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Renormalization-group functions and spectral functions

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The interplay of the spectral representation and the renormalization group is investigated from a global nonperturbative point of view. A simple physical behavior is strongly indicated for quantum electrodynamics at high energies, with features resembling strong-interaction duality and critical phenomena. Also included is a renormalization-group derivation of the Goldberger-Treiman relation in ps-ps theory.

I. INTRODUCTION

The last decade has witnessed a tremendous interest in the renormalization group (RG). Yet, there have been surprisingly few discussions of the relation between RG functions and spectral functions.[1] Our purpose is to fill this lacuna, and to trace the relation in its logical, mathematical, and physical aspects. On the one hand we shall illustrate how the spectral representation accommodates a RG fixed point, the concept which has proved so fruitful in the understanding of critical phenomena,[2] on the other hand we explore the implications of the RG for the structure of the spectral function in the high-energy region.[3] For the most part the discussion is restricted to the case of the photon propagator in quantum electrodynamics (QED). However, emphasis is on its global aspects rather than on perturbation theory[4]; the latter will be used as a guide only when not in conflict with general principles such as positivity and analyticity.

The organization of the paper is as follows. Section II deals with the fixed-point nature of the bare charge e_∞. We show that the fixed-point behavior may be inferred without use of asymptotic theorems in the infrared[5] or the ultraviolet,[6] even for cases where the Gell-Mann–Low function[7] $\psi(e_\lambda^2)$ or the Callan-Symanzik function[8] $\beta(e)$ fail to exist in a global sense. The only case in which the derivation breaks down turns out to be when the functions do exist but are identically zero.[15,38] Some evidence is provided in favor of the view that if the physical charge e approaches $e_\infty < \infty$ the theory becomes free. Also given is a simple bound on $\psi(e_\lambda^2)$.[40] Section III deals with the asymptotic behavior of the spectral function. Here, unlike the case of Green's functions, the differential form of the RG equation is not directly applicable. We find, however, that the integrated version of the RG equation is likely to apply; in that case a dualitylike[54,55] relation is obtained between the massive and the massless electron theory.[9,10] (A simple proof of the nonanalyticity[4,11] of ψ is also provided in this connection.) We proceed to argue that the scaling behavior of the photon propagator discovered by Gell-Mann and Low[7] is the precise analog of the scaling laws[62] in critical phenomena, with the mass of the polarization current taking the place of the inverse of the correlation length. Under this hypothesis, the β function is directly related to the first moment of the spectral weight of the inverse propagator. We also show that the second moment is infinite under more conventional assumptions. A corollary is that the Schwinger term[12] in the current-current commutator is divergent if $e_\infty = \infty$. Section IV is devoted to a general discussion, whereas questions of mathematical rigor are discussed in Appendix A. Appendix B contains a brief indication of the extension to theories other than QED, the example treated being the Goldberger-Treiman relation[13] in ps-ps theory.

[*Note added in proof.* The analyticity properties of ψ have been investigated in detail by N. N. Khuri, Phys. Rev. D 23, 2285 (1981).]

II. THE FIXED-POINT NATURE OF THE BARE CHARGE

Our purpose in this section is to provide a critical derivation of the following results:

(i) The bare charge e_∞ is independent of the physical charge e

and therefore[14]

(ii) $Z_3 = e^2/e_\infty^2 \to 0$ as $e^2 \to 0$.

Before giving our own, we shall have a brief look at the previous derivations.

The first one is due to Gell-Man and Low.[7] In their derivation, an effective charge e_λ associated with momentum λ is introduced, which interpolates between e and e_∞ for $\lambda = 0$ and $\lambda = \infty$. The charge e_λ is shown to satisfy an equation of the form

$$\lambda^2(de_\lambda^2/d\lambda^2) = \psi(1/\lambda^2, e_\lambda^2) \ . \qquad (2.1)$$

(We shall often set the electron mass m equal to

1.) A crucial approximation is then made in which the λ^2 dependence of ψ is neglected for large λ:

$$\lambda^2(de_\lambda{}^2/d\lambda^2)=\psi(e_\lambda{}^2),\quad \psi(e_\lambda{}^2)\equiv\psi(0,e_\lambda{}^2)\ . \tag{2.2}$$

With this approximation it follows immediately that $e_\infty{}^2$ is given by the zero of ψ independent of e, i.e., $e_\infty{}^2$ is a fixed point of Eq. (2.2).

Equation (2.2), however, is an autonomous differential equation in $\ln\lambda^2$, whereas the original Eq. (2.1) is not, so the global properties of their solutions can be quite different.[15] The derivation therefore left room for doubt, particularly in view of the novelty of the result.

A different approach was adopted by Johnson, Baker, and Willey,[9] who imposed the requirement of perturbational self-consistency. They have also arrived at result (i), but later it was shown by Adler[16] that a rearrangement of their perturbation series yields a different (although interesting) result, and the situation remained unclear.

Yet another approach[16-19] became possible with the discovery of the Callan-Symanzik equation[8]

$$[-\lambda(\partial/\partial\lambda)+\beta(e)(\partial/\partial e)]e_\lambda{}^{-2}=\Delta_s\Gamma^{(2)}(-\lambda^2)/e^2\lambda^2\ . \tag{2.3}$$

In the asymptotic region $\lambda\to\infty$, Weinberg's theorem[6] indicates that the right-hand side may be neglected order by order to give

$$[-\lambda(\partial/\partial\lambda)+\beta(e)(\partial/\partial e)]e_\lambda{}^{-2}=0\,, \tag{2.4}$$

and we recover the result that e_∞ is independent of e, this time being given by $\beta(e_\infty)=0$.

To infer the asymptotic behavior of a function from an order-by-order investigation of its perturbation series, however, can be quite dangerous.[20] Indeed, the infrared behavior of the electron propagator[21-24] provides an explicit example. The RG arguments themselves give[23,24,19]

$$(p^2-m^2)S_{Fc}(p)\sim(p^2-m^2)^{-(3-\alpha)e^2/8\pi^2}\,, \tag{2.5}$$

which has a quite different behavior from each term of its expansion

$$1-\frac{3-\alpha}{8\pi^2}\,e^2\ln(p^2-m^2)+\cdots\ . \tag{2.6}$$

The situation may be improved if the electron mass m is also renormalized multiplicatively,[25,26] treating it on the same footing as e. In particular, if a mass-independent renormalization scheme is used, the RG equations take the form[27]

$$[-\lambda(\partial/\partial\lambda)+\bar\beta(\bar e)(\partial/\partial\bar e)+\bar\gamma_s(\bar e)\bar m(\partial/\partial\bar m)]e_\lambda{}^{-2}=0\,, \tag{2.7}$$

with appropriate parameters $\bar e$ and $\bar m$. This time $e_\lambda{}^2$ will have a limit independent of its initial value whenever $\bar\gamma_s(\bar e_\infty)<1$, where $\bar e_\infty$ is the first positive zero of $\bar\beta$. Unfortunately, however, it is not known whether this condition holds or not.[28] Another difficulty is that the parameters $\bar e$, $\bar e_\infty$, and $\bar m$, in general, will have no direct physical significance; the equations become easier to solve technically, but harder to interpret physically. In particular, if dimensional regularization[29] with minimal subtraction is used, Z_3 as a function of space-time dimension d will have an essential singularity at $d=4$.[30]

With these remarks, let us turn to our derivation. We start with the photon propagator

$$D_{Fc}(k)_{\mu\nu}=\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)\frac{d_c(k^2,e^2)}{k^2}+\text{gauge terms}. \tag{2.8}$$

In terms of the proper photon self-energy $(-g_{\mu\nu}k^2+k_\mu k_\nu)\Pi_c(k^2)$ we have

$$d_c{}^{-1}(k^2,e^2)=1+e^2\Pi_c(k^2)\equiv\Gamma^{(2)}(k^2)/k^2\ . \tag{2.9}$$

Theory and experiment strongly support the spectral representation for $\Pi_c(k^2)$ in the once-subtracted form

$$\Pi_c(k^2)=\int\frac{dM^2}{M^2}\,\frac{k^2}{M^2-k^2}\,\rho(M^2,e^2)\,, \tag{2.10}$$

where gauge invariance and generalized unitarity require the spectral weight ρ to be positive.

At $k^2=0$, Π_c vanishes, and the residue of the photon pole in (2.8) is properly normalized,

$$d_c(0,e^2)=1\ . \tag{2.11}$$

For $k^2=-\lambda^2$ (spacelike), Π_c is a monotonically decreasing function of λ^2. Since a zero of $1+e^2\Pi_c(-\lambda^2)$ would imply a spacelike pole for the (transverse) photon propagator, we must have $Z_3\equiv1+e^2\Pi_c(-\infty)\geq0$ or (Refs. 31, 7, and 69)

$$0\leq Z_3=1-e^2\int\frac{dM^2}{M^2}\,\rho(M^2,e^2)<1\ . \tag{2.12}$$

For $k^2\to\infty$,

$$D_{Fc}(k)_{\mu\nu}\cong\frac{Z_3{}^{-1}}{k^2}\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)+\text{gauge terms}\,, \tag{2.13}$$

so Z_3 is indeed the charge renormalization constant,[83] and we may write down the formula for the bare charge

$$\frac{1}{e_\infty{}^2}=\frac{1}{e^2}-\int\frac{dM^2}{M^2}\,\rho(M^2,e^2)\ . \tag{2.14}$$

Barring pathological behavior, Eq. (2.12) requires that

$$\rho(M^2,e^2)\to0\quad(M^2\to\infty,\ e^2>0)\ . \tag{2.15}$$

It is well known that (2.12) and (2.15) are violated to any order in perturbation theory.[32] (A similar problem exists for vertex functions.[33]) In particular, $\Pi(0)$ appears as a divergent (cutoff-dependent) quantity. We also notice that an unsubtracted dispersion relation may be written down as

$$\Pi(z) = \int \frac{dM^2}{M^2 - z} \rho(M^2, e^2), \tag{2.16}$$

with

$$\Pi_c(z) = \Pi(z) - \Pi(0), \tag{2.17}$$

$$\Pi(0) = \int \frac{dM^2}{M^2} \rho(M^2, e^2) = (1 - Z_3)/e^2. \tag{2.18}$$

However, as first recognized by Gell-Mann and Low,[7] the multiplicative renormalizability of QED allows us to study the limit as the cutoff is removed. The first observation to be made is that the spectral representation itself provides a natural cutoff for $\Pi(0)$:

$$\Pi_\lambda(0) \equiv \int \frac{dM^2}{M^2} \frac{\lambda^2}{\lambda^2 + M^2} \rho(M^2, e^2). \tag{2.19}$$

Evidently

$$\Pi_\lambda(0) = -\Pi_c(-\lambda^2) = \Pi(0) - \Pi(-\lambda^2). \tag{2.20}$$

Comparison of (2.20) with (2.17) suggests a renormalization scheme in which subtractions are performed at a large spacelike momentum $k_2 = -\lambda^2$ rather than at the on-shell value $k^2 = 0$, e.g., we replace (2.17) with

$$\Pi_\lambda(k^2) = \Pi(k^2) - \Pi(-\lambda^2). \tag{2.21}$$

The condition corresponding to (2.11) for the new propagator function[16] $d_c(k^2, \lambda^2, e_\lambda^2) \equiv [1 + e_\lambda^2 \Pi_\lambda(k^2)]^{-1}$ is then

$$d_c(-\lambda^2, \lambda^2, e_\lambda^2) = 1. \tag{2.22}$$

Here we have written e_λ to indicate the new expansion parameter in perturbation theory, since under (2.22) it is no longer equal to the physical (on-shell) charge defined, for example, by Thomson scattering. However, the basic postulate of renormalization theory is that observable quantities in QED are finite and uniquely determined in terms of the physical charge e and the physical electron mass m; the amplitudes constructed from $d_c(k^2, \lambda^2, e_\lambda^2)$ and e_λ^2 should therefore be those for a certain value of e.[34] (We restrict the change in the renormalization scheme so that m is unaffected.) By observing that the structure of the Schwinger-Dyson equations[35] is invariant under the group of (finite) multiplicative renormalization[23, 36, 37]

$$d_c \to z_3^{-1} d_c, \quad e^2 \to z_3 e^2 \tag{2.23}$$

we may obtain the desired condition for physical equivalence between the different parametrizations (renormalization schemes) $(0, e^2)$ and (λ^2, e_λ^2),

$$z_3 = e^2/e_\lambda^2 = d_c(k^2, \lambda^2, e_\lambda^2)/d_c(k^2, e^2). \tag{2.24}$$

This is just Dyson's relation,[35] since we may identify $d_c(k^2, \lambda^2, e_\lambda^2)$ and e_λ^2 as the bare propagator and the bare charge associated with the cutoff λ.[25]

We emphasize that the relation between e^2 and e_λ^2 is required to be invertible. Indeed, for $k^2 = 0$ and $k^2 = -\lambda^2$, Eq. (2.24) reduces explicitly to

$$e^2 = e_\lambda^2 d_c(0, \lambda^2, e_\lambda^2), \quad e_\lambda^2 = e^2 d_c(-\lambda^2, e^2). \tag{2.25}$$

In other words, global invertibility is essential[38] if the multiplicative RG (2.23) is to be realized by a change in the subtraction point, or equivalently, a change in the cutoff.

Now the central observation of Gell-Mann and Low[7] is that when $d_c(k^2, \lambda^2, e_\lambda^2)$ is written in terms of the variables $s = -k^2/\lambda^2$, m^2/λ^2, and e_λ^2, it has negligible dependence on m^2/λ^2 when $\lambda \gg m$ and $s \gtrsim 1$:

$$d_c(-s\lambda^2, \lambda^2, e_\lambda^2) \equiv d(-s, m^2/\lambda^2, e_\lambda^2)$$
$$\cong d(-s, 0, e_\lambda^2). \tag{2.26}$$

In modern language, there are no infrared divergences as $m \to 0$ if the photon propagator is subtracted off-shell.[5] Therefore, for $\mu^2 = -k^2 = s\lambda^2$

$$e_\mu^2 = e_\lambda^2 d(-s, m^2/\lambda^2, e_\lambda^2) \cong e_\lambda^2 d(-s, 0, e_\lambda^2), \tag{2.27}$$

where we have used (2.22) and the invariance relation which follows from (2.24):

$$e_\mu^2 d(k^2/\mu^2, m^2/\mu^2, e_\mu^2) = e_\lambda^2 d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2). \tag{2.28}$$

Equation (2.27) says that for $\lambda \gg m$, the change in the normalization point $\lambda^2 \to \mu^2 = s\lambda^2$ induces a transformation on the effective charge T_s: $e_\lambda^2 \to e_\mu^2 = e_\lambda^2 d(-s, 0, e_\lambda^2)$, which is dependent only on $s = \mu^2/\lambda^2$. This is analogous to the situation in classical mechanics where time translation T_s: $(q(t), p(t)) \to (q(t+s), p(t+s))$ depends only on $s = (t+s) - t$ for closed systems. Stated more formally, Eq. (2.27) gives a nonlinear realization of the multiplicative real half-line, whereas a classical dynamical system gives a nonlinear realization of the additive real line.[26]

As usual, an essential role is played by the generator of infinitesimal transformations.[39] In this case it is obtained simply by differentiating by s and then setting $s = 1$:

$$\lambda^2 (de_\lambda^2/d\lambda^2) = (\partial/\partial s)\big|_{s=1} \, e_\lambda^2 d(-s, 0, e_\lambda^2). \tag{2.29}$$

Evidently, this is just the Gell-Mann–Low equation (2.2) with

$$\psi(e_\lambda{}^2)=(\partial/\partial s)\big|_{s=1}\; e_\lambda{}^2 d(-s,0,e_\lambda{}^2)\,. \qquad (2.30)$$

However, as mentioned before, it is questionable whether the λ dependence may be really neglected in (2.27). Therefore, let us consider what may be legitimately derived from the exact form (2.1). It will turn out that we must have result (ii) apart from the exceptional case mentioned in the Introduction.

To this end, we combine the spectral representation equations (2.11)–(2.18) with the RG equations (2.19)–(2.30). We immediately obtain

$$\frac{1}{e_\lambda{}^2}-\frac{1}{e_\infty{}^2}=\int\frac{dM^2}{M^2+\lambda^2}\rho(M^2,e^2)$$
$$\geq\frac{1}{\lambda^2}\int\frac{dM^2}{M^2+1}\rho(M^2,e^2)\quad(\lambda\geq 1)\,, \qquad (2.31)$$

which shows that $e_\lambda{}^{-2}$ indeed approaches $e_\infty{}^{-2}$ but with a rate not faster than λ^{-2}. Also,

$$\psi(1/\lambda^2,e_\lambda{}^2)=e_\lambda{}^4\int dM^2\frac{\lambda^2}{(M^2+\lambda^2)^2}\rho(M^2,e^2(\lambda^2,e_\lambda{}^2))$$
$$\leq e_\lambda{}^4\int\frac{dM^2}{M^2+\lambda^2}\rho(M^2,e^2(\lambda^2,e_\lambda{}^2))=e_\lambda{}^2-\frac{e_\lambda{}^4}{e_\infty{}^2}\,, \qquad (2.32)$$

i.e., $\psi(e_\lambda{}^2)$, if it exists, is positive and bounded by $e_\lambda{}^2$.[40,41]

Further properties follow in conjunction with invertibility [Eq. (2.25)]. When $e_\lambda{}^2$ is plotted against λ^2 (Fig. 1), it is easy to see that invertibility forbids different curves from intersecting. This result also follows immediately from (2.1) itself; integral curves of an ordinary differential equation must form a flow. Since $e_\lambda{}^2$ is an increasing function of λ^2 for given e^2, it follows that when e^2 is considered as a function of $e_\lambda{}^2$ and λ^2, it must decrease with increasing λ^2 (Fig. 2). Since e^2 is bounded from below, there must exist

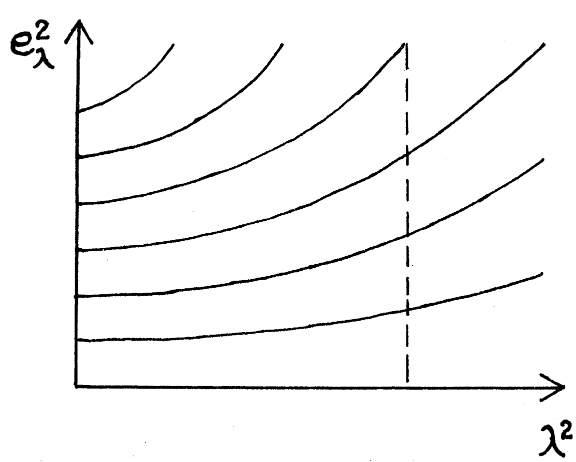

FIG. 1. The trajectories of $e_\lambda{}^2$ plotted vs λ^2.

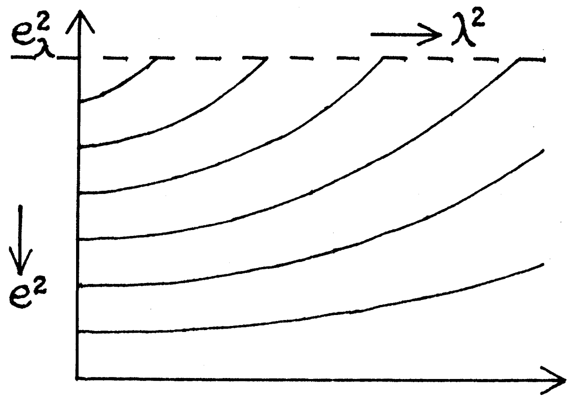

FIG. 2. The relation between e^2 and λ^2 for fixed $e_\lambda{}^2$.

a limit

$$e_*{}^2=\lim e^2(\lambda^2,e_\lambda{}^2)\quad(e_\lambda{}^2\text{ fixed})\,. \qquad (2.33)$$

(In the following, lim is to be understood as $\lim_{\lambda\to\infty}$ unless otherwise indicated.)

It is also seen that e^2 decreases with decreasing $e_\lambda{}^2$ for fixed λ^2 (Fig. 1). Therefore, there exist only two possibilities for $e_*{}^2$:

(1) $e_*{}^2=0$ for $e_\lambda{}^2$ below a certain value;
(2) $e_*{}^2>0$ for all $e_\lambda{}^2>0$.

As we shall see, possibility (1) is favored; in this case, result (ii) must hold since

$$0\leq Z_3(e^2)=\frac{e^2}{e_\infty{}^2}<\frac{e^2}{e_\lambda{}^2}\to 0\quad(e^2\to e_*{}^2=0\,,\ e_\lambda{}^2\text{ fixed})\,. \qquad (2.34)$$

As for possibility (2), there is a serious difficulty.[42] Let us assume $e_*{}^2>0$. Then it follows from (2.15) and (2.32) that for a fixed $e_\lambda{}^2$

$$\lim\psi(1/\lambda^2,e_\lambda{}^2)=\lim e_\lambda{}^4\int\frac{dM^2}{(M^2+1)^2}\rho[\lambda^2M^2,e^2(\lambda^2,e_\lambda{}^2)]$$
$$=e_\lambda{}^4\int\frac{dM^2}{(M^2+1)^2}\rho(\infty,e_*{}^2)=0\,, \qquad (2.35)$$

i.e., $\psi(e_\lambda{}^2)$ vanishes identically.[15]

It is evidently important that this reasoning fails to apply for $e_*{}^2=0$. This we may expect to be the case, since perturbation theory gives

$$\rho(M^2,0)=\frac{1}{12\pi^2}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2}\theta(M^2-4m^2)$$
$$\to\frac{1}{12\pi^2}\quad(M^2\to\infty)\,, \qquad (2.36)$$

in contrast with (2.15), i.e., the limit of ρ as $(M^2,e^2)\to(\infty,0)$ is dependent on the path of approach [see also (3.14)]. It should also be noticed that result (ii) must hold not only when $\psi(e_\lambda^2)$ is finite and nontrivial, but also when $\lim\psi(\lambda^2, e_\lambda^2)$ is infinite or nonexistent, since "(1)⇒(ii)" and "(2)⇒ψ existent but zero" imply that "ψ nonexistent⇒not (2)⇒(1)⇒(ii)." In any case, the Gell-Mann–Low equation (2.1) together with possibility (1) gives result (ii).[43]

Let us now turn to the implications of the Callan-Symanzik equation.[8] We may briefly recall its standard derivation.[44] The starting point is again the relation between the renormalized n-photon proper vertex and its bare counterpart

$$\Gamma^{(n)} = z_3(\lambda)^{n/2}\Gamma_b^{(n)}, \tag{2.37}$$

where λ is the cutoff; the bare quantities are denoted with the subscript b. The operation

$$m\frac{\partial}{\partial m}\bigg|_{e_b,\lambda} = m\frac{\partial}{\partial m}\bigg|_{e,\lambda} + \left(m\frac{\partial}{\partial m}\bigg|_{e_b,\lambda} e\right)\frac{\partial}{\partial e}\bigg|_{m,\lambda} \tag{2.38}$$

is applied and then the limit $\lambda\to\infty$ is taken (with e and m fixed) to give

$$[m(\partial/\partial m)+\beta(e)(\partial/\partial e)-n\beta(e)/e]\,\Gamma^{(n)} = \Delta_s\Gamma^{(n)}, \tag{2.39}$$

where[45]

$$\beta(e) \equiv \lim m\frac{\partial}{\partial m}\bigg|_{e_b,\lambda}, \quad e = -\lim\lambda\frac{\partial}{\partial\lambda}\bigg|_{e_b,m} e\,. \tag{2.40}$$

Implicit in the derivation is that the relation between e and the (cutoff) bare charge e_b is invertible, and that the limit in (2.40) exists. In perturbation theory, the first requirement is trivially satisfied (formal power series are always invertible), whereas the second one may be shown to be equivalent to the renormalizability of QED and hence also satisfied.

However, a global assessment is also necessary if the Callan-Symanzik equation is to be used for deriving the global properties of QED. We have already seen that invertibility, if taken in a global sense, is nontrivial. (As before, we may adjust the cutoff so that $e_b = e_\lambda$.) As for the existence of β, we rewrite (2.40) as

$$\beta(e) = \lim\beta(1/\lambda, e), \tag{2.41}$$

$$e\beta(1/\lambda, e) = \lambda^2\frac{\partial}{\partial\lambda^2}\bigg|_e \frac{1}{e_\lambda^2}\bigg/\frac{\partial}{\partial e^2}\bigg|_\lambda \frac{1}{e_\lambda^2}\,. \tag{2.42}$$

The numerator of (2.42) vanishes as $\lambda\to\infty$ (e is fixed now),

$$\lambda^2\frac{\partial}{\partial\lambda^2}\frac{1}{e_\lambda^2} = -\int dM^2\frac{\lambda^2}{(M^2+\lambda^2)^2}\rho(M^2, e^2)\to 0, \tag{2.43}$$

whereas the denominator goes as

$$\frac{\partial}{\partial e^2}\frac{1}{e_\lambda^2} = \frac{d}{de^2}\frac{1}{e_\infty^2} + \int\frac{dM^2}{M^2+\lambda^2}\frac{\partial\rho}{\partial e^2}(M^2, e^2)\to\frac{d}{de^2}\frac{1}{e_\infty^2}\,. \tag{2.44}$$

Therefore, among the following outcomes,

(1′) $\beta(e)$ is finite and nontrivial,
(2′) $\beta(e)$ is finite but identically zero,
(3′) $\lim\beta(1/\lambda, e)$ is nonexistent,

(1′) and (3′) are possible only if result (i) is true.

Finally, having seen that e_∞ is indeed likely to be independent of e,[46] we may inquire into its consequences. By taking the limit $e^2\uparrow e_\infty^2$ in (2.14), we obtain

$$\int\frac{dM^2}{M^2}\rho(M^2, e^2)\to 0 \quad (e^2\uparrow e_\infty^2), \tag{2.45}$$

which strongly suggests that

$$\rho(M^2, e^2)\to\rho(M^2, e_\infty^2) = 0 \quad (e^2\uparrow e_\infty^2) \tag{2.46}$$

since ρ is positive. Applying generalized unitarity to (2.46), we find that for the critical theory with $e = e_\infty$, the vertex functions for the process γ^* (virtual photon)→anything (on-shell) must vanish. (This may seem strange for $e_\infty = \infty$; however, recall that e is factored out from Π.) In particular, for $e = e_\infty < \infty$, there is no (virtual) photon-photon scattering[9,16] due to the Jost-Schroer-Federbush-Johnson (JSFJ) theorem.[47,48] Further application of crossing and analyticity[49] then yields the result that the theory is free.[50] (This conclusion for $e = e_\infty < \infty$, however, raises the paradoxical question as to what happens to Thomson scattering when $e\uparrow e_\infty$.)

III. THE STRUCTURE OF THE SPECTRAL FUNCTION

Let us now explore in more detail the connection between the RG functions and the spectral functions. (We assume the former to be existent and nontrivial.) We start with

$$\lambda^2(\partial/\partial\lambda^2)|_e e_\lambda^2 \cong \psi(e_\lambda^2), \tag{2.2}$$

$$\lambda(\partial/\partial\lambda)|_{e_\lambda} e \cong -\beta(e), \tag{2.40}$$

which integrates to

$$G(e_\lambda^2) \cong \phi(e^2)\lambda^2, \tag{3.1}$$

where

$$\ln\phi(e^2) \equiv \int\frac{de^2}{e\beta(e)}, \quad \ln G(e_\lambda^2) \equiv \int\frac{de_\lambda^2}{\psi(e_\lambda^2)}\,. \tag{3.2}$$

Since $\psi(e_\lambda^2) > 0$ for $0 < e_\lambda^2 < e_\infty^2$, $x = G(e_\lambda^2)$ is monotonic and may be inverted to give $e_\lambda^2 = F(x)$. In terms of F, (3.1) becomes

$$e_\lambda{}^2 = e^2 d_c(-\lambda^2, e^2) \cong F(\phi(e^2)\lambda^2), \qquad (3.3)$$

a well known result.[7,22]

We also have

$$[-\lambda(\partial/\partial\lambda) + \beta(e)(\partial/\partial e)]\Pi(-\lambda^2) \cong 0,$$
$$[-\lambda(\partial/\partial\lambda) + \beta(e)(\partial/\partial e)]\Pi_c(-\lambda^2) \cong 2\beta(e)/e^3, \qquad (3.4)$$

a result familiar from e^+-e^- annihilation.[51]

In perturbation theory[18]

$$\frac{\beta(e)}{e} = \frac{1}{3}\left(\frac{e^2}{4\pi^2}\right) + \frac{1}{4}\left(\frac{e^2}{4\pi^2}\right)^2 - \frac{121}{288}\left(\frac{e^2}{4\pi^2}\right)^3 + O(e^8). \qquad (3.5)$$

Therefore, to lowest order in e

$$\phi(e^2) \cong \exp(-12\pi^2/e^2 + \text{const}) \qquad (3.6)$$

and[22,26,52]

$$e^2 \cong \frac{12\pi^2}{\ln[\lambda^2/G(e_\lambda{}^2)] + \text{const}} \to 0 \quad (\lambda^2 \to \infty, \quad e_\lambda{}^2 \text{ fixed}). \qquad (3.7)$$

Given (3.3) and (3.6), it is not surprising that (2.12) and (2.15) should fail to hold in perturbation theory. We may also give a heuristic argument in favor of (2.12). It is well known that the lowest-order result for ρ [Eq. (2.36)] corresponds to the process where an e^+ and an e^- travel apart freely after they are created, leading to a divergent vacuum polarization. However, the Coulomb attraction should tend to suppress such polarization, particularly at short distances.[53]

In view of (3.3), it is natural to assume that

$$\rho(M^2, e^2) \cong r(\phi(e^2)M^2), \quad M^2 \gg 1, \qquad (3.8)$$

since ρ is the absorptive part of $(e^2 d_c)^{-1}$. Indeed, under (3.8), we may recover (3.3)

$$\frac{1}{e_\lambda{}^2} = \frac{1}{e_\infty{}^2} + \int \frac{dM^2}{M^2+1}\rho(\lambda^2 M^2, e^2)$$
$$\cong \frac{1}{e_\infty{}^2} + \int \frac{dM^2}{M^2+1} r(\phi(e^2)\lambda^2 M^2). \qquad (3.9)$$

(Note that $e_\infty{}^2$ is independent of e^2.)

A word, however, is in order. As mentioned in the Introduction, we cannot hope for

$$[-M(\partial/\partial M) + \beta(e)(\partial/\partial e)]\rho(M^2, e^2) \cong 0, \qquad (3.10)$$

since $\partial\rho/\partial M$ will have strong threshold singularities. Therefore, Eq. (3.8) is to be interpreted as in duality,[54] i.e., r describes the average behavior of ρ. In fact, if we introduce the quantities

$$\Delta\rho(M^2, e^2) \equiv \rho(M^2, e^2) - r(\phi(e^2)M^2), \qquad (3.11)$$

$$\Delta\beta(1/\lambda, e) \equiv \beta(1/\lambda, e) - \beta(e), \qquad (3.12)$$

we obtain the following relation between the falloff of the "bumps" $\Delta\rho$ and the "background" r,

$$-\lambda\frac{\partial}{\partial\lambda}\bigg|_{e_\lambda} \int \frac{dM^2}{M^2+1}\Delta\rho(\lambda^2 M^2, e^2)$$
$$= \frac{\Delta\beta}{\beta}\int \frac{dM^2}{(M^2+1)^2} r(\phi(e^2)\lambda^2 M^2). \qquad (3.13)$$

[The existence of such a correlation is somewhat reminicent of the situation observed in deep-inelastic scattering.[55] See also (3.44).]

To proceed further, we sharpen (3.1) and (3.8) into

$$\lim \rho(\lambda^2 M^2, e^2(\lambda^2, e_\lambda{}^2)) = \lim r(\phi[e^2(\lambda^2, e_\lambda{}^2)]\lambda^2 M^2)$$
$$= r(G(e_\lambda{}^2)M^2) \qquad (3.14)$$

for fixed $e_\lambda{}^2$ and $M^2 > 0$. In particular, since $G(0) = 0$ and $G(e_\infty{}^2) = \infty$,

$$r(0) = 1/12\pi^2, \quad r(\infty) = 0. \qquad (3.15)$$

Then, instead of (2.35), we have

$$\frac{\psi(e_\lambda{}^2)}{e_\lambda{}^4} = \int \frac{dM^2}{(M^2+1)^2} r(G(e_\lambda{}^2)M^2) \qquad (3.16)$$

and similarly as in (3.9)

$$\frac{1}{e_\lambda{}^2} - \frac{1}{e_\infty{}^2} = \int \frac{dM^2}{M^2 + G(e_\lambda{}^2)} r(M^2). \qquad (3.17)$$

These equations may also be recast as

$$\frac{1}{F(z)} - \frac{1}{F(\infty)} = \int \frac{dM^2}{M^2+z} r(M^2), \qquad (3.18)$$

$$\psi(F(z)) = zF'(z). \qquad (3.19)$$

Equation (3.16) shows explicitly that the region $e_\lambda{}^2 \downarrow 0$ corresponds to $M^2 \downarrow 0$ for $r(M^2)$, whereas $e_\lambda{}^2 \uparrow e_\infty{}^2$ corresponds to $M^2 \uparrow \infty$. In particular, it is not surprising that Landau and others[22,56] should have run into difficulties since their formula

$$e^2 \cong e_\lambda{}^2\left(1 + \frac{e_\lambda{}^2}{12\pi^2}\ln\lambda^2\right)^{-1} < \frac{12\pi^2}{\ln\lambda^2} \to 0 \qquad (3.20)$$

corresponds to taking the lowest order in the perturbation expansion of $\psi(e_\lambda{}^2)$ (Refs. 57 and 18)

$$\frac{\psi(e_\lambda{}^2)}{e_\lambda{}^2} = \frac{1}{3}\left(\frac{e_\lambda{}^2}{4\pi^2}\right) + \frac{1}{4}\left(\frac{e_\lambda{}^2}{4\pi^2}\right)^2$$
$$+ \left(\frac{\zeta(3)}{3} - \frac{101}{288}\right)\left(\frac{e_\lambda{}^2}{4\pi^2}\right)^3 + O(e_\lambda{}^8) \qquad (3.21)$$

and substituting it into (3.7).[58]

Although we have put $m = 1$ so far, it is easy to see that here we are dealing with the massless electron theory,[9,10] since for fixed $e_\lambda{}^2$, $m^2/\lambda^2 \to 0$ may be realized either by $\lambda \to \infty$ or $m \to 0$. (In the latter case, λ serves as the subtraction point as in Sec. II.) In fact, we may easily generalize (3.17) into

$$\frac{1}{e_\lambda^2 d(-s,0,e_\lambda^2)} = \frac{1}{e_\infty^2} + \int \frac{dM^2}{M^2+G(e_\lambda^2)s} r(M^2), \tag{3.22}$$

which identifies r as the spectral weight of the massless electron theory. In particular,[10] for $s=-k^2/\lambda^2 \to 0$

$$e_\lambda^2 d(-s,0,e_\lambda^2) \cong \frac{12\pi^2}{\ln[G(e_\lambda^2)s]+\text{const}}. \tag{3.23}$$

Also, by taking $\lambda \to 0$ in (3.22) keeping k^2 and e_λ^2 fixed, we find that on-shell normalization for d is possible in the massless electron theory if and only if $e_\infty^2 < \infty$. This result may also be obtained by taking $m \to 0$ in

$$e^2 d_c(k^2/m^2, e^2) \cong F[-\phi(e^2)k^2/m^2], \tag{3.24}$$

showing that the limits $\lambda \to 0$ and $m \to 0$ are commutative.

The fact that the photon propagator in the massless theory is essentially a function of $G(e_\lambda^2)k^2/\lambda^2$ only may also be expressed as a homogeneous RG equation characteristic of fully massless theories (perhaps we should call it the Coleman-Weinberg[59] equation),

$$[\lambda^2(\partial/\partial\lambda^2) + \psi(e_\lambda^2)(\partial/\partial e_\lambda^2)]\, e_\lambda^2 d(k^2/\lambda^2, 0, e_\lambda^2) = 0. \tag{3.25}$$

It is worth noting that (3.25) requires $\psi(e_\lambda^2)$ to be nonanalytic at the origin.[11] Let us assume the contrary. To be specific, we shall take the perturbation theory result (3.21). Then from (3.2) (Ref. 22)

$$G(\zeta) = \exp\left(-\frac{12\pi^2}{\zeta}\right) \times \zeta^{-9/4} \times \text{analytic function of } \zeta. \tag{3.26}$$

Now from (3.25)

$$e_\lambda^2 d(k^2/\lambda^2, 0, e_\lambda^2) = F(-G(e_\lambda^2)k^2/\lambda^2). \tag{3.27}$$

Since the left-hand side should be cut analytic in k^2 for $0 < e_\lambda^2 < e_\infty^2$, so must be the right-hand side, i.e., $F(z)$ is analytic in z except for a cut along the negative z axis. On the other hand, from the normalization conditions

$$d(-1,0,e_\lambda^2) = 1, \tag{3.28}$$

$$(\partial/\partial s)|_{s=1} e_\lambda^2 d(-s,0,e_\lambda^2) = \psi(e_\lambda^2), \tag{2.30}$$

and the principle of analytic continuation, we have

$$\zeta = F(z), \quad z = G(\zeta), \tag{3.29}$$

$$zF'(z) = \psi(\zeta). \tag{3.30}$$

However, Eq. (3.30) entails a difficulty, for let us take the limit $\zeta \to 0$. The right-hand side obviously goes to zero. But by (3.26), $G(\zeta)$ has an essential singularity at the origin, so z may be made to converge to any value z_0. This leads to the result that $z_0 F'(z_0) = 0$ for arbitrary z_0, i.e., $\psi(\zeta) \equiv 0$ again.

Evidently, the same reasoning applies to a zero of higher order for ψ, although it would be strange indeed if perturbation theory failed to give correct results for an *analytic* function. As for a first-order zero

$$\psi(\zeta) = a\zeta + O(\zeta^2), \tag{3.31}$$

$$G(\zeta) = \zeta^{1/a} \times \text{analytic function of } \zeta, \tag{3.32}$$

we may either rely on the argument above or we may use (3.16). (Take $\zeta = e_\lambda^2 \to 0$ to obtain a contradiction.) $\psi(0) \neq 0$ may be ruled out by (2.32) for one; therefore, $\psi(e_\lambda^2)$ must be nonanalytic at $e_\lambda^2 = 0$ as claimed.[60] [See *Note added in proof.*]

So far, we have concentrated on the relation of the Gell-Mann–Low function to the spectral function. Let us now proceed to that of the Callan-Symanzik function. Although (2.41) and (2.42) already give an implicit relation, a more explicit form may be obtained if we allow ourselves to assume that

$$\langle M^2 \rangle \equiv \int dM^2 \rho(M^2, e^2) < \infty. \tag{3.33}$$

Then the numerator and the denominator of (2.42) are, respectively,

$$-\frac{1}{\lambda^2}\int dM^2 \left(\frac{\lambda^2}{M^2+\lambda^2}\right)^2 \rho(M^2,e^2) \cong -\frac{\langle M^2\rangle}{\lambda^2}, \tag{3.34}$$

$$\frac{1}{\lambda^2}\int dM^2 \frac{\lambda^2}{M^2+\lambda^2}\frac{\partial\rho}{\partial e^2}(M^2,e^2) \cong \frac{1}{\lambda^2}\frac{d}{de^2}\langle M^2\rangle, \tag{3.35}$$

so

$$e\beta(e) = -\langle M^2\rangle de^2/d\langle M^2\rangle, \tag{3.36}$$

which we may write as[61]

$$\int dM^2 \rho(M^2,e^2) = 1/\phi(e^2). \tag{3.37}$$

Besides giving a direct relation between ρ and β, an appealing feature of (3.37) is that it allows us to rewrite (3.3) and (3.8) as

$$e^2 d_c(-\lambda^2, e^2) \cong F(\lambda^2/\langle M^2\rangle), \tag{3.38}$$

$$\rho(M^2, e^2) \cong r(M^2/\langle M^2\rangle), \tag{3.39}$$

i.e., asymptotically the scale is set by $\langle M^2\rangle$, which may be interpreted as the mass squared of the polarization current.

This is strikingly similar to the scaling laws[62] in critical phenomena,[2] particularly since in both cases the relevant length scales ξ and $\langle M^2\rangle^{-1/2}$ diverge as we approach the critical point ($e = e_\infty$ for the latter).[63] There is one difference; in critical phenomena, the length scale is given essentially by

$$\xi^{-2}=\Gamma^{(2)}(0)\,, \tag{3.40}$$

but in our case

$$\langle M^2\rangle = e^{-2}\Gamma^{(2)}(\infty) \tag{3.41}$$

if $e_\infty=\infty$. This is, however, to be expected; critical phenomena is related to infrared behavior, whereas we are dealing with ultraviolet behavior.

We also note that with (3.33)

$$\frac{1}{e_\lambda{}^2}-\frac{1}{e_\infty{}^2}\cong\frac{\langle M^2\rangle}{\lambda^2}=\frac{1}{\phi(e^2)\lambda^2}\cong\frac{1}{G(e_\lambda{}^2)}\,. \tag{3.42}$$

Comparison with (3.17) gives the normalization[61]

$$\int dM^2\gamma(M^2)=1\,, \tag{3.43}$$

which in turn leads to a superconvergence relation[64]

$$\int dM^2\Delta\rho(M^2,e^2)=0\,. \tag{3.44}$$

We may view (3.44) as an expression of global duality.[54] This leads us to expect that (3.44) may be valid even if $\langle M^2\rangle=\infty$.

Further information may be obtained from the Callan-Symanzik equation for the j-fold mass insertion term[65,8]

$$[m(\partial/\partial m)+\beta(e)(\partial/\partial e-2/e)-j(1+\gamma_s(e))]\Delta_s^j\Gamma^{(2)}(k^2)=\Delta_s^{j+1}\Gamma^{(2)}(k^2)\,. \tag{3.45}$$

Introducing the notation

$$\Delta_s^j e_\lambda{}^{-2}\equiv\Delta_s^j\Gamma^{(2)}(-\lambda^2)/e^2\lambda^2\,, \tag{3.46}$$

$$\mathfrak{D}\equiv-\lambda(\partial/\partial\lambda)+\beta(e)(\partial/\partial e)\,, \tag{3.47}$$

$$J(e^2)\equiv\int de\,\frac{1+\gamma_s(e)}{\beta(e)}\,, \tag{3.48}$$

we may rewrite (3.45) as[66]

$$\Delta_s^j e_\lambda{}^{-2}=e^{-jJ(e^2)}(e^{jJ(e^2)}\mathfrak{D})^j e_\lambda{}^{-2}\,. \tag{3.49}$$

In particular,

$$\Delta_s e_\lambda{}^{-2}=\mathfrak{D}e_\lambda{}^{-2}=-\Delta\beta\frac{\partial}{\partial e}e_\lambda{}^{-2}=\frac{2\Delta\beta(1/\lambda,e)}{\beta(1/\lambda,e)}\,\frac{\psi(1/\lambda^2,e_\lambda{}^2)}{e_\lambda{}^4}\,, \tag{3.50}$$

$$\frac{\Delta_s{}^2e_\lambda{}^{-2}}{\Delta_s e_\lambda{}^{-2}}=-(1+\gamma_s(e))+\mathfrak{D}\ln\frac{\Delta\beta}{\beta(e)}-\Delta\beta\frac{\partial}{\partial e}\ln\Delta\beta\frac{\partial}{\partial e}e_\lambda{}^{-2}\,. \tag{3.51}$$

To any order in perturbation theory[6]

$$\lim(\Delta_s^{j+1}e_\lambda{}^{-2}/\Delta_s^j e_\lambda{}^{-2})=0\quad(e\text{ fixed},\ j=0,1)\,. \tag{3.52}$$

We have seen that for $j=0$, Eq. (3.52) must be true for the exact sum if we are to obtain a nontrivial β. Therefore, let us *assume* that (3.52) is also true globally for $j=1$. Then it follows that

$$1+\gamma_s(e)=\lim\mathfrak{D}\ln[\Delta\beta(1/\lambda,e)/\beta(e)]\,. \tag{3.53}$$

Experience with $\langle M^2\rangle<\infty$ suggests that we inquire whether

$$\langle M^4\rangle\equiv\int dM^2M^2\rho(M^2,e^2)<\infty \tag{3.54}$$

may be true. If (3.54) does hold, so must (3.33); then a little calculation yields

$$1+\gamma_s(e)=\lim\mathfrak{D}\ln\frac{\phi(e^2)\lambda^2\Delta\beta}{\beta(e)}=\beta(e)\frac{d}{de}\left(\ln\beta(e)\frac{d}{de}\frac{\langle M^4\rangle}{\langle M^2\rangle^2}\right). \tag{3.55}$$

Without loss of generality, we may set

$$\frac{\langle M^4\rangle}{\langle M^2\rangle^2}=\pm\int\frac{de}{\beta(e)}e^{J(e^2)}\,. \tag{3.56}$$

The positivity of $\beta(e)$ and $\langle M^n\rangle$ for $0<e<e_\infty$ fixes the limits of integration to be either

$$\frac{\langle M^4\rangle}{\langle M^2\rangle^2}=\int_e^{e_\infty}\frac{d\eta}{\beta(\eta)}e^{J(\eta^2)} \tag{3.57}$$

or

$$\frac{\langle M^4\rangle}{\langle M^2\rangle^2}=\int_0^{e}\frac{d\eta}{\beta(\eta)}e^{J(\eta^2)}\,. \tag{3.58}$$

On the other hand, the Schwarz inequality gives

$$\langle M^4\rangle/\langle M^2\rangle^2\geq e^2/(1-Z_3)\geq e^2\,. \tag{3.59}$$

Therefore, we must have either

$$\int_e^{e_\infty}\frac{d\eta}{\beta(\eta)}e^{J(\eta^2)}\geq e^2 \tag{3.60}$$

or

$$\int_0^{e}\frac{d\eta}{\beta(\eta)}e^{J(\eta^2)}\geq e^2\,. \tag{3.61}$$

However, Eq. (3.60) is impossible since there is a contradiction as $e\uparrow e_\infty$. Equation (3.61) also has a serious difficulty: Perturbation theory gives

$$\gamma_s(e)=\frac{3e^2}{8\pi^2}+O(e^4)\,,\qquad J(e^2)=-\frac{6\pi^2}{e^2}+\tfrac{9}{8}\ln e^2+\cdots\,, \tag{3.62}$$

implying that the left-hand side vanishes faster than $\exp(-6\pi^2/e^2)$ as $e^2\to0$. Therefore, we conclude that $\langle M^4\rangle$ is unlikely to be finite, i.e., even if the average mass squared of the current is finite, its fluctuation is not.[67]

So far, we have not touched on the question of the finiteness of $e_\infty{}^2$ or Z_3^{-1}.[68] Since Z_3 may be interpreted as $|\langle\text{bare photon}\,|\,\text{physical photon}\rangle|^2$,[69,70] it may be said that it would be surprising if Z_3 were actually to decrease as the coupling is turned off, as is the case for $e_\infty^2<\infty$. However, in view of our meager knowledge concerning the true situation in quantum field theory,[71] surprises are not excluded[72] and we shall consider both $e_\infty^2<\infty$ and $e_\infty^2=\infty$.

(1″) $e_\infty^2<\infty$. In this case we have

$$\frac{d_c(k^2,e^2)}{k^2}=\frac{1}{k^2}+\int\frac{dM^2}{k^2-M^2}\sigma(M^2,e^2)\,, \tag{3.63}$$

$$Z_3^{-1}=1+\int dM^2\sigma(M^2,e^2)<\infty\,, \tag{3.64}$$

$$e^2D_{Fc}(x)_{\mu\nu}\cong\frac{e_\infty^2}{8\pi^2}\frac{g_{\mu\nu}x^2+2x_\mu x_\nu}{(x^2-i0)^2}+\text{gauge terms}\,, \tag{3.65}$$

where

$$\sigma(M^2,e^2)\equiv e^2|d_c(M^2,e^2)|^2M^{-2}\rho(M^2,e^2)\,. \tag{3.66}$$

If $\langle M^2\rangle<\infty$, then we also have

$$\mu^2\equiv Z_3\int dM^2M^2\sigma(M^2,e^2)=\frac{e_\infty^2}{\phi(e^2)}<\infty \tag{3.67}$$

since[73]

$$\langle M^2\rangle=\lim\lambda^2\left(\frac{1}{e^2d_c(-\lambda^2,e^2)}-\frac{1}{e_\infty^2}\right)=\frac{\mu^2}{e_\infty^2}\,. \tag{3.68}$$

Furthermore, (3.42) gives $\psi'(e_\infty^2)=-1$, saturating the bound (2.32). This is in contradiction with the result[16,9,76] that ψ has an infinite order zero at e_∞^2. Unfortunately, it is not clear whether this should be taken as a sign that e_∞^2 and $\langle M^2\rangle$ cannot be both finite, or that the expansion scheme employed in Refs. 9 and 16 is not valid.[60]

We may also rewrite the results in terms of the electromagnetic current using the relations

$$\int d^4x\,e^{ikx}\langle 0|T^*j_\mu(x)j_\nu(0)|0\rangle=i(-g_{\mu\nu}k^2+k_\mu k_\nu)\int dM^2\frac{k^2}{k^2-M^2}\sigma(M^2,e^2)\,, \tag{3.69}$$

$$\int d^4x\,e^{ikx}\langle 0|[j_\mu(x),j_\nu(0)]|0\rangle=i(-g_{\mu\nu}k^2+k_\mu k_\nu)k^2\sigma(k^2,e^2)2\pi\epsilon(k_0)\theta(k^2)\,. \tag{3.70}$$

In particular, (3.67) implies the Schwinger-Wilson operator-product expansion[74] for the commutator

$$[j_\mu(x),j_\nu(0)]=\frac{e_\infty^4}{e^2\phi(e^2)}\frac{i}{2\pi}\partial_\mu\partial_\nu\epsilon(x_0)\delta(x^2)+\cdots\,, \tag{3.71}$$

since the Schwinger term[12] is known to be a c number.[75]

(2″) $e_\infty^2=\infty$. In this case we have the sum rule[77]

$$\int\frac{dM^2}{M^2}e^2\rho(M^2,e^2)=1\,. \tag{3.72}$$

If $\langle M^2\rangle<\infty$, we further obtain the asymptotic behavior

$$\psi(e_\lambda^2)\cong G(e_\lambda^2)\cong e_\lambda^2\,, \tag{3.73}$$

$$e^2d_c(k^2,e^2)\cong-\phi(e^2)k^2\,, \tag{3.74}$$

$$e^2D_{Fc}(x)_{\mu\nu}\cong\phi(e^2)(-g_{\mu\nu}\Box+\partial_\mu\partial_\nu)\times\frac{1}{4\pi^2}\frac{1}{x^2-i0}+\text{gauge terms}\,. \tag{3.75}$$

Also, the spectral representation for the propagators involve an extra subtraction for $\langle M^2\rangle<\infty$:

$$\frac{d_c(k^2,e^2)}{k^2}=\frac{1}{k^2}-\frac{\phi(e^2)}{e^2}+\int\frac{dM^2}{k^2-M^2}\sigma(M^2,e^2)\,, \tag{3.76}$$

$$\int d^4x\,e^{ikx}\langle 0|T^*j_\mu(x)j_\nu(0)|0\rangle=i(-g_{\mu\nu}k^2+k_\mu k_\nu)\left(-\frac{\phi(e^2)}{e^2}k^2+\int dM^2\frac{k^2}{k^2-M^2}\sigma(M^2,e^2)\right)\,. \tag{3.77}$$

This would be a difficulty, if the covariant T^* product were simply the T product. However, it is well known that the T product is not covariant[78] in the presence of a Schwinger term, so it is quite possible that extra terms arise from the seagulls necessary for covariance.

On the other hand, since seagulls do not contribute to the absorptive part, Eqs. (3.66) and (3.70) continue to hold for $e_\infty^2=\infty$. However, this time the short-distance singularity is stronger than in (3.71) for both $\langle M^2\rangle<\infty$ and $\langle M^2\rangle=\infty$, since

$$\int dM^2\sigma(M^2,e^2)=\infty\,. \tag{3.78}$$

[For $\langle M^2\rangle=\infty$, Eq. (3.78) is evident since (3.63) continues to hold but (3.64) is now divergent; for $\langle M^2\rangle<\infty$, the result follows from (3.66) and (3.74) and $\langle M^4\rangle=\infty$.] To summarize, given (3.52), the Schwinger term is finite if and only if e_∞^2 and $\langle M^2\rangle$ are both finite.

Finally, we mention that although we have modified the number of subtractions in (3.76), it is not possible to modify our starting point (2.10) into

$$\Pi_c(k^2)=\sum_{i=0}^{n}a_i(-k^2)^i+\int\frac{dM^2}{M^2}\frac{k^2}{k^2-M^2}\rho(M^2,e^2)\,, \tag{3.79}$$

since $a_0\neq 0$ would violate (2.11), whereas $a_n\neq 0$ for

$n \geq 1$ would lead to $e_\lambda{}^2 \to 0$ as $\lambda \to \infty$. (For $a_n < 0$, there are spacelike poles as well.)

IV. DISCUSSIONS

We hope that our discussion of the RG based on the spectral representation has been instructive, even though much of the material in Sec. II was not essentially new. In particular, we hope we have clarified the reason why such a seemingly trivial group of transformations [Eq. (2.23)] should lead to such a nontrivial result as a fixed bare charge independent of the physical charge. The crucial aspect was how the group was realized, particularly its generators.

On the issue of the physical charge and the bare charge, there has been a proposal[16] that the physical charge rather than the bare charge should be the zero of ψ, assuming that such a zero exists. This possibility is attractive in a certain sense, since, in principle, it allows a theoretical determination of the fine structure constant. Unfortunately, we have seen that this solution is incompatible with the spectral representation and the customary assumptions concerning renormalization theory.

There has also been a suggestion[38] that e_∞ may be, in fact, a nontrivial function of e, as in superrenormalizable theories.[71] However, since this requires $\beta(e) \equiv 0$ globally, we would be forced to reject the perturbation series for β as a red herring. But then, why not the series for the anomalous magnetic moment?

Sometimes the results (i) and (ii) have been opposed on the ground that bare perturbation theory would be meaningless if the bare charge were indeed fixed. This argument we may counter with several replies. One is that in practice the expansion parameter of bare perturbation theory is the cutoff bare charge e_b, not e_∞. Another (deeper) one is that all the renormalized theories with $0 < e < e_\infty$ so solve the same bare Hamiltonian.[79]

However, perhaps the most compelling argument in favor of (i) and (ii) is that it leads to the picture of high-energy QED outlined in Sec. III: scaling, duality, similarity with critical phenomena, and the relevance of zero electron mass and "bare photon mass" μ.[80] Admittedly, much was conjectural. Also, from a logical point of view, the situation is always precarious: One abstracts from perturbation theory only to deny it. However, in our opinion, the simplicity and the consistency of the emergent picture provides ample justification. We hope the reader will agree.

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APPENDIX A

In the text, we have freely interchanged the order of limiting procedures. However, it is known that such procedures are often dangerous when spectral weights are involved,[3,16] a canonical example being the Lehmann-Symanzik-Zimmermann (LSZ) conditions[81]

$$\langle m | \varphi_f(t) | n\rangle \to Z^{1/2}\langle m | \varphi_f^{\rm in}(t) | n\rangle , \quad t \to -\infty \tag{A1}$$

but

$$\langle 0 | [\varphi_f(t), \varphi_g(t)] | 0\rangle = \langle 0 | [\varphi_f^{\rm in}(t), \varphi_g^{\rm in}(t)] | 0\rangle , \tag{A2}$$

i.e.,

$$\lim_{t\to-\infty} \sum_n [\langle 0 | \varphi_f(t) | n\rangle\langle n | \varphi_g(t) | 0\rangle - (f \leftrightarrow g)] \neq \sum_n \lim_{t\to-\infty} [\langle 0 | \varphi_f(t) | n\rangle\langle n | \varphi_g(t) | 0\rangle - (f \leftrightarrow g)] . \tag{A3}$$

Therefore, we shall give a more rigorous treatment below.[82] Fortunately, it turns out that the conditions required for such a justification are quite mild, although, of course, whether the conditions are actually satisfied or not is a question which cannot be answered at the present moment.

The positivity of ρ and the monotonic convergence theorem ensure (2.12) and (2.43) without any additional assumptions. For (2.35) to hold, it is sufficient that the convergence in (2.15) is locally uniform for $e^2 > 0$. Since by hypothesis $e_*{}^2 > 0$, the integrand is bounded by const/$(M^2+1)^2$, and Lebesgue's theorem is applicable.

More stringent conditions are required for (2.44). A sufficient one is that $\partial\rho/\partial e^2$ be continuous in M^2 and

$$\int \frac{dM^2}{M^2+\lambda^2} \left| \frac{\partial \rho}{\partial e^2}(M^2, e^2)\right| < \infty , \tag{A4}$$

the convergence being locally uniform for $e^2 > 0$.

As for (2.46), a weaker version

$$\liminf_{e^2 \uparrow e_\infty{}^2} \rho(M^2, e^2) = 0 \tag{A5}$$

follows rigorously from Fatou's lemma.

We may continue in a similar fashion for Sec. III. However, in view of its more heuristic nature, we shall leave the details to the interested reader.

APPENDIX B

Our analysis of the RG in the text was facilitated by the Ward identity,[83] which allowed us to work only with the photon propagator. For theories other than QED, a straightforward generalization of our analysis would require the analyticity property of vertex functions, which is considerably involved.[84] Therefore, we shall not pursue it further here. Instead, we shall show how it is still possible in other theories to use the RG with only propagators at disposal.

As mentioned in the Introduction, the example given will be the Goldberger-Treiman (GT) relation[13] in ps-ps theory. The Lagrangian is

$$\mathcal{L} = \bar{N}(i\gamma_\mu \partial^\mu - m)N - \tfrac{1}{2}\pi^a(\Box + \mu^2)\pi^a - ig\bar{N}\gamma_5\tau^a N\pi^a - \frac{h}{4!}(\pi^a\pi^a)^2 . \tag{B1}$$

We define the axial-vector current A^a_μ and the off-shell pion decay constant $f_\pi(k^2)$ by

$$A^a_\mu = \bar{N}\gamma_\mu\gamma_5(\tau^a/2)N , \tag{B2}$$

$$\int d^4x\, e^{ikx}(\Box + \mu^2)\langle 0 | T^* A^a_\mu(x)\pi^b(0) | 0\rangle = \delta^{ab}k_\mu f_\pi(k^2) . \tag{B3}$$

We may also introduce the proper $A^a_\mu - \pi^b$ part $-ik_\mu \delta^{ab}\Pi^{(11)}_c(k^2)$ and the pion proper self-energy $\delta^{ab}\Pi^{(02)}_c(k^2)$ so that

$$f_\pi(k^2) = \Pi^{(11)}_c(k^2)/[1 + \Pi^{(02)}_c(k^2)] . \tag{B4}$$

In perturbation theory, both satisfy once-subtracted dispersion relations

$$\Pi^{(02)}_c(k^2) = \int \frac{dM^2}{M^2 - \mu^2}\frac{k^2 - \mu^2}{M^2 - k^2}\rho^{(02)}(M^2, g, h) , \tag{B5}$$

$$\Pi^{(11)}_c(k^2) = \int \frac{dM^2}{M^2 + \lambda^2}\frac{k^2 + \lambda^2}{M^2 - k^2}\rho^{(11)}(M^2, g, h) , \tag{B6}$$

where the (arbitrary) subtraction point $-\lambda^2$ reflects the mixing of $\bar{N}\gamma_\mu\gamma_5(\tau^a/2)N$ and $\partial_\mu\pi^a$ under renormalization.

However, we know that $\Pi^{(02)}_c(k^2)$ must, in fact, obey an unsubtracted relation; let us assume that this is also true for $\Pi^{(11)}_c(k^2)$,

$$\Pi^{(11)}_c(k^2) = \Pi^{(11)}(k^2) - \Pi^{(11)}(-\lambda^2) , \tag{B7}$$

$$\Pi^{(11)}(k^2) = \int \frac{dM^2}{M^2 - k^2}\rho^{(11)}(M^2, g, h) . \tag{B8}$$

Then we may unambiguously define the on-shell decay constant f_π by

$$f_\pi = \Pi^{(11)}(\mu^2) . \tag{B9}$$

As before, we may view λ^2 as a cutoff; this leads us to the RG equation[51]

$$\lim[-\lambda(\partial/\partial\lambda) + \beta_g(\partial/\partial g) + \beta_h(\partial/\partial h) + \gamma]\Pi^{(11)}_c(\mu^2) = C , \tag{B10}$$

where to lowest order in perturbation theory

$$\beta_g = 9g^3/16\pi^2 , \quad \gamma = g^2/4\pi^2 , \quad C = -mg/2\pi^2 . \tag{B11}$$

If (B7) and (B8) are true, then (B10) reduces to

$$[\beta_g(\partial/\partial g) + \beta_h(\partial/\partial h) + \gamma]f_\pi = C \tag{B12}$$

or up to a solution of the homogeneous equation

$$f_\pi \cong 8m/5g , \tag{B13}$$

which is precisely the GT relation.

Again, given (B13), it is not surprising that (B8) should fail to hold in perturbation theory.[85] What is surprising[72] is that the derivation, although incomplete,[86] does not make any use of the smallness of the pion mass.[87] In a sense, we have returned to the original derivation of the GT relation, rather than to its successors based on PCAC.

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[59]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973). See also H. Yamagishi, *ibid.* 23, 1880 (1981).

[60]The nonanalyticity of ψ casts some doubt on the validity of the expansion scheme employed in Refs. 9 and 16, although many of the conclusions are in agreement with ours.

[61]Equations (3.33) and (3.1) fix the arbitrary constants in (3.2).

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[77]The equation should be considered as a dynamical constraint on the functional form of ρ, rather than a bootstrap condition on e, since this is what should replace the equal-time commutation relation of the canonical formalism.

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[79]K. G. Wilson, Phys. Rev. D 2, 1818 (1970).

[80]None of these ideas concerning high-energy behavior are new. However, their close connection seems to have gone unnoticed so far. This rather surprising, since all the relevant quantities e_∞^2, μ^2, F, and $\phi(e^2)$ already appear in Ref. 7.

[81]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 1425 (1955); K. Hepp, in *Particle Symmetries and Axiomatic Field Theory*, 1965, Brandeis Summer Institute in Theoretical Physics, edited by M. Chrétien and S. Deser (Gordon and Breach, New York, 1966), Vol. I.

[82]The relevant theorems may be found, for example, in E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. (Oxford University Press, Oxford, England, 1939).

[83]F. J. Dyson, Phys. Rev. 75, 1736 (1949); J. C. Ward, *ibid.* 78, 182 (1950).

[84]For example, G. Källen and A. S. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. 1, No. 6 (1958). See, however, K. Nishijima, Phys. Rev. 124, 255 (1961).

[85]The situation is quite similar to that encountered in the ξ-limiting procedure: T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962).

[86]It is not clear what to do with the solution of the homogeneous equation.

[87]For a different application of the RG to pion physics, see S. Weinberg, Physica (Utrecht) 96A, 327 (1979).