$ ($[\bar{k}]$) to represent the representation of $SU(n)$ consisting of completely antisymmetric tensors with k upper (lower) indices. Thus, the generators under discussion transform as $[2]$ and $[n-2]=[\bar{2}]$ and act by adding or removing two upper or lower $SU(n)$ indices. Therefore we conclude that the spinorial representations of $SO(2n)$ decomposes into

$$[0]+[2]+[4]+\cdots \text{ or } [1]+[3]+[5]+\cdots\ . \tag{A25b}$$

Using the identity

$$2^{n-1}=\sum_{l=0}^{n}\binom{n}{l}=\sum_{l=1}^{n}\binom{n}{l}$$

and the fact that the dimension of k in $SU(n)$ is

$$\binom{n}{k}$$

we can determine that the sequences in Eq. (A25) terminate with $[n]$ or $[n-1]$.

Once again, we see that $SO(2n)$ behaves quite differently according to whether n is even or odd. For n odd, the sequence $[0]+[2]+\cdots+[n-1]$ is conjugate to $[1]+[3]+\cdots+[n]$, since $[n-k]$ is equivalent to $[\bar{k}]$ the conjugate of $[k]$. On the other hand, for n even the sequence $[0]+[2]+\cdots+[n]$ is conjugate to itself (similarly for the sequence $[1]+[3]+\cdots+[n-1]$).

Thus, we have determined the following decomposition of spinors. For n odd,

$$S^+\to[0]+[2]+\cdots+[n-1]\ , \tag{A26a}$$

$$S^-\to[1]+[3]+\cdots+[n]\ . \tag{A26b}$$

For n even,

$$S^+\to[0]+[2]+\cdots+[n]\ , \tag{A27a}$$

$$S^-\to[1]+[3]+\cdots+[n-1]\ . \tag{A27b}$$

To identify which sequence corresponds to S^+ or S^- we could have invoked the fact that in the $|\epsilon\rangle$ notation $|{+}{+}{+}\cdots{+}\rangle$ and $|{-}{-}{-}\cdots{-}\rangle$ are $SU(n)$ singlets. (See later.)

Since $SO(N)$ is anomaly free (except for $N=6$), the $SU(n)$ anomalies of the representations appearing in the decomposition of $SO(2N)$ spinors necessarily sum up to zero. [For $SO(6)\to SU(3)$ we have $\underline{4}_s^{\pm}$ decompose into $\underline{1}+\overline{\underline{3}}$ and $\underline{1}+\underline{3}$, respectively, which are manifestly not anomaly free.]

As an illustration of the foregoing discussion, we have for the possibly physically relevant case of SO(10) → SU(5), the anomaly-free decomposition

$$\underline{16}^{+} \to [0]+[2]+[4]=\underline{1}+\underline{10}+\overline{\underline{5}}\,.$$

Mass pattern of SO(2n) gauge bosons

It is occasionally useful to have a more explicit notation. Referring to Eqs. (A22) and (A25), we can write the generator of SO(2n) as

$$M=\begin{pmatrix} A+C & B+S \\ B-S & A-C \end{pmatrix}, \tag{A28}$$

where A, B, C are $n\times n$ antisymmetric matrices while S is an $n\times n$ symmetric matrix.

The SU(n) subgroup is generated by A and the traceless part of S. The trace of S transforms as 1 under SU(n), while $C\pm iB$ transforms as $n(n-1)/2$ and $\overline{n(n-1)/2}$.

As an application of this notation let us consider the mass pattern of the gauge bosons upon breaking SO(2n) by a Higgs boson transforming as the antisymmetric two-indexed tensor, i.e., as the adjoint representation. Physically, we are motivated to consider this choice of Higgs mechanism since the breaking of SU(5) into SU(3)× SU(2)×U(1) is induced by the $\underline{24}$-adjoint representation of SU(5). When we extend the SU(5) theory to an SO(10) theory, it is natural to suppose that the symmetry breaking is induced by the $\underline{45}$-adjoint representation which contains $\underline{24}$, as was shown earlier. Thus, we suppose the Higgs vacuum expectation value to be

$$\langle\phi\rangle=\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}. \tag{A29}$$

Here ϕ is an antisymmetric matrix. Equation (A29) is written in the same basis as Eq. (A28). v denotes a diagonal traceless $n\times n$ matrix. In the physically relevant (possibly) case of SU(5) v would be

$$\begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{bmatrix} W \tag{A30}$$

with W some number.

The gauge-boson mass terms in the Lagrangian are proportional to

$$\begin{aligned}-\mathrm{Tr}[M,\phi]^2 &= -\mathrm{Tr}(\{B,v\}^2+\{C,v\}^2+[S,v]^2-[A,v]^2)\\ &=\sum_{\mu\nu}[(B_{\mu\nu}{}^2+C_{\mu\nu}{}^2)(v_\mu+v_\nu)^2+(A_{\mu\nu}{}^2+S_{\mu\nu}{}^2)(v_\mu-v_\nu)^2].\end{aligned} \tag{A31}$$

Our first remark is that the trace of S, namely the gauge boson transforming as a singlet [see Eq. (A25)], remains massless. This may have important physical consequences.

On the other hand, if the vacuum expectation value transforms as a singlet 1, or in other words, if v in Eq. (A29) is proportional to the unit matrix, the mass term would be such that only the bosons transforming as $n(n-1)/2$ and $\overline{n(n-1)/2}$ become massive.

Occasionally, we need to explicitly identify the SO(2n) gauge bosons according to their SU(n) transformation properties. Denote the SO(2n) gauge bosons by W^{ij} $(=-W^{ji})$. Introduce the notation

$$\hat\alpha \equiv \alpha + i(\alpha+n) \tag{A32}$$

corresponding to the n SU(n) indices. A lower index $\hat\alpha$ is equivalent to an upper index $\hat\alpha \equiv \alpha - i(\alpha+n)$. Thus, we have

$$\begin{aligned} &1:\ W^{\hat\alpha}_{\hat\alpha}\,,\\ &n^2-1:\ W^{\hat\alpha}_{\hat\beta}\ \text{traceless},\\ &\overline{n(n-1)/2}:\ W_{[\hat\alpha\hat\beta]}\,,\\ &\overline{n(n-1)/2}:\ W_{[\hat\alpha\hat\beta]}\,.\end{aligned} \tag{A33}$$

The square brackets mean antisymmetrization. Thus, for example, for SO(10)→ SU(5), the SU(5) singlet is the combination

$$W^{16}+W^{27}+W^{38}+W^{49}+W^{510}\,. \tag{A34}$$

Explicit identification of fermion states

If we assign fermions to the spinor representations, we would often find it useful to know to which fermion a given component of the spinor representation corresponds. In particular, we now embed SU(3)×SU(2)×U(1) into SU(5) and in turn into SO(10). In this connection we find it somewhat more convenient to use the basis

$$v=\begin{bmatrix} x_1\\ y_1\\ x_2\\ y_2\\ \cdot\\ \cdot\\ \cdot\\ x_n\\ y_n \end{bmatrix}, \tag{A35}$$

rather than the one in Eq. (A21) which we have used so far. In this basis $1=\hat{1}+i2$, $\hat{2}=3+i4$, etc., and the singlet boson is

$$W^{12}+W^{34}+W^{56}+W^{78}+W^{910}\,. \qquad \text{(A36)}$$

We wish to embed $SU(2)_{\text{weak}}$ into the SO(4) subgroup of SO(10) generated by σ_{ij}, $1\leq i,\ j\leq 4$. As is well known there are two inequivalent ways to embed SU(2) in SO(4): magnetic minus electric:

$$\tau_k \longleftrightarrow \epsilon_{kij}\sigma_{ij}-\sigma_{k4}\,,\quad i,j,k=1,2,3 \qquad \text{(A37)}$$

or magnetic:

$$\tau_k \longleftrightarrow \epsilon_{kij}\sigma_{ij}\,. \qquad \text{(A38)}$$

Here τ_k denote the generators of SU(2) and the double arrows indicate the SO(4) generators with which they are to be identified. We will see that, physically, the magnetic minus electric method corresponds to the Glashow-Weinberg-Salam-Ward theory, while the magnetic method leads to the Cheng-Li-Bilensky-Petkov theory.

For the moment, let us study the instructive $SU(2)\subset SO(4)$ example with the help of the cross-product notation introduced in Eqs. (A16)–(A18). Then the spinorial representation S^+ of SO(4) consists of the states $|++\rangle$ and $|--\rangle$, while S^- consists of $|+-\rangle$ and $|-+\rangle$. In the "$B-E$" embedding we have

$$\tau_3 \longleftrightarrow \sigma_{12}-\sigma_{34}=-\tau_3\times 1+1\times\tau_3\,. \qquad \text{(A39)}$$

(The Pauli matrices τ_i on the left- and right-hand side should not be confused.) Thus, under SO(4) $\to$ SU(2), S^+ decomposes into two singlets while S^- decomposes into a doublet:

$$\underline{2}^+\to\underline{1}+\underline{1}\,, \qquad \text{(A40)}$$

$$\underline{2}^-\to\underline{2}\,. \qquad \text{(A41)}$$

This fact that S^+ and S^- decomposes quite differently may have important physical consequences. It also confirms nicely the previously proven theorem [Eq. (A13)] that S^+ and S^- are conjugate of each other only for $SO(2n)$ with n odd. On the other hand, with the "B" embedding we have, under $SO(4)\to SU(2)$, both S^+ and S^- decomposing into doublets. Henceforth, we concentrate on the $B-E$ embedding unless otherwise stated.

For SO(6), S^+ consists of $|+++\rangle$, $|+--\rangle$, $|-+-\rangle$, and $|--+\rangle$. Under $SO(6)\to SU(3)$, it decomposes as

$$\underline{4}^+\to\underline{\bar{3}}+\underline{1}\,. \qquad \text{(A42)}$$

Obviously, the state $+++$ is the "odd man out" and transforms as 1 under SU(3). According to our general analysis, we must have

$$\underline{4}^-\to\underline{3}+\underline{1}\,. \qquad \text{(A43)}$$

[The identification of $\underline{3}$ and $\underline{\bar{3}}$ in (A42) (A43) is such as to conform to our convention below.]

Having learned how to embed SU(2) into SO(4) and SU(3) into SO(6), we are now ready to embed SU(5) into SO(10). The spinor representation S^+ of SO(10) contains 16 states: $|\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle$, $\epsilon_i=\pm 1$, and $\sum_{i=1}^5\epsilon_i=+1$. We let SU(2) "act on" ϵ_1 and ϵ_2, and SU(3) on ϵ_3, ϵ_4, and ϵ_5. From what we have learned in Eqs. (A40)–(A43) we can now just read off the physical particle states (all left-handed),

SU(2) doublets

$$\begin{aligned}
d&=|+-++-\rangle \text{ and color permutations}\\
&=|+-++-\rangle,\quad |+-+-+\rangle,\quad |+--++\rangle,\\
u&=|-+++-\rangle,\text{ and color permutations},\\
e^-&=|+----\rangle,\\
\nu&=|-+---\rangle;
\end{aligned} \qquad \text{(A44)}$$

SU(2) singlets

$$\begin{aligned}
u&=|+++--\rangle,\text{ and color permutations},\\
d&=|--+--\rangle,\text{ and color permutations},\\
e^+&=|--+++\rangle,\\
N&=|+++++\rangle.
\end{aligned}$$

[Incidentally, it is now clear that, if we had used the magnetic embedding of $SU(2)_w$, then we would not have SU(2) singlets. The theory contains the weak-interaction doublets

$$\begin{pmatrix}u\\ d\end{pmatrix}_L,\ \begin{pmatrix}\bar{d}\\ \bar{u}\end{pmatrix}_L,\ \begin{pmatrix}\nu\\ e^-\end{pmatrix}_L,\ \begin{pmatrix}e^+\\ N\end{pmatrix}_L \qquad \text{(A45)}$$

and is "vectorlike."]

With the assignment in Eq. (A43) we can read off the coupling of the SU(5) singlet gauge boson in Eq. (A36). It couples to

$$\begin{aligned}
-(\sigma_{12}+\sigma_{34}+\cdots+\sigma_{910})=\tau_3&\times 1\times 1\times 1\times 1+1\\
&\times\tau_3\times 1\times 1\times 1\times\cdots\\
&+1\times 1\times 1\times 1\times\tau_3
\end{aligned} \qquad \text{(A46)}$$

which acting on $|\epsilon_1\cdots\epsilon_5\rangle$ just gives $\sum_{i=1}^5\epsilon_i$. We will denote $\frac{1}{5}\sum\epsilon_i$ by X. Thus we find that acting on the SU(5) representations $\underline{1}$, $\underline{10}$, $\underline{5}$ the values of $5X$ are in the ratio $5:1:-3$. It is easy to see that in general, for the case $SO(2n)\to SU(n)$, the corresponding ratio for S^+ is

$$n:n-4:n-8:\cdots, \qquad \text{(A47)}$$

and for S^- is

$$n-2:n-6:n-10:\cdots. \qquad \text{(A48)}$$

Another way of deriving this is to go back to the notation of Eq. (A28) and compute the commutation relation between

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} C & \pm iC \\ \pm iC & -C \end{pmatrix},$$

where C denotes an $n\times n$ antisymmetric matrix. This is of course just the usual SU(2) algebra with the raising and lowering operators equal to $\tau_3 \pm i\tau_1$.

It turns out that in SO(10), baryon number minus lepton number is actually a gauge generator. By inspecting Eq. (A43) we see that

$$B-L=\tfrac{1}{3}(\epsilon_3+\epsilon_4+\epsilon_5)\,. \tag{A49}$$

That B-L is proportional to $(\epsilon_3+\epsilon_4+\epsilon_5)$ is obvious since $SU(2)_w$ commutes with baryon and lepton numbers. Similarly, since $\Delta Q=\pm 1$ under weak interactions, we find that electric charge is given by

$$Q=-\tfrac{1}{2}\epsilon_1+\tfrac{1}{6}(\epsilon_3+\epsilon_4+\epsilon_5)\,, \tag{A50}$$

while the weak hypercharge is given by

$$\frac{Y}{2}=Q-T_3=-\tfrac{1}{4}(\epsilon_1+\epsilon_2)+\tfrac{1}{6}(\epsilon_3+\epsilon_4+\epsilon_5)\,. \tag{A51}$$

Thus, the three generators B-L, $Y/2$, and X are not linearly independent,

$$B-L=X+\frac{4}{5}\left(\frac{Y}{2}\right). \tag{A52}$$

This fact is relevant to the discussion of B-L violation in the SU(5) and SO(10) unification schemes.

We now discuss the formation of mass terms for the fermions assigned to the spinorial representations. We could form the following invariants:

$$(\psi C_D C \Gamma_\kappa \psi)\phi\,. \tag{A53}$$

The notation is highly schematic. Γ_κ denotes an antisymmetric product of $\kappa\gamma$ matrices. ϕ is a scalar field transforming like an antisymmetric tensor with κ indices under SO($2n$). C and C_D denote the conjugation matrix under SO($2n$) and the Lorentz group respectively. C_D is antisymmetric.

Using the facts that $C\gamma_{\text{FIVE}}=(-)^n\gamma_{\text{FIVE}}C$ [Eq. (A12)] and that γ_{FIVE} is symmetric, we learn that for SO($2n$) the coupling in Eq. (A53) is allowed only for κ odd if n is odd (and for κ even if n is even), since ψ is an eigenstate of γ_{FIVE}. Next we consider the constraint imposed by the peculiar symmetry property of C [Eq. (A11)]. Since ψ is anticommuting, we have

$$\begin{aligned}\psi C_D C\Gamma_\kappa\psi &= -\psi C_D^T\Gamma_\kappa^T C^T\psi \\ &=(-)^{n(n+1)/2}\psi C_D C(C^{-1}\Gamma_\kappa^T C)\psi\,.\end{aligned} \tag{A54}$$

Now

$$\begin{aligned}C^{-1}(\gamma_\kappa\cdots\gamma_1)^T C &= C^{-1}(\gamma_1^T\cdots\gamma_\kappa^T)C=(-1)^{n\kappa}(\gamma_1\cdots\gamma_\kappa)\\ &=(-1)^{n\kappa}(-1)^{\kappa(\kappa-1)/2}(\gamma_\kappa\cdots\gamma_1)\,,\end{aligned}$$

where we have used Eqs. (A10) and (A11). Thus, the coupling in Eq. (A53) is allowed only if the combination

$$n(n+1)/2+n\kappa+\kappa(\kappa-1)/2=(n+\kappa)(n+\kappa+1)/2-\kappa \tag{A55}$$

is even. This peculiar requirement could be further analyzed as follows. For n odd, κ is required to be odd and so $(n+\kappa)(n+\kappa+1)/2=\sum_{l=1}^{n+\kappa} l$ is required to be odd. This fixes (for n odd)

$$n+\kappa=4k+2 \tag{A56a}$$

with k an integer. On the other hand, for n even we find

$$n+\kappa=4k \tag{A56b}$$

with k an integer. To give some examples, consider the cases mentioned in the text:

SO(10), $n=5$, $\kappa=4k+1=1,5$,

SO(14), $n=7$, $\kappa=4k+3=3,7$,

SO(16), $n=8$, $\kappa=4k=0,4,8$.

For instance, to give mass to fermions assigned to the spinorial representation of SO(10), the Higgs fields have to transform as [1] and [5], i.e., as $\underline{10}$ and $\underline{126}$. Here, as before, the notation l represents the antisymmetric tensor with l indices. Note that $\underline{5}$ can be taken to be either self-dual or anti-self-dual and thus has dimension $\frac{1}{2}[10!/(5!5!)]$.

More generally, fermions may not be assigned to a single spinorial representation. In the physically relevant (possibly) example of SO(10), fermions of a given family are assigned to a spinorial representation. In that case the coupling in Eq. (A53) has to be generalized to read

$$f_{ab}(\psi_a C_D C\Gamma_\kappa\psi_b)\phi\,, \tag{A57}$$

where a, b are family indices. Higgs fields satisfying the constraints in Eq. (A56) would thus contribute to the mass matrix a term symmetric in the family indices, while Higgs fields not satisfying these constraints induce a term antisymmetric in the family indices.

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