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Finite-temperature symmetry breaking as Bose-Einstein condensation

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The effects of a net background charge on ideal and interacting relativistic Bose gases are investigated. For a non-Abelian symmetry only chemical potentials that correspond to mutually commuting charges may be introduced. The symmetry-breaking pattern is obtained by computing a μ-dependent functional integral. We find that μ always raises the critical temperature and that below that temperature the existence of a ground-state expectation value for some scalar field produces Bose-Einstein condensation of a finite fraction of the net charge so as to keep the total charge fixed. (In the special, but familiar, case of total charge neutrality, the condensate contains equal numbers of particles and antiparticles.) There are four classes of results depending on whether volume or entropy is kept fixed and on whether the quadratic mass term m^2 is positive or negative.

I. INTRODUCTION

Many of the gauge and global symmetries which are broken in the currently observed universe are thought to have been exact symmetries in an earlier, hotter epoch. The critical temperature at which a particular phase transition occurs is determined (to leading order) by the vanishing of a scalar-boson effective mass of the form[1]

$$0=m^2+(C_1e^2+C_2f^2+C_3\lambda)T_C{}^2 \,, \tag{1.1}$$

where e, f, λ are typical gauge, Yukawa, and quartic couplings of the scalar bosons and where $m^2<0$ in order that there be a phase transition.

We shall investigate how these phase transitions could be affected by the presence of finite densities of the various conserved charges. A chemical potential $\mu(T)$ must be introduced into the functional integral for each type of background charge. We find it useful to characterize a charge as either bosonic or nonbosonic, according to whether it is carried by some boson in the theory or not. For example, in the standard $\mathrm{SU}(2)\times\mathrm{U}(1)$ model with one scalar doublet the bosonic charges are T_3 and Y (or, equivalently, $Q=T_3+Y$ and $Q^z=T_3\cot\theta_w-Y\tan\theta_w$); the nonbosonic charges are baryon number and the separate lepton number of each generation $(e,\mu,\tau,\ldots)$. Obviously gauge symmetries are always bosonic but, in general, a global symmetry may be either bosonic or nonbosonic.

The effect of a net background of nonbosonic charge was investigated by Harrington and Yildiz.[2] The effects are quite small; precisely because the scalar bosons do not carry the charge, the only way in which the chemical potential μ can contribute to the effective mass (1.1) is through fermion loops. Consequently the equation for the modified critical temperature T_C' becomes

$$0=m^2+(C_1e^2+C_2f^2+C_3\lambda)T_C'^2+C_4f^2\mu^2 \,, \tag{1.2}$$

with $C_4>0$.[2] This lowers the critical temperature $(T_C'<T_C)$ but the effect is usually negligible due the the smallness of the Yukawa couplings f. The nonbosonic charge (e.g., baryon or lepton number) remains conserved at $T<T_C'$ and the corresponding fermions remain in thermal equilibrium, essentially unaffected by the phase transition.[3]

In this paper we shall investigate the effects of bosonic charges and will find very different effects than those above. Our central results will be more obvious if we first discuss a toy model. The model consists of a two-component real scalar field ϕ_a $(a=1,\,2)$ with O(2)-invariant Hamiltonian and conserved charge Q:

$$\mathscr{H}=\tfrac{1}{2}\pi_a\pi_a+\tfrac{1}{2}(\vec{\nabla}\phi_a)\cdot(\vec{\nabla}\phi_a)+\tfrac{1}{2}m^2\phi_a\phi_a+\frac{\lambda}{4!}(\phi_a\phi_a)^2 \,, \tag{1.3}$$

$$Q=\int d^3x\,(\phi_1\pi_2-\phi_2\pi_1) \,. \tag{1.4}$$

The effects of a net background of this charge are computed in the grand canonical ensemble from $H-\mu Q$. The corresponding μ-dependent, Hamiltonian density may be thought of as

$$\mathscr{H}(\mu)\equiv\mathscr{H}-\mu(\phi_1\pi_2-\phi_2\pi_1) \,. \tag{1.5}$$

We will now show that the presence of μ in (1.5) actually modifies the classical potential. Applying Hamilton's equation yields

$$\dot{\phi}_a = \frac{\delta \mathscr{H}(\mu)}{\delta \pi_a} = \pi_a + \mu \epsilon_{ab} \phi_b \ , \tag{1.6}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$. A proper functional integral calculation requires integrating out the canonical momenta. In either approach one obtains a μ-dependent Lagrangian density

$$\begin{aligned} \mathscr{L}(\mu) &= \pi_a \dot{\phi}_a - \mathscr{H}(\mu) \\ &= (\partial_\nu \phi_a)(\partial^\nu \phi_a) - \mathscr{V}(\phi) \\ &\quad + \mu(\phi_1 \overleftrightarrow{\partial}_0 \phi_2) \ , \end{aligned} \tag{1.7}$$

where the classical potential that results is

$$\mathscr{V}(\phi) = \tfrac{1}{2}(m^2 - \mu^2)\phi_a \phi_a + \frac{\lambda}{4!}(\phi_a \phi_a)^2 \ . \tag{1.8}$$

From this potential it is already obvious that the critical temperature will be determined by a relation of the form

$$0 = m^2 - \mu^2 + (C_1 e^2 + C_2 f^2 + C_3 \lambda) T_C''^{\,2} \ . \tag{1.9}$$

There are a number of consequences: (i) if $m^2 < 0$ the background charge raises the critical temperature ($T_C'' > T_C$) and this shift may be appreciable since there is no coupling-constant suppression. (ii) For all $T < T_C''$ the ground-state expectation value $\langle 0 | \phi_a | 0 \rangle \equiv \hat{\phi}_a(T)$ is nonzero and larger in magnitude than when $\mu = 0$. (iii) For all $T_C < T_C''$ the canonical momenta defined by (1.6) have expectation values $\langle 0 | \pi_a | 0 \rangle = -\mu \epsilon_{ab} \hat{\phi}_b(T)$ and consequently there is a net charge

$$\frac{Q}{V} = \mu(\hat{\phi}_1{}^2 + \hat{\phi}_2{}^2) \tag{1.10}$$

stored in the ground state. (iv) Even if $m^2 > 0$ the symmetry will break once $\mu(T)$ becomes larger than m. What we find, in short, is that finite-temperature symmetry breaking in the presence of a background charge is just a form of Bose-Einstein condensation. Indeed, for the standard case of no background charge [$\mu(T) = 0$] the Bose-Einstein ground state is a superposition with equal probabilities for particles and antiparticles.

That nonrelativistic Bose-Einstein condensation is a broken-symmetry phenomenon was originally shown by Bogoliubov.[4,5] There one is interested in the behavior of a complex (nonrelativistic) field operator ψ with μ-dependent Hamiltonian density

$$\mathscr{H}(\mu) = \frac{1}{2m} | \vec{\nabla} \psi |^2 + \frac{\lambda}{4! m^2} (\psi^\dagger \psi)^2 - \mu_{\mathrm{NR}} \psi^\dagger \psi, \tag{1.11a}$$

$$\mu_{\mathrm{NR}} \equiv \mu - m \ . \tag{1.11b}$$

The conserved charge

$$Q = \int d^3x \, \psi^\dagger \psi \tag{1.12}$$

generates U(1) phase transformations and corresponds to particle number. Here the standard results are that at high temperatures $\mu < m$ and the symmetry is exact but at low temperatures $\mu > m$ and the symmetry is broken. The theory described by (1.5) is just the relativistic version of the Bogoliubov model[6]; the essential new feature that complicates the analysis of (1.5) is the option of taking $m^2 < 0$.

Our approach is as follows. In Sec. II we compute the grand partition function for the ideal Bose gas using a functional integral. We use (1.5) with $\lambda = 0$ and $m^2 > 0$. For small charge densities ($\rho << m^3$), Bose-Einstein condensation occurs at nonrelativistic temperatures ($T_C << m$) and the usual textbook results apply. For large charge densities ($\rho >> m^3$), Bose-Einstein condensation occurs at relativistic temperatures ($T_C >> m$). This section and the appendices on the high temperature expansion[7] are a fuller account of some results quoted in a recent letter.[8]

In Sec. III we examine the interacting Bose gas with an O(N) non-Abelian symmetry. We show that one cannot introduce a chemical potential for every group generator but only for those generators that are mutually commuting (i.e., the Cartan subalgebra).[9] To determine the pattern of symmetry breaking a μ-dependent effective potential is introduced and is explicitly computed in the large-N limit.[10] The potential depends on an effective mass function $M(T)$ that provides a close analogy to the ideal-gas case in that at high temperature $\mu(T) < M(T)$. However, at low temperatures $\mu(T) = M(T)$, the field operator acquires an expectation value, and a finite fraction of the charge is in the Bose-Einstein condensed ground state. For $m^2 < 0$ the previous high-temperature expansions apply because $T_C >> M(T_C)$ and $\rho >> M^3(T_C)$. We discuss the distinctions between keeping Q and V fixed or keeping Q and S fixed.

In Sec. IV we suggest various extension of these ideas, particularly the inclusion of fermions and gauge bosons.

II. THE IDEAL BOSE GAS WITH $\mu \neq 0$

A. Derivation via the functional integral

We analyze the model field theory (1.3) with conserved charge (1.4). To study the thermodynamics of these bosons at finite charge density and temperature, we compute the grand partition function

$$\exp(-\beta\Omega)=\mathrm{Tr}\{\exp[-\beta(H-\mu Q)]\}\ , \tag{2.1}$$

$$\beta\equiv 1/T\ , \tag{2.2}$$

where μ is the chemical potential and $\hbar=c=k=1$. The thermodynamic potential is denoted by $\Omega(T,V,\mu)$. Following Bernard,[11] we write $e^{-\beta\Omega}$ as a functional integral:

$$\exp(-\beta\Omega)=N\int[d\pi][d\phi]\exp\left\{\int_0^\beta d\tau\int d^3x[i\pi_a\dot{\phi}_a-\mathscr{H}(\pi,\phi)+\mu(\phi_1\pi_2-\phi_2\pi_1)]\right\}\ , \tag{2.3}$$

where τ is real Euclidean time and the integration variables ϕ_a are periodic in τ with period β.[12] Note that $\dot{\phi}_a\equiv\partial\phi_a/\partial\tau$.

We first derive the standard thermodynamics of an ideal Bose gas by putting $\lambda=0$ in (1.3). Then all the integrations in (2.3) are Gaussians and can be done exactly. First the integrations over π_1 and π_2 are performed. Next, we expand $\phi_a(\vec{x},\tau)$ in a Fourier series,

$$\phi_a(\vec{x},\tau)=\frac{1}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}e^{i(\vec{k}\cdot\vec{x}+\omega_n\tau)}\phi_a(\vec{k},n)\ , \tag{2.4}$$

where $\omega_n\equiv 2\pi n/\beta$, $n=0,\ \pm1,\ \pm2,\ldots$. We find

$$\exp(-\beta\Omega)=N'(\beta)\int[d\phi]\exp\left[\frac{-1}{2\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\phi_a(-\vec{k},-n)A_{ab}\phi_b(\vec{k},n)\right]\ , \tag{2.5}$$

where

$$A=\begin{bmatrix}\omega_n{}^2+\vec{k}^2+m^2-\mu^2 & -2\mu\omega_n\\ 2\mu\omega_n & \omega_n{}^2+\vec{k}^2+m^2-\mu^2\end{bmatrix}\ . \tag{2.6}$$

The sum over $a,b=1,2$ is implied and $N'(\beta)$ is a β-dependent factor arising from the integration over the π_a .[12]

We now make an important observation. If the integration in (2.5) is to be convergent, the argument of the exponential must be negative definite. Because the sum over n and integration over $\vec{k}$ include the points $n=0$ and $\vec{k}=0$ we must require that $m^2-\mu^2\geq 0$. Thus $|\mu|\leq m$ is required in order to obtain sensible results.

Performing the integration in (2.5) (and dropping β-independent constants), we find

$$\beta\Omega=\tfrac{1}{2}\ln\mathrm{Det}(A_{11}A_{22}-A_{12}A_{21})-\ln N'(\beta)\ . \tag{2.7}$$

The determinant is defined over the appropriate function space spanned by the $\phi(n,\vec{k})$. Using $\ln\mathrm{Det}X=\mathrm{tr}\ln X$ we obtain

$$\beta\Omega=\frac{V}{2}\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\{\beta^4[(\omega_n{}^2+\vec{k}^2+m^2-\mu^2)^2+4\mu^2\omega_n{}^2]\}\ , \tag{2.8}$$

where V is the volume of x space. As shown by Bernard,[12] the effect of the $\ln N'(\beta)$ term in (2.7) is to provide the appropriate factors of β inside the argument of the logarithm in (2.8) so that the argument is dimensionless. The sum over n in (2.8) can be performed after factoring the argument of the logarithm into two factors by solving for $\beta^2\omega_n{}^2$. The end result is

$$\Omega=V\int\frac{d^3k}{(2\pi)^3}\{E+T\ln[1-e^{-\beta(E-\mu)}]+T\ln[1-e^{-\beta(E+\mu)}]\}\ , \tag{2.9}$$

where $E=(\vec{k}^2+m^2)^{1/2}$. The factor of E is simply the sum of the two zero-point energies for the two charged particles. This piece is irrelevant for the discussions of thermodynamics (it is absent in the free-field theory if the Hamiltonian is normal ordered); so we will drop it for the remainder of this section. What remains is a result similar to the standard one given in textbooks of statistical mechanics.[13] The only difference is that in (2.9) (neglecting the zero-point energy) we have two terms rather than just one. The correct interpretation is to identify one term corresponding to particles (with charge + 1) and the second term corresponding to antipatricles (with charge − 1). Loosely speaking, we may say that particles and antiparticles have equal and opposite chemical potentials μ. More precisely, only one chemical potential μ decribes a system of bosons. As we have seen, μ is constrained such that $|\mu| \leq m$; the sign of μ indicates whether particles outnumber antiparticles or vice versa. The appearance of two terms related to each other by $\mu \rightarrow -\mu$ is a consequence of the fundamental structure of relativistic field theory which requires that the eigenvalues of the charge operator Q for particles and antiparticles are equal in magnitude and opposite in sign. This can be seen by realizing that the argument of the exponents in (2.9) are in fact $-\beta(E_\alpha - q_\alpha \mu)$ where q_α is the charge of particle α. If we change particle to antiparticle, E_α and μ are unchanged whereas q_α changes sign.

B. Calculation of the thermodynamic potential in the high-temperature limit

We now evaluate the integral in (2.9) (neglecting the zero-point energy). We define dimensionless variables $x = \beta k$, $\bar{m} = \beta m$, $r = \mu/m$ (note that $|r| \leq 1$). Integration by parts gives

$$\Omega(T,V,\mu) = \frac{-4VT^4}{\pi^2} H_5(\bar{m},r) , \tag{2.10}$$

where H_5 is one of a class of functions defined by

$$H_l(\bar{m},r) \equiv \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1}dx}{(x^2+\bar{m}^2)^{1/2}} \left[\frac{1}{\exp[(x^2+\bar{m}^2)^{1/2} - r\bar{m}] - 1} + (r \rightarrow -r) \right] . \tag{2.11}$$

It is of interest to study the high-temperature behavior of Ω. Naively, one would think that because $\bar{m} << 1$ in this limit, it would be sufficient to expand the integrand of (2.11) as a power series in $\bar{m}$ and integrate term by term. Unfortunately, such a power series about $\bar{m} = 0$ does not exist due to a branch point at $\bar{m} = 0$. (In practice, if one attempts such an expansion of H_l about $\bar{m} = 0$, one finds coefficients which are, in general, divergent integrals.) A high-temperature expansion for H_l can however be computed by a more sophisticated technique which is fully described in Appendix A.[7] First, we introduce a second class of functions:

$$G_l(\bar{m},r) \equiv \frac{1}{\Gamma(l)} \int_0^\infty x^{l-1} dx \left[\frac{1}{\exp[(x^2+\bar{m}^2)^{1/2} - r\bar{m}] - 1} - (r \rightarrow -r) \right] . \tag{2.12}$$

In Appendix A, we show how to calculate expansions for G_1 and H_1 for $\bar{m} << 1$. The results are given by (A28) and (A36), respectively. Using (A7) and (A8), we see that knowledge of G_1 and H_1 will give us G_l and H_l for all positive odd l. We therefore find the following result for Ω in the large-temperature limit:

$$\frac{\Omega(T,V,\mu)}{V} = \frac{-\pi^2 T^4}{45} + \frac{T^2(m^2 - 2\mu^2)}{12} - \frac{T(m^2-\mu^2)^{3/2}}{6\pi} - \frac{\mu^2(3m^2-\mu^2)}{24\pi^2} + \frac{m^4}{16\pi^2} \left[\ln\left[\frac{4\pi T}{m}\right] - \gamma + \frac{3}{4} \right] + O\left[\frac{m^6}{T^2}, \frac{m^4\mu^2}{T^2} \right] . \tag{2.13}$$

We remarked earlier that $|\mu| \leq m$ was required in order that the functional integral converge. In (2.13), we see that this requirement ensures that Ω is real.

C. Thermodynamics of the ideal Bose gas

Given the thermodynamic potential as a function of μ, V, and T, all the physical quantities may be calculated. For example, the pressure P, charge density $\rho \equiv Q/V$, entropy S, and energy U are given by

$$P = -\left[\frac{\partial\Omega}{\partial V}\right]_{T,\mu} , \tag{2.14}$$

$$\rho = -\frac{1}{V}\left[\frac{\partial\Omega}{\partial \mu}\right]_{T,V} , \tag{2.15}$$

$$S = -\left[\frac{\partial\Omega}{\partial T}\right]_{\mu,V} , \tag{2.16}$$

and $U = TS - PV + \mu\rho V$. In Ref. 8, we have given the explicit high-temperature expansions for the above quantities. They are easily derived using (2.13). Here, we wish to concentrate only on the expression for the charge density ρ. The exact expression is given by

$$\rho=\frac{T^3}{\pi^2}G_3(\bar{m},r)\ , \tag{2.17}$$

where $\bar{m}\equiv\beta m$, $r\equiv\mu/m$, and the function G_3 is defined in (2.12). In more familiar terms, we may write

$$\rho=\int\frac{d^3k}{(2\pi)^3}(n_k-\bar{n}_k)\ , \tag{2.18}$$

where $n_k=1/\{\exp[\beta(E_k-\mu)]-1\}$ and $\bar{n}_k$ is obtained from n_k by replacing μ by $-\mu$. Equation (2.18) illustrates that n_k, $\bar{n}_k$ may be interpreted as the number density for particles and antiparticles, respectively. Our previously obtained result $|\mu|\leq m$ is seen in this context to be a requirement that the number densities be non-negative. In the nonrelativistic limit, the presence of antiparticles should be negligible. This is indeed the case and in Appendix B we show that in this limit we regain the standard textbook results.

D. Bose-Einstein condensation

In Ref. 8 we discussed some properties of Bose-Einstein condensation in the case of a relativistic Bose gas and briefly summarize some of those remarks here. The question which we ask is this: What is the requirement for Bose-Einstein condensation to take place at relativistic temperatures (i.e., $T>>m$) and what is the nature of the phase transition?

Consider (2.18), which is an equation for the charge density ρ as a function of T and μ. Since ρ is a physical quantity and μ is a derived quantity, (2.18) is in fact an implicit formula for μ as a function of ρ and T. For T above some critical temperature T_C, we can always find a $\mu<m$ satisfying (2.18). Below T_C no such μ can be found and we interpret (2.18) as an expression for the charge density of the excited states. [Clearly the ground state $k=0$ is given zero weight in the integral (2.18).] The critical temperature T_C at which Bose-Einstein condensation occurs corresponds to $\mu=\pm m$ (the sign depending on the sign of ρ). For $T\leq T_C$, $|\mu|$ as shown in Fig. 1. The exact value of T_C is determined by setting $|\mu|=m$ in (2.18). For $T\geq T_C>>m$ we may use (2.13) and (2.15) to find

$$\rho=\tfrac{1}{3}\mu T^2\ . \tag{2.19}$$

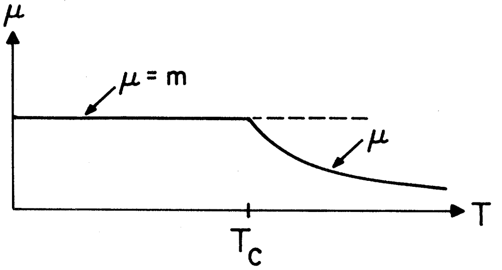


FIG. 1. The temperature dependence of the chemical potential for an ideal Bose gas of mass m with fixed Q and V.

Thus T_C is given by

$$\rho=\tfrac{1}{3}mT_C{}^2\ . \tag{2.20}$$

At temperatures $T<T_C$ the charge density in the $\vec{k}\neq0$ state is

$$\rho(k>0)=\tfrac{1}{3}mT^2=\rho\left[\frac{T}{T_C}\right]^2\ . \tag{2.21}$$

The remaining charge density in the $\vec{k}=0$ state is

$$\rho(k=0)=\rho\left[1-\left[\frac{T}{T_C}\right]^2\right]\ . \tag{2.22}$$

As the temperature is lowered the fraction of charge in the $\vec{k}=0$ state increases to unity.

It is perhaps worthwhile to note that the necessary condition for an ideal Bose gas of mass m to condense at a relativistic temperature ($T_C>>m$) is that $\rho>>m^3$. Conversely, in the nonrelativistic regime, we may apply the standard textbook results to see that $T_C<<m$ provided $\rho<<m^3$.

These results allow us to make some interesting observations about massless bosons with net charge. Since $|\mu|\leq m$, we see that $\mu=0$ for massless bosons; it appears that (2.18) implies that $\rho=0$. But using (2.20) we see that $T_C=\infty$ and hence all the net charge of an ideal gas of massless bosons resides in the Bose-Einstein condensed ground state. Of course in the case of a photon gas, no conserved quantum number exists; hence $\mu=0$ and no Bose-Einstein condensation takes place.

It is of interest of examine the adiabatics (lines of constant S in the T-V plane) with the total charge Q fixed. For an ideal nonrelativistic Bose gas, it is well known that the adiabatics satisfy $TV^{2/3}=\text{constant}$.[13] Furthermore, in the nonrelativistic limit, the temperature at which Bose-Einstein condensation first occurs is

$$T_C=\frac{2\pi}{m}\left[\frac{N}{V\zeta(\frac{3}{2})}\right]^{2/3} \qquad (2.23)$$

(where $N=Q$; see Appendix B). Therefore, it follows that one cannot change between the uncondensed and condensed phases by varying T and V adiabatically.

We find very different results for a ideal Bose gas in the ultrarelativistic regime. By examining the leading terms in the relativistic expressions for the entropy and charge density, we find that the adiabatics satisfy $TV^{1/3}=\text{constant}$. On the other hand, according to (2.20),

$$T_C=\left[\frac{3Q}{mV}\right]^{1/2}, \qquad (2.24)$$

so that if we begin in the uncondensed phase and *raise* the temperature adiabatically we will eventually enter the condensed phase at some very large, relativistic temperature. [In the presence of interactions, however, this does not occur. See Fig. 4(b).]

The formalism we have provided also allows us to discuss thermodynamics in the general case of d space dimensions. Here we summarize how the nature of the Bose-Einstein phase transition depends on d. The generalization of (2.17) to d space dimensions is straightforward:

$$\rho=\pi^{-(d+1)/2}\Gamma\left[\frac{d+1}{2}\right]T^d G_d(\bar{m},r)\ . \qquad (2.25)$$

If d is odd, then G_d can be obtained from G_1 and H_1 [(A28) and (A36)] by using the recursion relations [(A7) and (A8)]. In the case that d is even, it is G_2 and H_2 that are needed. These have also been calculated in Appendix A [see (A16) and (A18)]; using the recursion relations all the higher-(even-) order functions can be found.

We now note that Bose-Einstein condensation will not occur if the following condition is satisfied: for any fixed ρ and T it is possible to find a μ obeying $|\mu|\le m$ (i.e., an r such that $|r|\le 1$) which satisfies (2.25). The results of $d=1$ and 2 space dimensions are noteworthy: In two space dimensions, the Bose-Einstein phase transition occur for massless bosons but does *not* occur for massive bosons. In one space dimension there is no Bose-Einstein phase transition at all. These results have also been noted by other authors; for more details and references we refer the reader to Ref. 8.

III. INTERACTING BOSE GAS WITH $\mu\neq 0$

A. Incorporation of μ in an O(N) model

To investigate the effects of a chemical potential on an interacting system of bosons we will examine a field theory of N real scalar fields with the O(N)-invariant Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}\pi_a\pi_a+\tfrac{1}{2}(\vec{\nabla}\phi_a)\cdot(\vec{\nabla}\phi_a)+\tfrac{1}{2}m_0{}^2\phi^2+\frac{\lambda_0}{4!}(\phi^2)^2,$$
$$\phi^2\equiv\phi_a\phi_a,\quad a=1,2,\ldots,N \qquad (3.1)$$

where m_0 and λ_0 are the unrenormalized parameters. This theory has $N(N-1)/2$ conserved charges

$$Q_{ab}=\int d^3x\, j^0_{ab}\ , \qquad (3.2a)$$

$$j^0_{ab}=\phi_a\pi_b-\phi_b\pi_a,\quad j^i_{ab}=\phi_a\nabla^i\phi_b-\phi_b\nabla^i\phi_a\ , \qquad (3.2b)$$

$$\partial_\nu j^\nu_{ab}=0\ . \qquad (3.2c)$$

A quantum state of the system is specified by the eigenvalues of a complete set of mutually commuting charges. The maximum number of mutually commuting charges is $N/2$ for N even and for definiteness we will always take N even.

A gas of these bosons in thermal and chemical equilibrium is specified by the temperature T, the volume V, and the expectation value of various charges. The existence of such a net charge prevents the system from being invariant under the full non-Abelian group. [For example, an ensemble with a net value for $\langle Q_{12}\rangle\neq 0$ is only invariant under O(2)× O($N-2$).] To describe such a system in the grand canonical emsemble one introduces a chemical potential μ_{ab} for each charge whose expectation value is specified. Thus (3.1) is replaced by

$$\mathcal{H}-\tfrac{1}{2}\mu_{ab}j^0_{ab}\ , \qquad (3.3)$$

where $\mu_{ab}=-\mu_{ba}$. The chemical potentials enter in the canonical momenta,

$$\pi_a=\dot{\phi}_a-\mu_{ac}\phi_c \qquad (3.4)$$

and modify the equations of motion so that (3.2c) is replaced by

$$\partial_\nu j^\nu_{ab}=\mu_{ac}j^0_{cb}-j^0_{ac}\mu_{cb}\ . \qquad (3.5)$$

This equation has several implications which we would like to discuss. First, suppose that the only non-zero chemical potential is $\mu_{12}=-\mu_{21}\neq 0$. The conserved currents are j^ν_{ab} for a, $b\geq 3$, and j^ν_{12}. This just reflects the fact that an ensemble with $\langle Q_{12}\rangle\neq 0$ and all other charges zero is only invariant under $\mathrm{O}(2)\times(N-2)$. As a next step one might try to have two nonzero chemical potentials, $\mu_{12}=-\mu_{21}\neq 0$ and $\mu_{13}=-\mu_{31}\neq 0$, corresponding to some fixed values of $\langle Q_{12}\rangle$ and $\langle Q_{13}\rangle$. This fails, however, because in this case neither j^ν_{12} nor j^ν_{13} are conserved. Thus it is not possible to specify the expectation value of two noncommuting charges.[9] (An alternative argument is to note that the partition function of the grand canonical ensemble is a weighted sum over partition functions of the canonical ensemble. All the quantum states which contribute to a particular canonical partition function must be eigenstates of the charge operators. Consequently only a mutually commuting set of operators can be used.)

In our model the maximum possible number nonzero chemical potentials in $N/2$. (Recall N is even.) A convenient set of mutually commuting charges is Q_{12}, Q_{34}, Q_{56},... which we label[14]

$$Q_\alpha=Q_{2\alpha-1,2\alpha}=\int d^3x\, j^0_\alpha\;,$$
$$j^\nu_\alpha\equiv j^\nu_{2\alpha-1,2\alpha}\quad(\alpha=1,2\,,\ldots,\,N/2)\;. \tag{3.6}$$

The only nonzero chemical potentials are

$$\mu_\alpha\equiv\mu_{2\alpha-1,2\alpha}\quad(\alpha=1,2\,,\ldots,\,N/2)\;. \tag{3.7}$$

If none of these vanish then the only conserved currents are those displayed in (3.6) and the full symmetry of the ensemble is $[\mathrm{O}(2)]^{N/2}$. By using (3.4) these currents can be written more explicitly as

$$j^\nu_\alpha=\phi_{2\alpha-1}\overleftrightarrow{\partial}^\nu\phi_{2\alpha}+\delta^{\nu 0}\mu_\alpha(\phi^2_{2\alpha-1}+\phi_{2\alpha}{}^2)\;,$$
$$\partial_\nu j^\nu_\alpha=0\quad(\alpha=1,2\,,\ldots,\,N/2)\;. \tag{3.8}$$

We will compute the grand partition function appropriate to these charges,

$$\exp(-\beta\Omega)=\mathrm{Tr}\{\exp[-\beta(H-\mu_\alpha Q_\alpha)]\}\;, \tag{3.9}$$

from which we can obtain all thermodynamic quantities by differentiating the thermodynamic potential $\Omega(T,V,\mu)$. [See (2.14)–(2.16).] At low temperatures we will find that the ground state has the property

$$\langle 0|\phi_a|0\rangle\equiv\hat{\phi}_a\neq 0\;. \tag{3.10}$$

The symmetry associated with Q_α is then spontaneously broken because the ground state is not an eigenstate of Q_α. If we shift the field operators

$$\phi_a=\hat{\phi}_a+\phi'_a\;,$$

then the charge operator contains a c-number piece

$$Q_\alpha=V\mu_\alpha(\hat{\phi}^2_{2\alpha-1}+\hat{\phi}_{2\alpha}{}^2)+Q'_\alpha\;,$$

where Q'_α contains the terms either linear or quadratic in the operators ϕ'. (Note that Q'_α still depends on μ_α.) We emphasize that in a finite volume V the spatial integral which defines Q_α converges and the charge is fully conserved[15]:

$$\frac{dQ_\alpha}{dt}=0\;.$$

The symmetry breaking occurs in the ground state [characterized by (3.10)], which is an infinite superposition of states with different numbers of charged, zero-momentum excitations. The ground-state expectation value of the charge is

$$\langle 0|Q_\alpha|0\rangle=V\mu_\alpha(\hat{\phi}^2_{2\alpha-1}+\hat{\phi}_{2\alpha}{}^2)\;.$$

This contributes to the thermal average charge density:

$$\rho_\alpha\equiv\left(\frac{1}{V}\right)\frac{\mathrm{Tr}\{Q_\alpha\exp[-\beta(H-\mu_\lambda Q_\lambda)]\}}{\mathrm{Tr}\{\exp[-\beta(H-\mu_\lambda Q_\lambda)]\}}$$
$$=\mu_\alpha(\hat{\phi}^2_{2\alpha-1}+\hat{\phi}_{2\alpha}{}^2)+\left(\frac{1}{V}\right)\frac{\mathrm{Tr}\{Q'_\alpha\exp[-\beta(H-\mu_\lambda Q'_\lambda)]\}}{\mathrm{Tr}\{\exp[-\beta(H-\mu_\lambda Q'_\lambda)]\}}\;. \tag{3.11}$$

We will later show that the transition to the broken symmetric phase is just the onset of Bose-Einstein condensation in the ultrarelativistic regime. The first term in (3.11) is the net charge density in the Bose-Einstein condensed (zero-momentum) ground state.[16] The second term is the net charge density in the excited (nonzero-momentum) states. Both contributions depend on temperature; their sum remains constant. No charge disappears from the system despite the spontaneous symmetry breaking. This is precisely analogous to the more familiar fact in the superfluid phase of He^4, although the global phase symmetry associated with baryon number is spontaneously broken, the net baryon number of the system never changes.[17]

B. The μ-dependent effective potential

For $m_0{}^2<0$ the ground-state expectation values $\langle 0\,|\,\phi_a\,|\,0\rangle$ all vanish at high temperature but not at low temperature. With no chemical potentials this pattern of symmetry breaking is determined by the value of the constant c-number $\hat{\phi}$ which minimizes the effective potential $\mathscr{V}(\hat{\phi})$:

$$\exp[-\beta V\mathscr{V}(\hat{\phi})]=\exp\left[W(J)-\int dx\,\hat{\phi}_a J_a\right]\,, \tag{3.12}$$

$$\int dx \equiv \int_0^\beta d\tau \int_V d^3x\,. \tag{3.13}$$

Here $\hat{\phi}_a=\delta W/\delta J_a$ and $W(J)$ is the generating functional for the connected, finite-temperature Green's functions:

$$\exp[W(J)]=N\int[d\pi][d\phi]\exp\left\{\int dx[i\pi_a\dot{\phi}_a-\mathscr{H}(\pi,\phi)+\phi_a J_a]\right\}. \tag{3.14}$$

The normalization constant N is the same as in (2.3). If we substitute the definition (3.14) into (3.12) it is convenient to shift the integration variable from ϕ to $\hat{\phi}+\phi$ and use $J_a=d\mathscr{V}/d\hat{\phi}_a$ to obtain[18]

$$\exp[-\beta V\mathscr{V}(\hat{\phi})]=N\int[d\pi][d\phi]\exp\left\{\int dx\left[i\pi_a\dot{\phi}_a-\mathscr{H}(\pi,\hat{\phi}+\phi)+\phi_a\frac{d\mathscr{V}}{d\hat{\phi}_a}\right]\right\}. \tag{3.15}$$

That part of $\mathscr{H}(\pi,\hat{\phi}+\phi)$ which is quadratic in ϕ defines a $\hat{\phi}$-dependent, effective propagator from which $\mathscr{V}(\hat{\phi})$ may be computed perturbatively.

To investigate how the introduction of chemical potentials $\mu_1, \mu_2, \ldots, \mu_{N/2}$ in the high-temperature phase will affect the transition to the less symmetric, low-temperature phase we compute the μ-dependent effective potential

$$\exp[-\beta V\mathscr{V}(\hat{\phi})]=N\int[d\pi][d\phi]\exp\left\{\int dx\left[i\pi_a\dot{\phi}_a-\mathscr{H}(\pi,\hat{\phi}+\phi)+\phi_a\frac{d\mathscr{V}}{d\hat{\phi}_a}+\mu_\alpha j^0_\alpha\right]\right\}. \tag{3.16}$$

In the above (and subsequent) equations a is summed from 1 to N and α is summed fro 1 to $N/2$ [see (3.6)]. The value of $\hat{\phi}$ which minimizes $\mathscr{V}(\hat{\phi})$ is the ground-state expectation value of the quantum field and determines the phase structure of the system. At that value of $\hat{\phi}$ the right-hand side of (3.16) is precisely the grand partition function. The thermodynamic potential (3.9) is therefore the minimum value of $\mathscr{V}$:

$$\frac{\Omega(T,V,\mu)}{V}=\mathscr{V}(\hat{\phi})\Big|_{d\mathscr{V}/d\hat{\phi}=0}\,. \tag{3.17}$$

For the N-component model the integrals over π_a in (3.16) are Gaussian and yield

$$\exp[-\beta V\mathscr{V}(\hat{\phi})]=N'(\beta)\int[d\phi]\exp\left\{\int dx\left[\mathscr{L}(\hat{\phi}+\phi)+\phi_a\frac{d\mathscr{V}}{d\hat{\phi}_a}\right]\right\}, \tag{3.18}$$

$$\begin{aligned}\mathscr{L}(\phi)\equiv{}&-\tfrac{1}{2}\dot{\phi}_a\dot{\phi}_a-\tfrac{1}{2}(\vec{\nabla}\phi_a)\cdot(\vec{\nabla}\phi_a)-\tfrac{1}{2}m_0{}^2\phi^2-\frac{\lambda_0}{4!}(\phi^2)^2\\ &-i\mu_\alpha(\dot{\phi}_{2\alpha-1}\phi_{2\alpha}-\dot{\phi}_{2\alpha}\phi_{2\alpha-1})+\tfrac{1}{2}\mu_\alpha{}^2(\phi^2_{2\alpha-1}+\phi_{2\alpha}{}^2)\,.\end{aligned} \tag{3.19}$$

The functional $\mathscr{L}(\phi)$ is essentially the Euclidean Lagrangian except that it depends on the external medium via μ. The corresponding classical potential is

$$\mathscr{V}_0(\hat{\phi})=\tfrac{1}{2}m_0{}^2\hat{\phi}^2+\frac{\lambda_0}{4!}(\hat{\phi}^2)^2-\tfrac{1}{2}\mu_\alpha{}^2(\hat{\phi}^2_{2\alpha-1}+\hat{\phi}_{2\alpha}{}^2)\,. \tag{3.20}$$

In $\mathscr{L}(\hat{\phi}+\phi)$ we separate out terms independent of ϕ and linear in ϕ by defining $\mathscr{L}(\hat{\phi};\phi)$ to contain all the terms quadratic, cubic, and quartic in ϕ:

$$\mathscr{L}(\hat{\phi}+\phi)=-\mathscr{V}_0(\hat{\phi})-\phi_a\frac{d\mathscr{V}_0}{d\hat{\phi}_a}-i\mu_\alpha(\dot{\phi}_{2\alpha-1}\hat{\phi}_{2\alpha}-\dot{\phi}_{2\alpha}\hat{\phi}_{2\alpha-1})+\mathscr{L}(\hat{\phi};\phi)\ . \tag{3.21}$$

The μ-dependent term is a total time derivative and can contribute only a surface term to (3.18), but due to the periodic boundary conditions $\phi(\tau+\beta)=\phi(\tau)$ the surface terms all cancel. The term in (3.21) that is linear in ϕ_a cancels when substituted into (3.18) and yields

$$\exp[-\beta V\mathscr{V}'(\hat{\phi})]=N'(\beta)\int[d\phi]\exp\left\{\int dx\left[\mathscr{L}(\hat{\phi};\phi)+\phi_a\frac{d\mathscr{V}'}{d\hat{\phi}_a}\right]\right\}, \tag{3.22}$$

$$\mathscr{V}(\hat{\phi})\equiv\mathscr{V}_0(\hat{\phi})+\mathscr{V}'(\hat{\phi})\ . \tag{3.23}$$

Thus $\mathscr{V}'$ is the sum of all the quantum corrections to the effective potential.

One might think that the way to proceed is to use all the terms in the truncated Lagrangian $\mathscr{L}(\hat{\phi};\phi)$ that are quadratic in ϕ to define a $\hat{\phi}$-dependent propagator and perturb in the cubic and quartic terms. For $\mu_\alpha=0$ it is well known that if $m_0{}^2<0$ and $\hat{\phi}$ is small the $\hat{\phi}$-dependent mass is negative and prevents each functional integral in the perturbation series from converging. To avoid this it is necessary to include more of the dynamics in the "unperturbed" part of $\mathscr{L}(\hat{\phi};\phi)$. We have found it very convenient to do this by including an arbitrary self-interaction term $\frac{1}{2}\Sigma_{ab}\phi_a\phi_b$ as follows:

$$\mathscr{L}(\hat{\phi};\phi)\equiv\mathscr{L}_0(\hat{\phi};\phi)+\mathscr{L}_I(\hat{\phi};\phi)\ , \tag{3.24}$$

$$\begin{aligned}\mathscr{L}_0(\hat{\phi};\phi)=&-\tfrac{1}{2}\dot{\phi}_a\dot{\phi}_a-\tfrac{1}{2}(\vec{\nabla}\phi_a)\cdot(\vec{\nabla}\phi_a)-\tfrac{1}{2}M^2{}_{ab}\phi_a\phi_b\\&-i\mu_\alpha(\dot{\phi}_{2\alpha-1}\phi_{2\alpha}-\dot{\phi}_{2\alpha}\phi_{2\alpha-1})+\tfrac{1}{2}\mu_\alpha{}^2(\phi_{2\alpha-1}^2+\phi_{2\alpha}{}^2)\ ,\end{aligned} \tag{3.25}$$

$$\mathscr{L}_I(\hat{\phi};\phi)=\tfrac{1}{2}\Sigma_{ab}\phi_a\phi_b-\frac{\lambda_0}{6}(\hat{\phi}_a\phi_a)\phi^2-\frac{\lambda_0}{4!}(\phi^2)^2\ , \tag{3.26}$$

where we have introduced an effective mass term defined as

$$\begin{aligned}M^2{}_{ab}=&\left[m_0{}^2+\frac{\lambda_0}{6}\hat{\phi}^2\right]\delta_{ab}\\&+\frac{\lambda_0}{3}\hat{\phi}_a\hat{\phi}_b+\Sigma_{ab}\ .\end{aligned} \tag{3.27}$$

We will use $\mathscr{L}_0(\hat{\phi};\phi)$ to define the $\hat{\phi}$-dependent, μ-dependent propagators D_{ab}:

$$(D^{-1})_{ab}=i(\Gamma_{ab}+M^2{}_{ab})\ , \tag{3.28}$$

where the matrix Γ_{ab} is block diagonal and each block is 2×2 of the form

$$\begin{aligned}&\begin{bmatrix}\Gamma_{2\alpha-1,2\alpha-1} & \Gamma_{2\alpha-1,2\alpha}\\ \Gamma_{2\alpha,2\alpha-1} & \Gamma_{2\alpha,2\alpha}\end{bmatrix}\\&=\begin{bmatrix}\omega_n{}^2+\vec{k}^2-\mu_\alpha{}^2 & -2\mu_\alpha\omega_n\\ 2\mu_\alpha\omega_n & \omega_n{}^2+\vec{k}^2-\mu_\alpha{}^2\end{bmatrix},\end{aligned} \tag{3.29}$$

where $\omega_n=2\pi nT$ [compare (2.6)]. Using this propagator we may compute $\mathscr{V}'$ as the sum of all closed-loop, connected Feynman diagrams. The term linear in ϕ in (3.22) requires that each diagram be one-particle irreducible. The momentum integrations take the form

$$\int dk\equiv iT\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\ . \tag{3.30}$$

Vertices are determined by $\mathscr{L}_I(\hat{\phi};\phi)$.

C. Computation of the effective mass $M^2(T)$

For small $\hat{\phi}$ (even though $m_0{}^2<0$) the mass term (3.27) will be non-negative at high temperature provided Σ accounts for the higher-order dynamics properly. Thus Σ should be determined by a self-consistent criterion. Any such criterion is equivalent to summing an infinite class of diagrams in perturbation theory. We require that this infinite class be renormalizable by itself. Since the N-component theory is renormalizable for any N, the diagrams at a fixed order in λ which must be renormalized together are those with the same N dependence. Among the diagrams with l loops the largest N dependence is N^l.

Thus we are led to investigate the leading N behavior (at fixed λN) of our model.[10,18] In the general expression (3.27) for $M^2{}_{ab}$ the off-diagonal terms $(\lambda_0/3)\hat{\phi}_a\hat{\phi}_b$ are negligible for large N. It is convenient to choose Σ_{ab} diagonal,

$$\Sigma_{ab}=\delta_{ab}\Sigma_0\,, \tag{3.31}$$

so that the effective mass is also diagonal (in the large-N limit):

$$\begin{aligned} &M^2{}_{ab}=\delta_{ab}M^2\,,\\ &M^2=m_0{}^2+\frac{\lambda_0}{6}\hat{\phi}^2+\Sigma_0\,. \end{aligned} \tag{3.32}$$

Inversion of (3.28) gives a block-diagonal propagator with each block 2×2 of the form

$$\begin{aligned} &\begin{bmatrix} D_{2\alpha-1,2\alpha-1} & D_{2\alpha-1,2\alpha}\\ D_{2\alpha,2\alpha-1} & D_{2\alpha,2\alpha}\end{bmatrix}\\ &\quad=\frac{-i}{A_\alpha{}^4+(2\mu_\alpha\omega_n)^2}\begin{bmatrix} A_\alpha^2 & 2\mu_\alpha\omega_n\\ -2\mu_\alpha\omega_n & A_\alpha{}^2\end{bmatrix},\\ &A_\alpha{}^2\equiv\omega_n{}^2\vec{k}{}^2+M^2-\mu_\alpha{}^2\,. \end{aligned} \tag{3.33}$$

To compute the full, radiatively corrected propagator we must compute the proper self-energy to all orders. For large N the cubic part of $\mathscr{L}_I(\hat{\phi};\phi)$ in (3.26) is negligible; we keep only ϕ^4 interactions and quadratic insertions of $\frac{1}{2}\phi^2\Sigma_0$. Figure 2 displays all contributions to the proper self-energy through three-loop order. Each ϕ^4 vertex contributes a factor $\lambda_0/6$ and the group indices do not couple the loops. Thus, for example, the contribution of Fig. 2(d) is

$$\left[\frac{\lambda_0}{6}\right]^2\delta_{ab}\int dk\,D_{cd}(k)D_{cd}(k)\int dk'D_{ee}(k')\,. \tag{3.34}$$

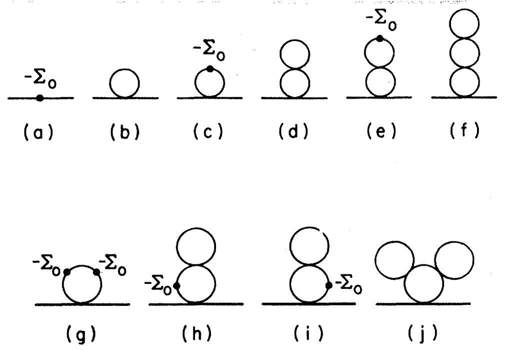


FIG. 2. All contributions to the proper self-energy through three-loop order. Because of the self-consistent choice (3.35) for Σ_0, all diagrams cancel ($a+b=0$, $c+d=0$, $e+f=0$, $g+h+i+j=0$) so that the full propagator is given by (3.33).

The sum of all the contributions to the proper self-energy is completely independent of the choice of Σ_0 since it cancels between $\mathscr{L}_0(\hat{\phi};\phi)$ and $\mathscr{L}_I(\hat{\phi};\phi)$. We can make our job easy by choosing Σ_0 to satisfy the self-consistent equation

$$\Sigma_0=\frac{\lambda_0}{6}\int dk\,D_{aa}(k)\,. \tag{3.35}$$

With this choice all the self-energy diagrams generated by $\mathscr{L}_I(\hat{\phi};\phi)$ (3.26) cancel among themselves. Therefore in the large-N limit the full self-energy is simply Σ_0 and the full propagator is (3.33). It is convenient to use (3.32) to write (3.35) as a self-consistent equation for M^2:

$$M^2=m_0{}^2+\frac{\lambda_0}{6}\hat{\phi}^2+\frac{\lambda_0}{6}\int dk\,D_{aa}(k)\,. \tag{3.36}$$

The trace of the propagator (3.33) yields

$$\begin{aligned} D_{aa}(k)=-i\sum_{\alpha-1}^{N/2}\frac{1}{E}&\left[\frac{E-\mu_\alpha}{\omega_n{}^2+(E-\mu_\alpha)^2}\right.\\ &\left.+\frac{E+\mu_\alpha}{\omega_n{}^2+(E+\mu_\alpha)^2}\right],\\ E\equiv(\vec{k}{}^2+M^2)^{1/2}\,.& \end{aligned}$$

When substituted into (3.36) the summation over frequencies ω_n must be performed and gives

$$M^2=m_0{}^2+\frac{\lambda_0}{6}\hat{\phi}^2+\frac{N\lambda_0}{12}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{E}\right]+\frac{\lambda_0}{3}J'\,, \tag{3.37}$$

$$J'(M,\mu)\equiv\frac{\partial J(M,\mu)}{\partial M^2}\,, \tag{3.38}$$

$$\begin{aligned} J(M,\mu)\equiv\sum_{\alpha=1}^{N/2}T\int\frac{d^3k}{(2\pi)^3}\{&\ln[1-e^{-\beta(E-\mu_\alpha)}]\\ &+\ln[1-e^{-\beta(E+\mu_\alpha)}]\}\,. \end{aligned} \tag{3.39}$$

All explicit dependence on temperature and chemical potential is contained in the finite integral J'. The divergent integral in (3.37) may be regularized in $d=3-2\epsilon$ spatial dimensions. To keep λ_0 dimensionless in d dimensions it is necessary to replace λ_0 by $\lambda_0\rho^{2\epsilon}$ where ρ is an arbitrary mass scale. The regularized integral is

$$\rho^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left[\frac{1}{E}\right]=\frac{M^2}{4\pi^2}\left[\ln\left[\frac{M}{4\pi\rho}\right]-\frac{1}{\epsilon}+\gamma-\frac{1}{2}\right] \tag{3.40}$$

so that (3.37) becomes

$$M^2=m_0{}^2+\frac{\lambda_0}{6}\hat{\phi}^2+\lambda_0\tilde{N}M^2\left[\ln\left[\frac{M}{4\pi\rho}\right]-\frac{1}{\epsilon}+\gamma-\frac{1}{2}\right]+\frac{\lambda_0}{3}J' , \tag{3.41}$$

$\tilde{N}\equiv N/48\pi^2$.

To renormalize (3.41) we use a modified minimal-subtraction scheme[19] to define finite parameters λ_R and $m_R{}^2$:

$$\begin{aligned}\frac{1}{\lambda_R}&\equiv\frac{1}{\lambda_0}+\tilde{N}\left[\frac{1}{\epsilon}+\ln(4\pi)-\gamma+\tfrac{1}{2}\right] ,\\ \frac{1}{m_R{}^2}&\equiv\frac{1}{m_0{}^2}+\frac{\lambda_0\tilde{N}}{m_0{}^2}\left[\frac{1}{\epsilon}+\ln(4\pi)-\gamma+\tfrac{1}{2}\right] .\end{aligned} \tag{3.42}$$

We may define a running coupling constant and running mass

$$\begin{aligned}\frac{1}{\lambda(\sigma)}&=\frac{1}{\lambda_R}+\tilde{N}\ln\left[\frac{\rho}{\sigma}\right] ,\\ \frac{1}{m^2(\sigma)}&=\frac{1}{m_R{}^2}+\frac{\lambda_R\tilde{N}}{m_R{}^2}\ln\left[\frac{\rho}{\sigma}\right]\end{aligned} \tag{3.43}$$

so that (3.41) becomes

$$M^2=m^2(\sigma)+\frac{\lambda(\sigma)}{6}\hat{\phi}^2+\lambda(\sigma)\tilde{N}M^2\ln\left[\frac{M}{\sigma}\right]+\frac{\lambda(\sigma)}{3}J' . \tag{3.44}$$

We now have a finite equation for M^2. (In subsequent equations we will not display the argument σ of the running parameters.) The apparent dependence on σ is illusory because from

$$\frac{d\lambda}{d\sigma}=\left[\frac{\lambda\tilde{N}}{\sigma}\right]\lambda , \tag{3.45}$$

$$\frac{dm^2}{d\sigma}=\left[\frac{\lambda\tilde{N}}{\sigma}\right]m^2 \tag{3.46}$$

it is easy to check that the solution to (3.44) satisfies $dM^2/d\sigma=0$.

Comparison of (3.39) with (2.9) shows that J is equivalent to the thermodynamic potential for $N/2$ ideal gases. The high-temperature expansion is given by (2.13):

$$J(M,\mu)=\frac{-N\pi^2T^4}{90}+\frac{NT^2M^2}{24}-\sum_{\alpha=1}^{N/2}\left[\frac{T^2\mu_\alpha{}^2}{6}+\frac{T(M^2-\mu_\alpha{}^2)^{3/2}}{6\pi}+\frac{\mu_\alpha{}^2(3M^2-\mu_\alpha{}^2)}{24\pi^2}\right]+\frac{NM^4}{32\pi^2}\left[\ln\left[\frac{4\pi T}{M}\right]-\gamma+\tfrac{3}{4}\right]+\cdots . \tag{3.47}$$

Differentiating with respect to M^2 gives J' and hence the high-temperature expansion of (3.44):

$$M^2=m^2+\frac{\lambda}{6}\hat{\phi}^2+\frac{N\lambda T^2}{72}-\frac{\lambda}{12\pi}\sum_{\alpha=1}^{N/2}[T(M^2-\mu_\alpha{}^2)^{1/2}+\tfrac{1}{2}\mu_\alpha{}^2]+\lambda\tilde{N}M^2\left[\ln\left[\frac{4\pi T}{\sigma}\right]-\gamma+\tfrac{1}{2}\right]+\cdots . \tag{3.48}$$

We postpone the application of this equation until we have computed the effective potential and its minimum.

D. Computation of $\mathscr{V}(\hat{\phi})$

To calculate the effective potential in the large-N limit[10] requires summing an infinite number of graphs with μ-dependent propagators (3.33) and effective mass (3.44). We can avoid this difficulty by using a trick suggested by Schnitzer.[18,20] In the present context we note that $\mathscr{L}(\hat{\phi};\phi)$ as given by the sum of (3.25) and (3.26) depends on $\hat{\phi}$, only through the term $(\lambda_0/6)\hat{\phi}^2$ in the large-N limit. Thus

$$2\frac{d\mathscr{L}(\hat{\phi};\phi)}{d\hat{\phi}^2}=-\frac{\lambda_0}{6}\phi^2(x) . \tag{3.49}$$

In the functional integral representation (3.22) for the quantum corrections $\mathscr{V}'$ we may neglect the term linear in ϕ_a for large N and obtain

$$2\frac{d\mathscr{V}'(\hat{\phi})}{d\hat{\phi}^2}=\frac{\frac{\lambda_0}{6}\int[d\phi]\phi^2\exp\left[\int dx\,\mathscr{L}(\hat{\phi};\phi)\right]}{\int[d\phi]\exp\left[\int dx\,\mathscr{L}(\hat{\phi};\phi\right]} \tag{3.50}$$

$$=\frac{\lambda_0}{6}\int dK\,D_{aa}(k)\ . \tag{3.51}$$

We emphasize that the right-hand side contains the full, not the bare, propagator. Because we have summed the bubbles in Fig. 2 to obtain the proper self-energy this propagator is known. Comparison with (3.35) shows that

$$2\frac{d\mathscr{V}'(\hat{\phi})}{d\hat{\phi}^2}=\Sigma_0(\hat{\phi})\ . \tag{3.52}$$

Thus we can sum all the leading N contributions to the effective potential by solving this differential equation.

The solution $\mathscr{V}'$ is of course divergent since Σ_0 is divergent. If we write (3.52) in terms of the finite function M^2,

$${m_0}^2+\frac{\lambda_0}{6}\hat{\phi}^2+2\frac{d\mathscr{V}'(\hat{\phi})}{d\hat{\phi}^2}=M^2(\hat{\phi})\ , \tag{3.53}$$

then it is useful to define

$$\overline{\mathscr{V}}(\hat{\phi})\equiv\tfrac{1}{2}{m_0}^2\hat{\phi}^2+\frac{\lambda_0}{4!}(\hat{\phi}^2)^2+\mathscr{V}'(\hat{\phi}) \tag{3.54}$$

so that (3.53) becomes

$$2\frac{d\overline{\mathscr{V}}(\hat{\phi})}{d\hat{\phi}^2}=M^2(\hat{\phi})\ . \tag{3.55}$$

This is the fundamental equation which determines the finite function $\overline{\mathscr{V}}$. To obtain the full potential $\mathscr{V}$ we need only add the finite μ terms in (3.20):

$$\mathscr{V}(\hat{\phi})=\overline{\mathscr{V}}(\hat{\phi})-\tfrac{1}{2}{\mu_\alpha}^2(\hat{\phi}_{2\alpha-1}^2+{\hat{\phi}_{2\alpha}}^2)\ . \tag{3.56}$$

The $\overline{\mathscr{V}}$ which solves the differential equation (3.55) is only determined to within a $\hat{\phi}$-independent constant. This constant is irrelevant for determining the pattern of symmetry breaking, but is necessary for the thermodynamic properties (e.g., the pressure). The boundary condition we impose is that for $\lambda=0$ the solution should reduce to the known ideal-gas result:

$$\overline{\mathscr{V}}(\hat{\phi})\Big|_{\lambda=0}=\tfrac{1}{2}m^2\hat{\phi}^2+J(m,\mu)\ . \tag{3.57}$$

To solve (3.55) we note that $\overline{\mathscr{V}}$ can depend on $\hat{\phi}^2$ either explicitly or implicitly through M^2 so that (3.55) actually means

$$\frac{\partial\overline{\mathscr{V}}}{\partial\hat{\phi}^2}+\frac{dM^2}{d\hat{\phi}^2}\frac{\partial\overline{\mathscr{V}}}{\partial M^2}=\tfrac{1}{2}M^2\ . \tag{3.58}$$

By differentiating (3.44) we know that

$$\frac{dM^2}{d\hat{\phi}^2}=\frac{\lambda}{6}\left[1-\frac{\lambda\tilde{N}}{2}-\lambda\tilde{N}\ln\left[\frac{M}{\sigma}\right]-\frac{\lambda}{3}J''\right]^{-1}\ ,$$
$$J''(M,\mu)\equiv\frac{\partial^2J(M,\mu)}{\partial M^2\partial M^2}\ . \tag{3.59}$$

The complete solution to (3.58) is then[21]

$$\overline{\mathscr{V}}(\hat{\phi})=\tfrac{1}{2}m^2\hat{\phi}^2+\frac{\lambda}{4!}(\hat{\phi}^2)^2+\frac{3\tilde{N}}{2}M^4\left[-\lambda\tilde{N}\ln^2\left[\frac{M}{\sigma}\right]+\ln\left[\frac{M}{\sigma}\right]-\frac{1}{4}\right]$$
$$+J-\frac{\lambda}{6}(J')^2-\lambda\tilde{N}J'M^2\ln\left[\frac{M}{\sigma}\right]+C_1\ , \tag{3.60}$$
$$C_1\equiv\frac{3\tilde{N}}{2}m^4\left[\ln\left[\frac{|m|}{\sigma}\right]-\frac{1}{4}\right]\ .$$

To verify that this is the solution it is necessary to use the implicit equation (3.44) for M^2 as well as (3.59). This is the exact solution in the large-N limit. All dependence on μ and T is contained in the known function J (3.39). One additional check of (3.60) is that although the subtraction mass σ appears explicitly as well as implicitly via $\lambda(\sigma)$ and $m^2(\sigma)$ one can show that $d\overline{\mathscr{V}}/d\sigma=0$ by using the self-consistent equation (3.44) for M^2.

E. Some properties of $\mathscr{V}$

To determine the pattern of symmetry breaking we must minimize $\mathscr{V}$ with respect to all the $\hat{\phi}_a$. It is convenient to introduce a bracket notation as follows:

$$[a/2] \equiv \begin{cases} \dfrac{a+1}{2} & \text{if } a = \text{odd} \\ \dfrac{a}{2} & \text{if } a = \text{even}\,. \end{cases} \tag{3.61}$$

This notation allows (3.56) to be written

$$\mathscr{V}(\hat{\phi}) = \overline{\mathscr{V}}(\hat{\phi}) - \tfrac{1}{2}\sum_{a=1}^{N} \mu_{[a/2]}^2 \hat{\phi}_a{}^2 \,. \tag{3.62}$$

Since $\overline{\mathscr{V}}$ depends on $\hat{\phi}$ only through $\hat{\phi}^2 = \hat{\phi}_a \hat{\phi}_a$ the extrema of the potential are determined from the vanishing of

$$\frac{d\mathscr{V}}{d\hat{\phi}_a} = (M^2 - \mu_{[a/2]}^2)\hat{\phi}_a \,. \tag{3.63}$$

The scalar-boson mass matrix is the value of the second derivative

$$\frac{d^2\mathscr{V}}{d\hat{\phi}_a d\hat{\phi}_b} = (M^2 - \mu_{[a/2]}^2)\delta_{ab} + \frac{dM^2}{d\hat{\phi}^2}\hat{\phi}_a \hat{\phi}_b \,, \tag{3.64}$$

evaluated at the minimum of the potential.

At very high temperatures we will find $M^2 > \mu_\alpha{}^2$ (for all α) so that the minimum of V occurs when all $\hat{\phi}_a = 0$. In this region we may put $\hat{\phi}^2 = 0$ in (3.44) and determine M^2 from

$$M^2 = m^2 + \lambda \tilde{N} M^2 \ln \left[\frac{M}{\sigma}\right] + \frac{\lambda}{3} J' \,. \tag{3.65}$$

By contrast, we will later use (3.63) to show that at low temperatures the potential minimum occurs when one of the chemical potentials becomes equal to M and $\hat{\phi}^2 \neq 0$. In this region (3.44) determines $\hat{\phi}^2$ as

$$\hat{\phi}^2 = \frac{6}{\lambda}\left[\mu^2 - m^2 - \lambda \tilde{N} \mu^2 \ln \left[\frac{\mu}{\sigma}\right] - \frac{\lambda}{3} J'\right] . \tag{3.66}$$

This will be our general method of evaluating $\mathscr{V}$ at its minimum.

We will be interested in computing the thermodynamic functions ρ_α and S in the various phases and must therefore differentiate the minimum value of $\mathscr{V}$ with respect to μ_α and T [see (2.14)–(2.16) and (3.17)]. It is useful to first use (3.44) to rewrite $\overline{\mathscr{V}}$ in (3.60) as

$$\begin{gathered} \overline{\mathscr{V}}(\hat{\phi}) = M^2 \left[\frac{3m^2}{\lambda} + \frac{\hat{\phi}^2}{2}\right] + \tfrac{3}{2} M^4 \left[\tilde{N} \ln \left[\frac{M}{\sigma}\right] - \frac{\tilde{N}}{4} - \frac{1}{\lambda}\right] + J + C_2 \,, \\ C_2 \equiv -\tfrac{3}{2} m^4 \left[\tilde{N} \ln \left[\frac{|m|}{\sigma}\right] - \frac{\tilde{N}}{4} + \frac{1}{\lambda}\right] . \end{gathered} \tag{3.67}$$

This may depend on a particular μ_α via $\hat{\phi}$, M^2, or J:

$$\frac{dJ}{d\mu_\alpha} = \frac{dM^2}{d\mu_\alpha} J' + \frac{\partial J}{\partial \mu_\alpha} \quad (V, T \text{ fixed}) \,. \tag{3.68}$$

When (3.67) is differentiated the coefficient of $dM^2/d\mu_\alpha$ cancels because of (3.44). A simple calculation gives

$$\begin{aligned} \frac{d\mathscr{V}}{d\mu_\alpha} = {} & -\frac{1}{2}\frac{d}{d\mu_\alpha}\left[\sum_{a=1}^{N} \mu_{[(a/2)]}^2 \hat{\phi}_a{}^2\right] \\ & + \tfrac{1}{2} M^2 \frac{d\hat{\phi}^2}{d\mu_\alpha} + \frac{\partial J}{\partial \mu_\alpha} \quad (V, T \text{ fixed}) \,. \end{aligned} \tag{3.69}$$

We will also need to compute the T derivative of $\mathscr{V}$. This dependence can occur in $\hat{\phi}$, M^2, and J:

$$\frac{dJ}{dT} = \frac{dM^2}{dT} J' + \frac{\partial J}{\partial T} \quad (V, \mu_\alpha \text{ fixed}) \,. \tag{3.70}$$

In differentiating (3.67) all dependence on the quantity dM^2/dT again cancels so that

$$\begin{aligned} \frac{d\mathscr{V}}{dT} = {} & -\tfrac{1}{2}\sum_{a=1}^{N} \mu_{[a/2]}^2 \frac{d\hat{\phi}_a{}^2}{dT} \\ & + \tfrac{1}{2} M^2 \frac{d\hat{\phi}^2}{dT} + \frac{\partial J}{\partial T} \quad (V, \mu_\alpha \text{ fixed}) \,. \end{aligned} \tag{3.71}$$

F. The symmetric phase

Before analyzing in detail the various cases of spontaneous symmetry breaking it is helpful to first summarize the common features of the high-temperature $\hat{\phi}=0$ phase. We remind the reader that because of (3.5) the number of conserved currents depends on how many nonzero chemical potential are present. If there are p nonzero chemical potentials, the symmetry of the high-temperature phase is $O(N-2p)\times[O(2)]^p$.

The thermodynamic potential Ω is obtained in this region by putting $\hat{\phi}_a=0$ in $\mathscr{V}$. Consequently the charge density and entropy obtained by setting $\hat{\phi}_a=0$ in (3.69) and (3.71) are

$$\rho_\alpha=-\frac{\partial J}{\partial\mu_\alpha}\,, \tag{3.72}$$

$$S=-\frac{\partial J}{\partial T}\,. \tag{3.73}$$

Since J in (3.39) is just a sum over $N/2$ ideal gas contributions we have essentially incorporated all interactions (in the leading-N approximation) into one function M^2. However, the interactions appear nontrivially in the pressure and energy,

$$\begin{gathered} P=-\mathscr{V}\,, \\ \frac{U}{V}=\mathscr{V}+\frac{TS}{V}+\mu_\alpha\rho_\alpha\,, \end{gathered} \tag{3.74}$$

because all terms in (3.67) contribute, not just the ideal-gas function J.

For $m^2<0$ and $N\lambda$ small[22] it is clear from (3.48) that we are always in the region $T>>M$. The ultrarelativistic expansions of (3.72) and (3.73) are the same as in Sec. II:

$$\rho_\alpha\approx\tfrac{1}{3}T^2\mu_\alpha+\cdots\,, \tag{3.75}$$

$$\frac{S}{V}\approx\frac{2N\pi^2T^3}{45}+\cdots\,, \tag{3.76}$$

$$\frac{Q_\alpha}{S}\approx\frac{15}{2N\pi^2}\left[\frac{\mu_\alpha}{T}\right]+\cdots\,, \tag{3.77}$$

with the last result being the quotient of the previous two. If the system is cooled with Q_α and V fixed then (3.75) requires that $\mu_\alpha(T)$ increase as $1/T^2$ as T is lowered. If, on the other hand, the system is cooled with Q_α and S fixed (as is the case in cosmological applications) then $\mu_\alpha(T)$ decrease linearly with T according to (3.77).

G. Spontaneous symmetry breaking without μ

To illustrate the use of (3.65) and (3.66) we first discuss the standard situation in which all $\mu_\alpha=0$ (i.e., no net charges). At high tempertatures (3.63) shows that the minimum of $\mathscr{V}$ occurs when all $\hat{\phi}_a=0$ because the M^2 which solves (3.65) is positive. At low temperatures (3.63) shows that the minimum of V occurs when $M^2=0$ and $\hat{\phi}^2\neq0$. For simplicity we may choose $\hat{\phi}_2=\hat{\phi}_3=\cdots=\hat{\phi}_N=0$ and use (3.66) to compute

$$\begin{aligned} \hat{\phi}_1{}^2&=\frac{6}{\lambda}\left[-m^2-\frac{\lambda}{3}J'(0,0)\right] \\ &=\frac{6}{\lambda}\left[|m^2|-\frac{N\lambda T^2}{72}\right] \\ &=\frac{N}{12}(T_C{}^2-T^2)\,, \end{aligned} \tag{3.78}$$

where the critical temperature at which the symmetry breaks is[1]

$$T_C{}^2=\frac{72|m^2|}{N\lambda}\,. \tag{3.79}$$

Associated with this breaking are $N-1$ massless Goldstone bosons $(\phi_2,\phi_3,\ldots,\phi_N)$ and one massive boson (ϕ_1).

H. Spontaneous symmetry breaking with one μ

We now investigate what happens to a system with one nonzero charge Q_1, as T is lowered. In the high-temperature phase (3.64) demonstrates that two bosons have mass $M^2-\mu^2$ and the remaining $N-2$ bosons have mass M^2. The symmetry is thus $O(2)\times O(N-2)$ as claimed.

(1) Q_1 and V fixed

We first discuss the T evolution with fixed charge density ρ_1. At very high T there is no spontaneous symmetry breaking so that according to (3.75)

$$\mu_1(T)\simeq3\rho_1/T^2\,. \tag{3.80}$$

In this region $\mu_1(T)<M(T)$, where

$$M(T)\approx\left[m^2+\frac{N\lambda T^2}{72}\right]^{1/2} \tag{3.81}$$

from (3.48) (recall $m^2<0$). As the temperature is reduced there is a value T_{C1} at which (3.80) becomes equal to (3.81):

$$\frac{3\rho_1}{T_{C1}^2}=\left[m^2+\frac{N\lambda T_{C1}^2}{72}\right]^{1/2} . \tag{3.82}$$

As illustrated in Fig. 3(a), this temperature is always larger than the usual critical temperature T_C (3.79) which applies when $\rho_1=0$. How much larger T_{C1} is than T_C depends on how large the background charge density ρ_1 is. The value of the chemical potential at T_{C1} is the fixed value

$$\mu_C\equiv\left[m^2+\frac{N\lambda T_{C1}^2}{72}\right]^{1/2} \tag{3.83}$$

and satisfies

$$\rho_1=\frac{24\mu_C}{N\lambda}(\mu_C{}^2-m^2) . \tag{3.84}$$

At the temperature T_{C1} the transition to the broken-symmetric phase occurs. From (3.63) the minimum of $\mathscr{V}$ in the region $T\leq T_{C1}$ is at $M(T)=\mu_1(T)$ and $\hat{\phi}_3=\hat{\phi}_4=\cdots=\hat{\phi}_N=0$ but $\hat{\phi}_1{}^2+\hat{\phi}_2{}^2\neq 0$. For simplicity we choose $\hat{\phi}_2=0$ so that (3.66) determines

$$\hat{\phi}_1{}^2=\frac{6}{\lambda}\left[\mu_1{}^2-m^2-\lambda\tilde{N}\mu_1{}^2\ln\left[\frac{\mu_1}{\sigma}\right]-\frac{\lambda}{3}J'(\mu_1,\mu_1)\right] . \tag{3.85}$$

This spontaneous breakdown of the Q_1 symmetry is the ultrarelativistic form of Bose-Einstein condensation; to show this we compute ρ_1. According to (2.15) and (3.17) we must differentiate the minimum value of $\mathscr{V}$ with respect to μ_1 keeping T and V fixed. Using (3.69) with $\hat{\phi}_1\neq 0$ gives

$$\rho_1=\mu_1\hat{\phi}_1{}^2-\left[\frac{\partial J}{\partial\mu_1}\right]_{M=\mu_1} . \tag{3.86}$$

This is an implicit equation for μ_1 in terms of known quantities. Comparison with (3.11) shows that the first term is the charge density in the Bose-Einstein condensed ($k=0$) ground state,[16]

$$\rho_1(k=0)=\mu_1\hat{\phi}_1{}^2 , \tag{3.87}$$

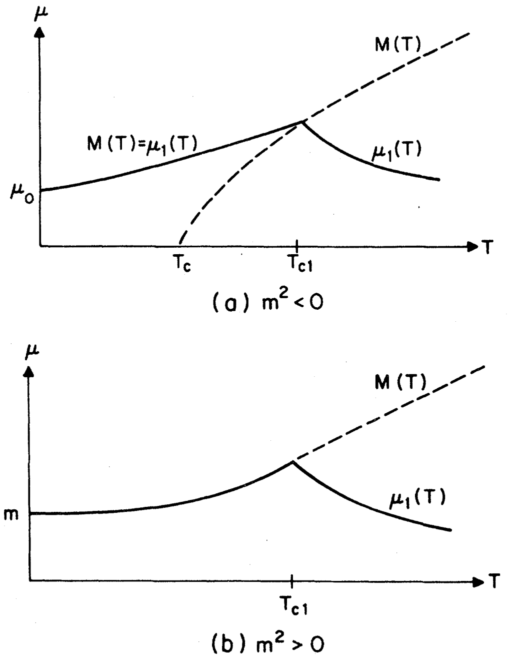


FIG. 3. The temperature dependence of $\mu_1(T)$ with Q_1 and V held fixed. For $T<T_{C1}$ the effective potential minimum occurs at $M(T)=\mu_1(T)$ and $\mu_1(T)$ is determined by (3.86). In (a) where $m^2<0$ the symmetry would break at T_C if there were no Q_1. In (b), the symmetry breaks even though $m^2>0$.

and the second term is the charge density in the excited ($k>0$) states,

$$\rho_1(k>0)=-\left[\frac{\partial J}{\partial\mu_1}\right]_{M=\mu_1} . \tag{3.88}$$

For $T\leq T_{C1}$ the entropy is similarly obtained by differentiating the minimum value of $\mathscr{V}$ with respect to T keeping μ_1 and V fixed. From (3.71) this gives

$$\frac{S}{V}=-\left[\frac{\partial J}{\partial T}\right]_{M=\mu_1} . \tag{3.89}$$

Before evaluating these expressions it is useful to use the original integral representation (3.39) for J to display

$$\rho_1(k>0)=\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\exp[\beta(E-\mu_1)]-1}-\frac{1}{\exp[\beta(E+\mu_1)]-1}\right]_{M=\mu_1} . \tag{3.90}$$

The ideal-gas form of this result confirms our interpretation of (3.88) as the charge density in the excited states.

To compute the charge densities in the high-temperature limit we use (3.47) to obtain

$$\rho_1(k=0)\approx\frac{6\mu_1}{\lambda}\left[\mu_1{}^2-m^2-\frac{N\lambda T^2}{72}\right], \qquad (3.91)$$

$$\rho_1(k>0)\approx\tfrac{1}{3}T^2\mu_1 .$$

Although each of these changes with T their sum is the fixed value ρ_1. This gives a cubic equation for $\mu_1(T)$. For temperatures just below T_{C1} the approximate solution is

$$\mu_1(T)\approx\mu_C-\frac{(N-4)\lambda\mu_C(T_{C1}{}^2-T^2)}{144[\mu_C{}^2+(\lambda/36)T_C{}^2]} . \qquad (3.92)$$

Substituting this into (3.91) and using (3.84) gives

$$\rho_1(k=0)=\rho_1\left[1-\left[\frac{T}{T_{C1}}\right]^2\right] \quad (0<<T\leq T_{C1}) . \qquad (3.93)$$

This is the same behavior as found in (2.22) for the ground-state charge density of the ideal gas. In the present context it corresponds to a temperature-dependent vacuum expectation value

$$\hat{\phi}_1{}^2\approx\tfrac{1}{3}(T_{C1}{}^2-T^2) \quad (0<<T\leq T_{C1}) . \qquad (3.94)$$

As T is reduced further $\mu_1(T)$ continues to decrease as shown in Fig. 3(a), although (3.92) is no longer a good approximation. At $T=0$ the value of the chemical potential is $\mu_0\equiv\mu_1(T=0)$ and satisfies

$$\rho_1\approx\frac{6\mu_0}{\lambda}(\mu_0{}^2-m^2)-6\widetilde{N}\mu_0{}^3\ln\left[\frac{\mu_0}{\sigma}\right] . \qquad (3.95)$$

We emphasize here that in the case under discussion (i.e., $m^2<0$ and $\lambda N<<1$), condensation always occurs at relativistic temperatures. This is true because $T_{C1}>>M$ as can be seen from (3.81). In addition, we note that for $T\geq T_{C1}$, $\rho_1>>M^3$ in accord with the conditions noted in Sec. II D.

Although we have concentrated on the case $m^2<0$, it is important to note that spontaneous symmetry breaking also occurs where $m^2>0$. If $\rho_1>>m^3$ the critical temperature will be ultrarelativistic and (3.82) still applies. The temperature dependence of $\mu(T)$ is shown in Fig. 3(b). If $\rho_1<<m^3$ then the nonrelativistic reduction of (3.65) and (3.66) leads to the standard results for interacting Bose gases.[4,5]

(2) Q_1 and S fixed

For most cosmological applications V changes with temperature but Q_1 and S are fixed. In the high-temperature region (3.77) requires that $\mu_1(T)$ decrease linearly with T:

$$\mu_1(T)\approx\frac{2N\pi^2}{15}\left[\frac{Q_1}{S}\right]T . \qquad (3.96)$$

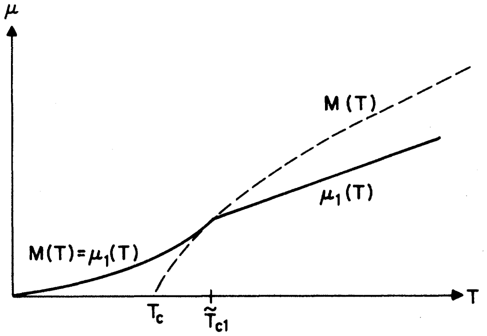
In this region $\mu_1(T)<M(T)$, where

$$M(T)\approx\left[m^2+\frac{N\lambda T^2}{72}\right]^{1/2} . \qquad (3.97)$$

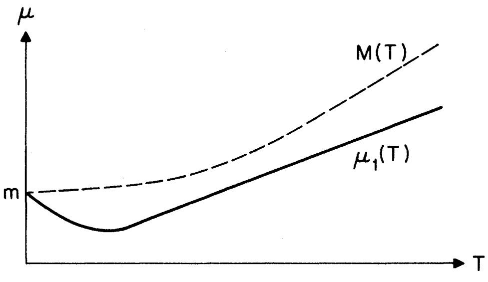
For $m^2<0$ there is a temperature $\widetilde{T}_{C1}$ at which $\mu_1(T)$ becomes equal to $M(T)$:

$$\frac{2N\pi^2}{15}\left[\frac{Q_1}{S}\right]\widetilde{T}_{C1}=\left[m^2+\frac{N\lambda\widetilde{T}_{C1}{}^2}{72}\right]^{1/2} . \qquad (3.98)$$

Again $\widetilde{T}_{C1}$ is greater than the temperature T_C (3.79) at which the symmetry would break if there were no net charge. See Fig. 4(a). We have assumed that Q_1/S is small enough so that (3.98) has



(a) $m^2<0$



(b) $m^2>0$

FIG. 4. The temperature dependence of $\mu_1(T)$ with Q_1 and S fixed. In (a), $m^2<0$, the symmetry breaks at $\widetilde{T}_{C1}$ and for $T<\widetilde{T}_{C1}$ the effective-potential minimum occurs at $M(T)=\mu_1(T)$ with $\mu_1(T)$ determined from (3.86) and (3.89). In (b), $m^2>0$, and the symmetry never breaks.

a real solution. [However, see (4.7).]

At $\tilde{T}_{C1}$ the broken-symmetry phase ensues. The minimum of the potential is computed as in the previous discussion. In particulat (3.85)–(3.90) still apply and $\mu_1(T)=M(T)$. At high temperatures, but below $\tilde{T}_{C1}$, the charge in the Bose-Einstein condensed ground state and the excited state becomes

$$\frac{Q_1(k=0)}{S} \approx \frac{6\mu_1}{\lambda}\left[\mu_1{}^2-m^2-\frac{N\lambda T^2}{72}\right]\frac{45}{2N\pi^2T^3}\,,$$
$$\frac{Q_1(k>0)}{S} \approx \left[\frac{\mu_1}{3}T^2\right]\frac{45}{2N\pi^2T^3}\,. \tag{3.99}$$

The requirement that the sum of these terms be T independent gives a cubic equation for $\mu_1(T)$. If $\tilde{\mu}_C$ is the value (3.98) at $\tilde{T}_{C1}$ then just below $\tilde{T}_{C1}$ the approximate solution is

$$\mu_1(T) \approx \tilde{\mu}_C - \frac{(N+2)\lambda\tilde{\mu}_C}{144B^2}(\tilde{T}_{C1}{}^2-T^2) \quad (0<<T\leq\tilde{T}_{C1})\,,$$
$$B^2 \equiv \tilde{\mu}_C{}^2+\frac{\lambda\tilde{T}_{C1}{}^2}{36}\,. \tag{3.100}$$

The ground-state charge is then given by

$$\frac{Q_1(k=0)}{S} \approx \frac{|m^2|}{2B^2}\left[\frac{Q_1}{S}\right]\left[1-\left[\frac{T}{\tilde{T}_{C1}}\right]^2\right] \quad (0<<T\leq\tilde{T}_{C1})\,. \tag{3.101}$$

As the temperature drops well below $\tilde{T}_{C1}$ (3.101) ceases to apply. In the region $T \to 0$, $\mu_1(T)$ behaves differently than was the case in (3.95). We find that $T >> M(T)=\mu_1(T)$ throughout the condensed region. To keep Q_1/S fixed it is necessary that $\mu_1(T)\to 0$, with the specific form being

$$\mu_1(T) \approx \frac{N\lambda\pi^2}{135}\left[\frac{Q_1}{S}\right]\frac{T^3}{|m^2|}\,. \tag{3.102}$$

This behavior is shown in Fig. 4(a). Near $T \approx 0$ the ground-state charge becomes

$$\frac{Q_1(k=0)}{S} = \frac{Q_1}{S}\left[1-\left[\frac{T}{\tilde{T}_0}\right]^2\right] \quad (0\leq T<<\tilde{T}_0)\,,$$
$$\tilde{T}_0{}^2 \equiv \frac{18|m^2|}{\lambda}\,. \tag{3.103}$$

The above discussion pertains to $m^2<0$. When $m^2>0$, Fig. 4(b) shows that adiabatic cooling will not lead to symmetry breaking at all (assuming Q_1/S is small). Note that as $T\to 0$, we enter the nonrelativistic regime (unlike the case where $m^2<0$). In this limit, we find that

$$\mu_1(T) \approx m - \frac{S}{Q}T\,. \tag{3.104}$$

I. Spontaneous symmetry breaking with two μ's

The behavior of a system with two net charges $Q_1>Q_2>0$ is now easy to surmise. For brevity we discuss only the case of fixed V and only for $m^2<0$.

At very high temperatures $\mu_1(T)>\mu_2(T)$ since $Q_1>Q_2$. Consequently Fig. 3(a) applies and the symmetry is broken at T_{C1}. At temperatures just below T_{C1} the Q_2 symmetry is still good; in this region $\mu_2(T)<\mu_1(T)=M(T)$ and $\mu_2(T)$ increases as T falls. Some of the Q_1 charge has Bose-Einstein condensed but none of the Q_2 charge has condensed.

At $T_{C2}<T_{C1}$, μ_2 becomes equal to μ_1 and M as shown in Fig. 5. The Q_2 symmetry breaks; some of the Q_2 charge begins to condense, and the Q_1 charge continues to condense. For $T\leq T_{C2}$, μ_1, μ_2, and M are all the same functions of T, e.g., $\mu(T)$. The charge densities in the Bose-Einstein condensed ground state $(k=0)^{16}$ and the excited states $(k>0)$ for $T\leq T_{C2}<T_{C1}$ are

$$\rho_1(k=0)=\mu\hat{\phi}_1{}^2\,, \tag{3.105a}$$
$$\rho_2(k=0)=\mu\hat{\phi}_3{}^2\,, \tag{3.105b}$$
$$\rho_1(k>0)=\rho_2(k>0)\approx\tfrac{1}{3}T^2\mu\,, \tag{3.106}$$

where for simplicity we have $\hat{\phi}_2=0$ and $\hat{\phi}_4=0$.

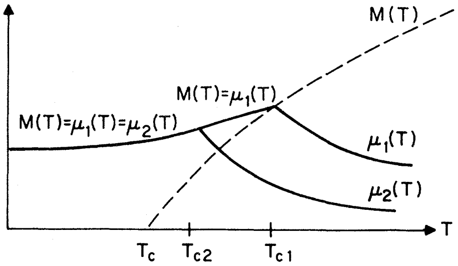


FIG. 5. The temperature dependence of two chemical potentials with Q_1, Q_2, and V held fixed assuming $Q_1>Q_2$ and $m^2<0$. The Q_1 symmetry breaks at T_{C1} and the Q_2 symmetry breaks at T_{C2}.

The vacuum expectation values must satisfy (3.66):

$$\hat{\phi}_1{}^2+\hat{\phi}_2{}^2 \approx \frac{6}{\lambda}\left[\mu^2-m^2-\frac{N\lambda T^2}{72}\right] . \qquad (3.107)$$

Furthermore, since the total charge densities ρ_1 and ρ_2 are constant their differences must also be constant. Thus

$$\mu(\hat{\phi}_1{}^2-\hat{\phi}_3{}^2)=C . \qquad (3.108)$$

The condensed charges may therfore be expressed as

$$\rho_1(k=0)\approx\frac{3\mu}{\lambda}\left[\mu^2-m^2-\frac{N\lambda T^2}{72}\right]+\frac{C}{2} , \qquad (3.109a)$$

$$\rho_2(k=0)\approx\frac{3\mu}{\lambda}\left[\mu^2-m^2-\frac{N\lambda T^2}{72}\right]-\frac{C}{2} , \qquad (3.109b)$$

where C satisfies

$$C=\left[\frac{6\mu}{\lambda}\left[\mu^2-m^2-\frac{N\lambda T^2}{72}\right]\right]\Bigg|_{T=T_{C2}} . \qquad (3.110)$$

The condition that the sum of (3.106) and (3.109a) be fixed at ρ_1 gives a cubic equation determining $\mu(T)$. [The condition that the sum of (3.106) and (3.109b) be fixed at ρ_2 gives the same cubic equation after using $\rho_1-\rho_2=C$.] The behavior of $\mu(T)$ is illustrated in Fig. 5.

At this point it is clear how the inclusion of a third charge density $\rho_3<\rho_2<\rho_1$ should proceed. In particular for $T<T_{C3}$, we see from (3.106) that the excited charge densities will all be the same and from (3.109) that the condensed charge densities will differ only by additive constants.

IV. DISCUSSION

We have demonstrated the equivalence of finite temperature symmetry breaking and Bose-Einstein condensation. For charges associated with a global symmetry there is no particular reason why the universe should be neutral. For charges associated with a local gauge symmetry the universe must be neutral only if it is gravitationally closed. Observational evidence favors an open universe and we therefore need experimental tests to measure any net charges.

A. Inclusion of fermions and gauge bosons

In a gauge theory the thermodynamic potential $\Omega(T,V,\mu)$ is still obtained by performing a function integral with μQ in the exponent. Since the allowed charges must be mutually commuting (as shown in Sec. III A), we may take them as simultaneously diagonal. Each real boson or chiral fermion denoted by $|\,i\,\rangle$ is thus an eigenstate of Q,

$$Q\,|\,i\,\rangle=q_i\,|\,i\,\rangle , \qquad (4.1)$$

with eigenvalue q_i. In calculating Ω, each such species behaves as if it had chemical potential μq_i. Of course, the charge density is still obtained by computing the derivative with respect to μ [as in (2.15)]. In the ultrarelativistic limit (neglecting all masses) this gives

$$\frac{\langle Q\rangle}{V}\approx\tfrac{1}{6}T^2\mu\sum_i q_i{}^2 , \qquad (4.2)$$

where the sum is over each polarization of fermion and real boson.

The phenomena we have described may be applicable in the breaking of chiral symmetry, of SU(5), of horizontal flavor symmetries, etc. For definiteness we concentrate on the SU(2)×U(1) electroweak symmetry. One must first fix the choice of two mutually commuting charges. If we choose electric charge and weak isospin, the functional integral contains

$$\mu_e Q_e+\mu_3 T_3 \qquad (4.3)$$

and an electrically neutral configuration implies $\mu_e(T)=0$. However, using $T_3=Q_e-Y$ we may rewrite this as

$$(\mu_e+\mu_3)Q_e-\mu_3 Y \qquad (4.4)$$

so that in this basis an electrically neutral configuration implies $\mu_e(T)=-\mu_3(T)$. Thus the numerical value of one chemical potential conveys no physical information unless the complete set of charge operators is also specified.

We shall adopt the basis (4.3). Then as shown in Fig. 4(a), the presence of a net value for $\langle T_3\rangle\neq0$ necessarily raises the temperature at which the electroweak symmetry breaks. At any $T<\tilde{T}_C$ the Higgs vacuum expectation value is automically larger [see (3.66)] and the Fermi coupling G automatically smaller than if $\langle T^3\rangle=0$. Consequently as the universe cools the neutrinos decouple from matter at a slightly higher temperature than in the standard scenario. At the temperatures of nucleosynthesis ($T\approx$ MeV) $\mu(T)$ is falling as T^3

[see (3.102)] and the effect of the modified value of the Fermi coupling on the helium abundance is quite negligible.

B. Remarks

There are several aspects of this problem that we have not pursued and which may merit further investigation. First, there is a problem of examining more realistic Higgs potentials. We have used the large-N limit in order to obtain a self-consistent equation for $M^2(T) \geq 0$. It would be desirable to develop a self-consistent scheme that does not require N large.[10] This would allow study of more complicated models with several Higgs multiplets. A particularly interesting case is that of Langacker and Pi[23] in which the U(1) of electromagnetism is broken during a finite temperature interval but is restored at small temperature. In such a system the effect of a net background charge and the phenomena of Bose-Einstein condensation is obviously more complicated.

Second, we have mostly ignored gauge bosons. It is possible for the gauge bosons to undergo Bose-Einstein condensation just as the scalars do. For example, in QCD without colored scalars a system with net color (e.g., as labeled by λ_3 and λ_8) will necessarily lead to condensation of color gluons. There may also be a technical problem when gauge bosons are present. For a non-Abelian group an ensemble with net charge cannot be invariant under all the group operations. As emphasized in Sec. III A, the presence of a chemical potential μ associated with a particular charge Q explicitly breaks the conservation of those charges which do not commute with Q. The corresponding gauge bosons have masses proportional to μ even at very high temperatures. It is not clear if this endangers the renormalizability of the theory.

Third, we note an intriguing property of the Bose-Einstein condensation when Q and S are fixed (as in the early universe). From (3.96) and (3.97) we see that at very high temperatures both $\mu(T)$ and $M(T)$ are linear in T. The condition that at very high temperatures $\mu(T) < M(T)$ therefore puts a bound on Q/S:

$$\frac{Q}{S} < \frac{5}{4\pi^2}\left[\frac{\lambda}{2N}\right]^{1/2} . \tag{4.7}$$

We have previously assumed that this bound was satisfied and found that for $m^2 < 0$ Bose-Einstein condensation eventually occurs [see Fig. 4(a)] and for $m^2 > 0$ Bose-Einstein condensation never occurs [Fig. 4(b)]. However, if Q/S is so large that (4.7) is not satisfied, then the charge is Bose-Einstein condensed even at infinite temperature (regardless of the sign of m^2) and remains condensed at all lower temperatures. This could modify the standard cosmology.

Note Added in Proof. After completion of this work we became aware of a paper by J. Kapusta, Phys. Rev. D 24, 426 (1981), treating the same subject. Our conclusions agree qualitatively; however, we do differ on a number of technical details [e.g., compare Kapusta's (1.29) with our (2.20)]. In particular, we emphasize that the chemical potential is a dependent parameter to be calculated from the known charge density. We explicitly do this in our computations.

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APPENDIX A: THE ULTRARELATIVISTIC LIMIT OF G_l AND H_l

In this appendix we evaluate the integrals G_l and H_l used in the text.[7] The appropriate dimensionless variables are $\bar{m} = m/T$ and $r = \mu/m$; the physical region is $\bar{m} \geq 0$ and $|r| \leq 1$. We define

$$g_l(\bar{m},r) \equiv \frac{1}{\Gamma(l)} \int_0^\infty x^{l-1}\,dx \left[\frac{1}{\exp[(x^2+\bar{m}^2)^{1/2} - r\bar{m}] - 1}\right] , \tag{A1}$$

$$h_l(\bar{m},r) \equiv \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1}dx}{(x^2+\bar{m}^2)^{1/2}} \left[\frac{1}{\exp[(x^2+\bar{m}^2)^{1/2} - r\bar{m}] - 1}\right] . \tag{A2}$$

The functions of interest are

$$G_l(\bar{m},r) = g_l(\bar{m},r) - g_l(\bar{m},-r) , \tag{A3}$$

$$H_l(\bar{m},r) = h_l(\bar{m},r) + h_l(\bar{m},-r) \tag{A4}$$

and satisfy

$$G_l(0,0) = 0 \quad (l > 0) , \tag{A5}$$

$$H_l(0,0) = \frac{2\zeta(l-1)}{l-1} \quad (l > 2) . \tag{A6}$$

We may derive recursion relations among the G_l and H_l by taking derivatives with respect to $\bar{m}$ and integrating by parts:

$$\frac{dG_{l+1}}{d\bar{m}}=lrH_{l+1}-\frac{\bar{m}}{l}G_{l-1}+\frac{\bar{m}^2 r}{l}H_{l-1}\,, \tag{A7}$$

$$\frac{dH_{l+1}}{d\bar{m}}=\frac{r}{l}G_{l-1}-\frac{\bar{m}}{l}H_{l-1}\,. \tag{A8}$$

Consequently, knowledge of G_1 and H_1 [plus the initial conditions (A5) and (A6)] will yield G_l and H_l for all positive odd l. Similarly, given G_2 and H_2 we can compute G_l and H_l for all positive even l. We remind the reader that the functions with l odd (even) are the ones relevant for thermodynamics in an odd (even) number of spatial dimensions.

A. The small $\bar{m}$ expansion of G_2 and H_2

It is fairly simple to obtain closed expressions for G_2 and H_2. In the expressions for g_2 and h_2 we substitute $t=\exp[-(x^2+\bar{m}^2)^{1/2}]$ and find

$$\begin{aligned} g_2(\bar{m},r)&=-\int_0^{\exp(-\bar{m})}\frac{\ln t\, dt}{\exp(-r\bar{m})-t}\\ &=\mathrm{Li}_2[e^{(r-1)\bar{m}}]-\bar{m}\ln[1-e^{(r-1)\bar{m}}]\,, \end{aligned} \tag{A9}$$

$$\begin{aligned} h_2(\bar{m},r)&=\int_0^{\exp(-\bar{m})}\frac{dt}{\exp(-r\bar{m})-t}\\ &=-\ln[1-e^{(r-1)\bar{m}}]\,, \end{aligned} \tag{A10}$$

where $\mathrm{Li}_l(z)$ is the polylogarithm function[24]

$$\mathrm{Li}_l(z)\equiv\frac{-1}{\Gamma(l)}\int_0^1\frac{(-\ln t)^{l-1}dt}{t-z^{-1}}=\sum_{n=1}^{\infty}\frac{z^n}{n^l}\,. \tag{A11}$$

By successive integration of the identity

$$\frac{1}{\exp(y)-1}=\frac{1}{y}-\tfrac{1}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}y^{2k-1}}{(2k)!} \tag{A12}$$

(B_{2k} are the Bernoulli numbers) and using

$$\zeta(2k)=\frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2(2k)!}\,, \tag{A13}$$

where k is a positive integer, we find

$$\begin{aligned} \ln[1-e^{-y}]&=\ln y-\frac{y}{2}\\ &\quad+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\zeta(2k)\left[\frac{y}{2\pi}\right]^{2k}, \end{aligned} \tag{A14}$$

$$\begin{aligned} \mathrm{Li}_2[e^{-y}]&=\frac{\pi^2}{6}+y\ln y-y-\frac{y^2}{4}\\ &\quad+2\pi\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{k(2k+1)}\left[\frac{y}{2\pi}\right]^{2k+1}. \end{aligned} \tag{A15}$$

Substituting these expansions in (A9) and (A10), we end up with

$$G_2(\bar{m},r)=2r\bar{m}\left[1-\tfrac{1}{2}\ln[\bar{m}^2(1-r^2)]+\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{k(2k+1)}C_k\left[\frac{\bar{m}}{2\pi}\right]^{2k}\right], \tag{A16}$$

$$C_k\equiv r^{2k}-(2k+1)!\sum_{j=0}^{k-1}\frac{(2k-2j-1)r^{2j}}{(2j+1)!(2k-2j)!}\,, \tag{A17}$$

$$H_2(\bar{m},r)=-\ln[\bar{m}^2(1-r^2)]+\bar{m}+\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{k}[(1+r)^{2k}+(1-r)^{2k}]\left[\frac{\bar{m}}{2\pi}\right]^{2k}. \tag{A18}$$

B. The small $\bar{m}$ expansion of G_1 and H_1

Unlike the previous case there are no closed forms available for the odd-l functions. Our strategy here will be totally different and based on an identity previously used by Dolan and Jackiw[1]:

$$\frac{1}{\exp(y)-1}=\frac{1}{y}-\tfrac{1}{2}+2\sum_{p=1}^{\infty}\frac{p}{y^2+(2\pi p)^2}\,. \tag{A19}$$

We substitute the identity into the integrands of G_1 and H_1 and integrate term by term. Actually, the integration over x must be performed with a convergence factor $x^{-\epsilon}$ $(0<\epsilon<1)$ in order to ensure that all integrations are finite. At the end of the calculation we may safely take $\epsilon \to 0$.

Applying this technique to G_1 yields

$$G_1(\overline{m},r)=I+2\sum_{p=1}^{\infty} L_p \ , \tag{A20}$$

where

$$I=2r\overline{m}\int_0^{\infty}\frac{x^{-\epsilon}dx}{x^2+\overline{m}^2(1-r^2)}=\frac{\pi r}{(1-r^2)^{1/2}}+O(\epsilon)\ , \tag{A21}$$

$$L_p=2r\overline{m}\int_0^{\infty}x^{-\epsilon}dx\frac{x^2+\overline{m}^2(1-r^2)-(2\pi p)^2}{[x^2+\overline{m}^2(1-r^2)+(2\pi p)^2]^2+(4\pi p r\overline{m})^2}\ . \tag{A22}$$

We expand the integrand of (A22) as a power series in $\overline{m}$ and integrate term by term. For illustration, consider the term linear in $\overline{m}$:

$$L_p^{(1)}=2r\overline{m}\int_0^{\infty}x^{-\epsilon}dx\frac{x^2-(2\pi p)^2}{[x^2+(2\pi p)^2]^2} \tag{A23}$$

$$=\frac{r\overline{m}}{(2\pi p)^{1+\epsilon}}\left[\Gamma\left[\frac{3-\epsilon}{2}\right]\Gamma\left[\frac{1+\epsilon}{2}\right]-\Gamma\left[\frac{3+\epsilon}{2}\right]\Gamma\left[\frac{1-\epsilon}{2}\right]\right] \tag{A24}$$

$$=\frac{-r\overline{m}\pi\epsilon}{(2\pi p)^{1+\epsilon}}+O(\epsilon^2)\ . \tag{A25}$$

Summing over p gives

$$2\sum_{p=1}^{\infty}L_p^{(1)}=\frac{-r\overline{m}2\pi\epsilon}{(2\pi)^{1+\epsilon}}\zeta(1+\epsilon)-O(\epsilon^2) \tag{A26}$$

$$=-r\overline{m}+O(\epsilon)\ . \tag{A27}$$

Higher-order terms in the $\overline{m}$ expansion are straightforward (one can safely set $\epsilon=0$ in the remaining terms). The integration over x can be done easily and the sum over p just gives Riemann's zeta function. We present the final result:

$$G_1(\overline{m},r)=\frac{\pi r}{(1-r^2)^{1/2}}-r\overline{m}+2\pi r\times\sum_{k=1}^{\infty}(-1)^{k+1}\zeta(2k+1)a_k\left[\frac{\overline{m}}{2\pi}\right]^{2k+1}\ , \tag{A28}$$

where $a_1=1$, $a_2=2r^2+\frac{3}{2}$, $a_3=3r^4+\frac{15}{2}r^2+\frac{15}{8}$. Note one apparently puzzling feature of (A28): from its definition we expect $G_l(m,r)$ to be an odd function in $\overline{m}$, whereas (A28) does not satisfy this property. The resolution of this puzzle is to note that we assumed $\overline{m}>0$ in computing (A21). For $\overline{m}<0$ one would obtain the opposite sign for I. Therefore, we conclude that $G_1(\overline{m},r)$ is in fact an odd function in $\overline{m}$ but it is not continuous at the point $\overline{m}=0$. However, $G_1(\overline{m},r)$ does have a definite limit as $\overline{m}\to 0$ from the positive side. Because $\overline{m}\geq 0$ corresponds to the physical region, it is clear that the appropriate zero-mass limit involves the limiting procedure just described.

We now turn to the computation of H_1. Using (A2), (A4), and (A19) we find:

$$H_1(\overline{m},r)=I'+I''+2\sum_{p=1}^{\infty}M_p\ , \tag{A29}$$

where

$$I'=2\int_0^{\infty}\frac{x^{-\epsilon}dx}{x^2+\overline{m}^2(1-r^2)}=\frac{\pi}{\overline{m}(1-r^2)^{1/2}}+O(\epsilon)\ , \tag{A30}$$

$$I''=-\int_0^{\infty}\frac{x^{-\epsilon}dx}{(x^2+\overline{m}^2)^{1/2}}=-\frac{1}{\epsilon}+\ln\left[\frac{\overline{m}}{2}\right]+O(\epsilon)\ , \tag{A31}$$

$$M_p=2\int_0^{\infty}x^{-\epsilon}dx\frac{x^2+\overline{m}^2(1-r^2)+(2\pi p)^2}{[x^2+\overline{m}^2(1-r^2)+(2\pi p)^2]^2+(4\pi p r\overline{m})^2}\ . \tag{A32}$$

We expand the integrand of (A32) as a power series in $\bar{m}$ and integrate term by term. As before, we may set $\epsilon=0$ in all but the term leading in $\bar{m}$. The leading term gives

$$M_p^{(1)}=2\int_0^\infty \frac{x^{-\epsilon}dx}{x^2+(2\pi p)^2}$$

$$=\frac{1}{2p^{1+\epsilon}}[1-\epsilon\ln 2\pi+O(\epsilon^2)]\ . \qquad \text{(A33)}$$

Summing over p and using

$$\zeta(1+\epsilon)=\frac{1}{\epsilon}+\gamma+O(\epsilon)\ , \qquad \text{(A34)}$$

where γ is Euler's constant, we find

$$2\sum_{p=1}^{\infty}M_p^{(1)}=\frac{1}{\epsilon}+\gamma-\ln(2\pi)+O(\epsilon)\ . \qquad \text{(A35)}$$

Notice that when we add (A31) and (A35) the $\epsilon=0$ singularity cancels and we may safely take $\epsilon\to 0$. The computation of higher orders in $\bar{m}$ of (A32) is straightforward and the end result is

$$H_1(\bar{m},r)=\frac{\pi}{\bar{m}(1-r^2)^{1/2}}+\ln\left[\frac{\bar{m}}{4\pi}\right]+\gamma + \sum_{k=1}^{\infty}(-1)^k\zeta(2k+1)b_k\left[\frac{\bar{m}}{2\pi}\right]^{2k}\ , \qquad \text{(A36)}$$

where $b_1=r^2+\frac{1}{2}$, $b_2=r^4+3r^2+\frac{3}{8}$, $b_3=r^6+\frac{15}{2}r^4+\frac{45}{8}r^2+\frac{5}{16}$. Note that (A36) is valid for $\bar{m}>0$; actually $H_1(\bar{m},r)$ is an even function in $\bar{m}$.

APPENDIX B: THE NONRELATIVISTIC LIMIT OF G_l AND H_l

Large $\bar{m}$ corresponds to the low-temperature (nonrelativistic) limit. We may derive asymptotic expansions for g_l and h_l directly as follows. Substituting $w=\exp[\bar{m}-(x^2+\bar{m}^2)^{1/2}]$ into (A1) and (A2) yields

$$g_l(\bar{m},r)=\frac{1}{\Gamma(l)}\int_0^1\frac{(-\ln\omega)^{l/2-1}(2\bar{m}-\ln\omega)^{l/2-1}(\bar{m}-\ln\omega)}{\exp[(1-r)\bar{m}]-\omega}d\omega\ , \qquad \text{(B1)}$$

$$h_l(\bar{m},r)=\frac{1}{\Gamma(l)}\int_0^1\frac{(-\ln\omega)^{l/2-1}(2\bar{m}-\ln\omega)^{l/2-1}}{\exp[(1-r)\bar{m}]-\omega}d\omega\ . \qquad \text{(B2)}$$

Writing

$$(2\bar{m}-\ln\omega)^{l/2-1}=(2\bar{m})^{l/2-1}\sum_{k=0}^{\infty}\frac{\Gamma(l/2)}{\Gamma(l/2-k)k!}\left[\frac{-\ln\omega}{2\bar{m}}\right]^k \qquad \text{(B3)}$$

and using (A11) to perform the w integrations, we find

$$g_l(\bar{m},r)=\frac{\Gamma(l/2)}{\Gamma(l)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(l/2-k)k!}\left[\frac{1}{2\bar{m}}\right]^{k+1-l/2}\left[\bar{m}\Gamma\left[\frac{l}{2}+k\right]\mathrm{Li}_{k+l/2}(e^{(r-1)\bar{m}}) + \Gamma\left[\frac{l}{2}+k+1\right]\mathrm{Li}_{k+l/2+1}(e^{(r-1)\bar{m}})\right]\ , \qquad \text{(B4)}$$

$$h_l(\bar{m},r)=\frac{\Gamma(l/2)}{\Gamma(l)}\sum_{k=0}^{\infty}\frac{\Gamma(l/2+k)}{\Gamma(l/2-k)k!}\mathrm{Li}_{k+l/2}(e^{(r-1)\bar{m}})\left[\frac{1}{2\bar{m}}\right]^{k+1-l/2}\ . \qquad \text{(B5)}$$

It is now easy to obtain the familiar results in the large-$\bar{m}$ limit. For example, as noted in (2.17) the charge density is

$$\frac{Q}{V}=\frac{T^3}{\pi^2}[g_3(\bar{m},r)-g_3(\bar{m},-r)]\ . \qquad \text{(B6)}$$

From (B4) the leading term in the large-$\bar{m}$ expansion of g_3 is

$$g_3(\bar{m},r)\approx\tfrac{1}{4}(2\pi\bar{m}^3)^{1/2}\mathrm{Li}_{3/2}(e^{(r-1)\bar{m}})\ . \qquad \text{(B7)}$$

By definition $(r-1)\bar{m}=(\mu-m)/T$, where the μ that appears in this paper is related to the chemical potential of nonrelativistic thermodynamics by $\mu_{\mathrm{NR}}=\mu-m$. Thus, $|\mu|\le m$ is equivalent to $-2mc^2\le\mu_{\mathrm{NR}}<0$ (where we have put back the speed of light). In the nonrelativistic limit $(c\to\infty)$

we regain the familiar result $-\infty < \mu_{\mathrm{NR}} < 0$. More precisely, the nonrelativistic limit corresponds to $T << m$ (i.e., $\bar{m} >> 1$). We define

$$z_{\mathrm{NR}} \equiv e^{\beta\mu_{\mathrm{NR}}} \equiv e^{(r-1)\bar{m}} . \tag{B8}$$

Then,

$$\begin{aligned} g_3(\bar{m},-r) &\approx \tfrac{1}{4}(2\pi\bar{m}^3)^{1/2}\mathrm{Li}_{3/2}(z_{\mathrm{NR}}{}^{-1}e^{-2\bar{m}}) \\ &\approx \frac{1}{4z_{\mathrm{NR}}}(2\pi\bar{m}^3)^{1/2}e^{-2\bar{m}} \end{aligned} \tag{B9}$$

is exponentially small. That is, the contribution of the antiparticles is exponentially small in the nonrelativistic limit. Furthermore, in this limit $Q=N$ (number of particles) since pair creation is suppressed. The end result is

$$\frac{N}{V} \approx \left[\frac{mT}{2\pi}\right]^{3/2} \mathrm{Li}_{3/2}(z_{\mathrm{NR}}) , \tag{B10}$$

which is the standard result.

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[1] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974); S. Weinberg, *ibid.* 9, 3357 (1974).

[2] B. J. Harrington and A. Yildiz, Phys. Rev. Lett. 33, 324 (1974).

[3] It is possible to evade this coupling-constant suppression in gauge theories. Linde has shown that a gauge field coupling to a net background of nonbosonic charge may develop a vacuum expectation value. In this case the critical temperature is raised by an amount independent of the coupling constant. Because we have restricted the scope of this paper mostly to non-gauge field theories, we will not discuss the above possibility further. The reader may consult the excellent review on phase transitions in gauge theories by A. D. Linde, Rep. Prog. Phys. 42, 389 (1979).

[4] N. N. Bogoliubov, J. Phys. (Moscow) 11, 23 (1947), treats $T=0$. For $T>0$ see A. E. Glassgold, A. N. Kaufman, and K. M. Watson, Phys. Rev. 120, 660 (1960).

[5] A modern, functional integral approach is used in F. W. Wiegel and J. B. Jalickee, Physica 57, 317 (1972) and in F. W. Wiegel, Phys. Rep. 16C, 57 (1975).

[6] The nonrelativistic limit of (1.5) is precisely (1.11). To see this first express (1.5) in terms of $\psi=(2m)^{-1/2} \times(m\phi+i\pi^+)$ and $\chi=(2m)^{-1/2}(m\phi-i\pi^+)$ where $\phi\equiv2^{-1/2}(\phi_1+i\phi_2)$ and $\pi^+=2^{-1/2}(\pi_1+i\pi_2)$. Then ψ and χ correspond to the particle and antiparticle, respectively. In the nonrelativistic regime all terms involving χ are negligible and (1.11) results.

[7] An alternative method (using Mellin transforms) for obtaining these high-temperature expansions is contained in H. E. Haber and H. A. Weldon, J. Math. Phys. (to be published).

[8] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. 46, 1497 (1981).

[9] See also K. Redlich and L. Turko, Z. Phys. C 5, 201 (1980); in Proceedings of the International Symposium on Statistical Mechanics of Quarks and Hadrons, Bielefeld, Germany, 1980, edited by H. Satz (North Holland, Amsterdam, to be published); L. Turko, Phys. Lett. 104B, 153 (1981).

[10] The effective potential in the large-N limit (at $T=0$) was studied by S. Coleman, R. Jackiw, and H. D. Politzer, Phys. Rev. D 10, 2491 (1974); L. F. Abbott, J. S. Kang, and H. J. Schnitzer, *ibid.* 13, 2212 (1976); M. Kobayashi and T. Kugo, Prog. Theor. Phys. 54, 1537 (1975) and was shown to have a pathology. Even at $T=0$ the symmetric vacuum ($\hat{\phi}=0$) is a lower energy minimum than the asymmetric vacuum ($\hat{\phi}\neq0$); furthermore, the Green's functions contain tachyon poles when computed in the asymmetric vacuum but not when computed in the symmetric vacuum. Abbott *et al.* point out that if the asymmetric vacuum is to be the true ground state for any O(N) model then there must be a critical value of N below which this pathology vanishes. Coleman *et al.* suggest that the leading $1/N$ approximation might remain valid for thermodynamic quantities since there are no internal inconsistencies in the thermodynamics. We expect that the qualitative features of our thermodynamics results do not depend on the large-N approximation.

[11] C. W. Bernard, Phys. Rev. D 9, 3312 (1974).

[12] The normalization constants N and $N'(\beta)$ are discussed in detail in Ref. 11.

[13] See K. Huang, *Statistical Mechanics* (Wiley, New York, 1963) or L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1968).

[14] This set is the canonical Cartan subalgebra.

[15] For example, periodic boundary conditions ensure this conservation in a finite volume. See G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, *Advances in Particle Physics, Vol. 2,* edited by R. D. Cool and R. G. Marshak (Wiley, New York, 1968).

[16] Here $\hat{\phi}^2$ is the exact vacuum expectation value and we refer to particles in the ground state of the interacting theory. In nonrelativistic treatments (Refs. 4 and 5) it is common to separate $\hat{\phi}^2$ into its tree value and the radiative corrections and consequently interpret these as separate contributions to the charge from the unperturbed ground state and from radiative excitations.

[17] A lucid discussion is contained in D. ter Haar, *Lectures on Selected Topics in Statistical Mechanics* (Pergamon, Oxford, 1977).

[18] R. Jackiw, Phys. Rev. D 9, 1686 (1974); H. J.

Schnitzer, *ibid.* 10, 1800 (1974); 10, 2042 (1974).

[19] Note that because of the extra $\frac{1}{2}$ included in (3.42), our subtraction scheme differs slightly from the $\overline{\text{MS}}$ scheme used in W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D 18, 3998 (1978).

[20] See also Refs. 1 and 10.

[21] The boundary condition (3.57) determines C_1 only for $m^2>0$. We assume C_1 is unchanged when $m^2<0$.

[22] Recall that we take $N\lambda$ fixed in the large-N limit.

[23] P. Langacker and S. Y. Pi, Phys. Rev. Lett. 45, 1 (1980).

[24] L. Lewin, *Dilogarithms and Associated Functions* (MacDonald, London, 1958); K. S. Kolbig, J. A. Mignaco, and E. Remiddi, B.I.T. 10, 38 (1970).