Effect of renormalization on the large-order behavior of weak- and strong-coupling perturbation theory

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We investigate the large-order behavior of weak- and strong-coupling perturbation series for a $g\phi^4$ theory in zero space-time dimensions. Both the unrenormalized and renormalized weak-coupling expansions are divergent but Borel summable. The unrenormalized strong-coupling expansion is convergent, but its renormalized form is divergent and not even Borel summable.

I. INTRODUCTION

Recently there has been much interest in strong-coupling expansions and their renormalization. There is a profound difference between the renormalization of weak- and strong-coupling expansions. Weak-coupling expansions can be renormalized order by order; the structure of the resulting renormalized series is the same as that of the unrenormalized series. By contrast, strong-coupling expansions cannot be renormalized order by order and the form of the renormalized series differs significantly from that of the unrenormalized series.

In this paper we illustrate these differences using a physically trivial but mathematically interesting model; namely, a $g\phi^4$ field theory in zero space-time dimensions. We examine four different expansions for the four-point Green's function: (i) the weak-coupling expansion in terms of the unrenormalized mass, (ii) the strong-coupling expansion in terms of the unrenormalized mass, (iii) the weak-coupling expansion in terms of the renormalized mass, and (iv) the strong-coupling expansion in terms of the renormalized mass.

We find the explicit large-order behavior of the coefficients in the series (i) and (iii). These are both divergent Borel-summable series with very similar large-order behaviors. On the other hand, we show that series (ii) has a nonzero radius of convergence. Nevertheless, by finding the precise large-order behavior we determine that the series in (iv) is a divergent series which is not even Borel summable.

The differences between the effects of renormalizing weak- and strong-coupling perturbation series arise from the behavior of the unrenormalized mass for fixed renormalized mass. In the weak-coupling theory, as the coupling constant g approaches zero the unrenormalized mass approaches the renormalized mass. For this reason, renormalization can be performed order by order in the perturbation series. In the strong-coupling case the square of the unrenormalized mass tends to $-\infty$ as $g\to+\infty$ for fixed renormalized mass. It is this singular behavior which causes expansions in the unrenormalized mass to change their character completely when reexpressed in terms of the renormalized mass.

II. THE MODEL

The Green's functions for a $g\phi^4$ quantum field theory are expressed as functional derivatives of the generating functional $Z[J]$, which can be represented as a functional integral in d-dimensional Euclidean space-time:

$$Z[J]=\int \mathscr{D}\phi \exp\left[-\int d^dx(\partial_\mu\phi\partial_\mu\phi/2+m^2\phi^2/2+g\phi^4/4-J\phi)\right] . \tag{2.1}$$

m and g denote the unrenormalized mass and coupling constant. The connected $2n$-point Green's functions W_{2n} are defined by

$$W_{2n}(x_1,x_2,\ldots,x_{2n})\equiv\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{2n})}\ln Z[J]\Big|_{J=0}\ . \tag{2.2}$$

In zero space-time dimensions the functional integral in (2.1) reduces to an ordinary integral and the Green's functions are moments:

$$Z[J]=\int_{-\infty}^{\infty}dx\, e^{-m^2x^2/2-gx^4/4+Jx}\ , \tag{2.3}$$

$$W_2=\int_{-\infty}^{\infty}x^2dx\, e^{-m^2x^2/2-gx^4/4}\Big/Z[0]\ , \tag{2.4}$$

$$W_4=\int_{-\infty}^{\infty}x^4dx\, e^{-m^2x^2/2-gx^4/4}\Big/Z[0]-3W_2{}^2\ , \tag{2.5}$$

and so on.

Ordinarily, the renormalized mass is defined as the pole of the two-point function. However, when the space-time dimension $d=0$ we use the simple definition

$$\frac{1}{M^2}\equiv W_2\ , \tag{2.6}$$

where we have absorbed the wave-function renormalization constant Z into M^2.

The renormalized coupling constant G is defined as the negative of the amputated connected four-point Green's function:

$$G\equiv-W_2{}^{-4}W_4\ . \tag{2.7}$$

In this paper we study perturbation expansions of the *dimensionless* coupling constant $\widetilde{G}$ defined by

$$\widetilde{G}\equiv GM^{-4}\ . \tag{2.8}$$

The analysis of this paper consists of expanding the integral representation for $\widetilde{G}$,

$$\widetilde{G}=3-\frac{\int_{-\infty}^{\infty}x^4dx\,\exp(-m^2x^2/2-gx^4/4)\int_{-\infty}^{\infty}dx\,\exp(-m^2x^2/2-gx^4/4)}{\left[\int_{-\infty}^{\infty}x^2dx\,\exp(-m^2x^2/2-gx^4/4)\right]^2} \tag{2.9}$$

in four different perturbation series. The unrenormalized weak-coupling perturbation series has the form

$$\widetilde{G}\sim\sum_{n=1}^{\infty}a_n(g/m^4)^n \tag{2.10}$$

and the unrenormalized strong-coupling perturbation series has the form

$$\widetilde{G}=\sum_{n=0}^{\infty}b_n(m^2/\sqrt{g}\,)^n\ . \tag{2.11}$$

The renormalized perturbation series for $\widetilde{G}$ are found by using

$$\frac{1}{M^2}=\frac{\int_{-\infty}^{\infty}x^2dx\,\exp(-m^2x^2/2-gx^4/4)}{\int_{-\infty}^{\infty}dx\,\exp(-m^2x^2/2-gx^4/4)} \tag{2.12}$$

to replace m by M. The renormalized weak-coupling perturbation series now has the form

$$\widetilde{G}\sim\sum_{n=1}^{\infty}c_n(g/M^4)^n \tag{2.13}$$

and the renormalized strong-coupling perturbation series has the form

$$\widetilde{G}\sim\sum_{n=0}^{\infty}d_n(M^4/g)^n\ . \tag{2.14}$$

We find that in large order

$$a_n\sim\frac{(-1)^{n+1}}{\pi\sqrt{2}}4^{n+2}(n+1)!\ \ (n\to\infty)\ , \tag{2.15}$$

$$c_n\sim\frac{(-1)^{n+1}}{\pi\sqrt{2}}4^{n+2}(n+1)!e^{-3/2}\ \ (n\to\infty)\ , \tag{2.16}$$

$$d_n\sim\frac{-4^n}{\pi\sqrt{2e}}(n-1)!\left[1-\frac{11}{16n}-\frac{743}{512n^2}+O\left[\frac{1}{n^3}\right]\right]\ \ (n\to\infty)\ . \tag{2.17}$$

The coefficients $b_n\to 0$ fast enough so that the series (2.11) converges for small enough $m^2/\sqrt{g}$. These results are established in the following sections.

III. UNRENORMALIZED WEAK-COUPLING PERTURBATION SERIES

The scaling transformation $y=mx$ transforms the expression for $\tilde{G}$ in (2.9) into

$$\tilde{G}=3-I_0(\epsilon)I_4(\epsilon)I_2(\epsilon)^{-2}, \tag{3.1}$$

where

$$I_{2n}=\int_0^\infty y^{2n}dy\exp(-y^2/2-\epsilon y^4/4) \tag{3.2}$$

and $\epsilon=g/m^4$. For each integral I_n we obtain the weak-coupling series in powers of ϵ by expanding $\exp(-\epsilon y^4/4)$ as a series in powers of ϵ and integrating term by term:

$$I_{2n}\sim 2^{n-1/2}\sum_{k=0}^{\infty}(-\epsilon)^k\Gamma(2k+n+\tfrac{1}{2})/k!\,. \tag{3.3}$$

Note that for large k the terms in the series (3.3) grow roughly like $k!$ For such series it is easy to determine the large-order behavior of products and ratios. For example, if a_k and b_k grow roughly like $k!$, then if

$$\sum_0^\infty a_k\epsilon^k\sum_0^\infty b_k\epsilon^k=\sum_0^\infty c_k\epsilon^k,$$

we have

$$c_k\sim a_0b_k+b_0a_k\quad(k\to\infty)\,. \tag{3.4}$$

Also, if

$$\frac{1}{\sum_0^\infty a_k\epsilon^k}=\sum_0^\infty c_k\epsilon^k,$$

we have

$$c_k=-a_0^{-2}a_k\,. \tag{3.5}$$

Using (3.3) in combination with the rules in (3.4) and (3.5) gives the large-order behavior for the weak-coupling expansion for $\tilde{G}$:

$$\tilde{G}=\sum_{n=1}^{\infty}a_n\epsilon^n, \tag{3.6}$$

$$a_n\sim\frac{(-1)^{n+1}}{\pi\sqrt{2}}4^{n+2}(n+1)!\quad(n\to\infty)\,. \tag{3.7}$$

In Table I we list the first 15 coefficients a_n in the expansion (3.6). The growth of these coefficients agrees well with the predicted asymptotic behavior in (3.7).

IV. UNRENORMALIZED STRONG-COUPLING PERTURBATION SERIES

The scaling transformation $y=g^{1/4}x$ transforms $\tilde{G}$ in (2.9) into a function of the strong-coupling

TABLE I. The first 15 coefficients a_n in the renormalized weak-coupling perturbation expansion for $\tilde{G}$ in (2.10) compared with the leading asymptotic approximation to a_n in (2.15).

Order n	Exact a_n	Asymptotic approximation to a_n in (2.15)
1	6	28.8
2	−90	−346
3	1800	553×10
4	−43 470	-111×10^3
5	1 215 000	267×10^4
6	−38 402 100	-743×10^5
7	1 352 473 200	238×10^7
8	−52 526 697 750	-856×10^8
9	2 231 765 314 200	343×10^{10}
10	−103 065 713 455 500	-151×10^{12}
11	5 144 893 088 706 600	724×10^{13}
12	−276 267 644 959 567 500	-376×10^{15}
13	15 888 974 353 860 150 000	211×10^{17}
14	−974 940 184 679 407 137 000	-126×10^{19}
15	63 597 030 590 709 945 900 000	809×10^{20}

expansion variable $m^2/\sqrt{g}$. Thus, the strong-coupling expansion has the form

$$G=\sum_{n=0}^{\infty} b_n\left[\frac{m^2}{\sqrt{g}}\right]^n . \tag{4.1}$$

This series has a finite radius of convergence because each integral in (2.9) is an entire function of $m^2/\sqrt{g}$. The radius of convergence of (4.1) is determined by the location of the zero of

$$\int_{-\infty}^{\infty} x^2 dx \exp(-zx^2/2-x^4/4)$$

nearest the origin in the complex z plane. The radius of convergence is about 5.2.

V. MASS-RENORMALIZED WEAK-COUPLING PERTURBATION SERIES

To derive the mass-renormalized weak-coupling perturbation series in (2.13), we recall that we have already found that

$$\widetilde{G}=\sum_{n=1}^{\infty} a_n\epsilon^n \tag{3.6}$$

and

$$a_n\sim\frac{(-1)^{n+2}4^{n+2}(n+1)!}{\pi\sqrt{2}}\quad (n\to\infty)\ , \tag{3.7}$$

where $\epsilon=g/m^4$.

The renormalized mass is defined by (2.12) as

$$\frac{1}{M^2}=\frac{1}{m^2}\frac{I_2(\epsilon)}{I_0(\epsilon)}\ , \tag{5.1}$$

where $I_{2n}(\epsilon)$ is defined in (3.2). Squaring both sides of (5.1) and multiplying by g gives

$$\frac{g}{M^4}=\epsilon\left[\frac{I_2(\epsilon)}{I_0(\epsilon)}\right]^2 \tag{5.2}$$

and the desired renormalized perturbation expansion

$$\widetilde{G}=\sum_{n=1}^{\infty} c_n\left[\frac{g}{M^4}\right]^n \tag{5.3}$$

comes from reverting the series in (5.2) and inserting into (3.5). From the techniques used in Sec. III, it is easy to establish from (5.2) that

$$\frac{g}{M^4}=\sum_{n=1}^{\infty}\epsilon^n\alpha_n\ , \tag{5.4}$$

where

$$\alpha_n=(-1)^{n-1}\frac{4^{n+1}n!}{\pi\sqrt{2}}\quad (n\to\infty)\ ,$$
$$\alpha_1=1\ ,\quad \alpha_2=-6\ . \tag{5.5}$$

To solve for c_n in (5.3) we conjecture that

$$c_n\sim K(-4)^n(n+1)! \tag{5.6}$$

and write

$$\widetilde{G}=\sum_{n=1}^{\infty} a_n\epsilon^n$$
$$=\sum_{n=1}^{\infty} c_n\epsilon^n(1+\alpha_2\epsilon+\alpha_3\epsilon^2+\cdots)^n\ . \tag{5.7}$$

Expanding the nth power in (5.7), and recalling that a_n and c_n both dominate α_n by a factor of n for large n one easily derives

$$a_n=\sum_{p=0} c_{n-p}(\alpha_2)^p\begin{bmatrix} n-p\\ p\end{bmatrix}\ . \tag{5.8}$$

Taking the large-n limit, and using (5.6) we find that

$$a_n=Ke^{-\alpha_2/4}(-4)^n(n+1)!\ . \tag{5.9}$$

Comparing (5.9) with (3.6) we find, using (5.5), that

$$K=-\frac{16e^{-3/2}}{\pi\sqrt{2}}\ . \tag{5.10}$$

Thus

$$c_n\sim-\frac{(-4)^{n+2}(n+1)!e^{-3/2}}{\pi\sqrt{2}}\quad (n\to\infty) \tag{5.11}$$

and our conjecture in (5.6) is consistent.

In Table II, we list the first 15 coefficients c_n in the expansion (2.13). Their growth with n agrees well with the predicted asymptotic behavior (5.11).

This analysis confirms a formal argument of Bender, Mandula, and McCoy[1] that renormalization does not affect the divergence of weak-coupling perturbation theory. Here, its effect is to reduce the asymptotic behavior of the coefficients by a factor of $e^{3/2}$.

VI. MASS-RENORMALIZED STRONG-COUPLING PERTURBATION SERIES

To derive the mass-renormalized strong-coupling series in (2.14) we begin by recognizing that the integrals in (2.9) and (2.12) define parabolic cylinder functions[2]

TABLE II. The first 15 coefficients c_n in the renormalized weak-coupling perturbation expansion for $\tilde{G}$ in (5.13) compared with the leading asymptotic approximation to c_n in (2.16).

Order n	Exact c_n	Asymptotic approximation to c_n in (2.16)
1	6	6.43
2	−54	−77.1
3	810	123×10
4	−16 362	-247×10^{2}
5	406 782	592×10^{3}
6	−11 872 494	-166×10^{5}
7	395 434 386	531×10^{6}
8	−14 752 605 330	-191×10^{8}
9	608 433 692 022	764×10^{9}
10	−27 468 612 893 862	-336×10^{11}
11	1 347 067 421 442 234	161×10^{13}
12	−71 308 295 362 534 266	-839×10^{14}
13	4 053 152 328 184 608 750	470×10^{16}
14	−246 247 460 613 952 843 230	-282×10^{18}
15	15 927 423 964 753 440 452 130	181×10^{20}

$$\tilde{G}=3-\frac{3}{2}\frac{M^4}{g}\frac{D_{-5/2}(m^2/\sqrt{2g}\,)}{D_{-1/2}(m^2/\sqrt{2g}\,)}\,, \tag{6.1}$$

$$\frac{1}{M^2}=\frac{1}{\sqrt{2g}}\frac{D_{-3/2}(m^2/\sqrt{2g}\,)}{D_{-1/2}(m^2/\sqrt{2g}\,)}\,, \tag{6.2}$$

so that

$$\frac{2M^4}{g}=4\left[\frac{D_{-1/2}(m^2/\sqrt{2g}\,)}{D_{-3/2}(m^2/\sqrt{2g}\,)}\right]^2 . \tag{6.3}$$

Let us define

$$z=m^2/\sqrt{2g}\ . \tag{6.4}$$

The limit $M^4/g\to 0$ corresponds to the limit $z\to-\infty$. The asymptotic behavior of the parabolic cylinder function on the negative real axis is[3]

$$D_{-\nu}(z)\sim\frac{-(2\pi)^{1/2}}{\Gamma(-\nu)}e^{i\nu\pi}z^{-\nu-1}e^{z^2/4}\times\sum_{n=0}^{\infty}\frac{(1+\nu/2)_n(\frac{1}{2}+\nu/2)_n}{n!(z^2/2)^n}\,, \tag{6.5}$$

where $(A)_n$ means $\Gamma(A+n)/\Gamma(n)$.

We show easily that for $z\to-\infty$,

$$\frac{2M^4}{g}=\frac{1}{z^2}+\frac{1}{z^4}+\cdots+\frac{\beta_n}{(z^2)^n}+\cdots \tag{6.6}$$

with

$$\beta_n\sim\frac{2^n(n-2)!}{\pi\sqrt{2}}\quad(n\to\infty)\ . \tag{6.7}$$

From (6.1) we know that

$$\tilde{G}=\sum_{n=0}^{\infty}\gamma_n\left[\frac{1}{z^2}\right]^n , \tag{6.8}$$

where

$$\gamma_n\sim\frac{-2^n(n-1)!}{\pi\sqrt{2}}\ . \tag{6.9}$$

As in the previous section, the problem is to revert (6.6) and substitute into (6.8). The argument is exactly as in the previous section. We write

$$\tilde{G}=\sum e_n\left[\frac{2M^4}{g}\right]^n \tag{6.10}$$

and assume

$$e_n\sim K2^n(n-1)!\ \ (n\to\infty)\ . \tag{6.11}$$

Again e_n and γ_n dominate β_n by a power of n. Consequently we find that

$$K=-\frac{1}{\pi\sqrt{2e}} \tag{6.12}$$

so that

$$\tilde{G}=\sum d_n\left[\frac{M^4}{g}\right]^n \tag{6.13}$$

with

$$d_n\sim-4^n(n-1)!/(\pi\sqrt{2e}\,)\ . \tag{6.14}$$

Here is another way[4] to derive the result in

(6.14) using a dispersion relation representation for $\tilde{G}$. We call the dimensionless strong-coupling expansion parameter

$$\beta = M^4/g\ . \tag{6.15}$$

In terms of the variables z and β, (6.1) and (6.2) become

$$\tilde{G}(\beta) = 3 - \beta + z(\beta)\sqrt{2\beta} \tag{6.16}$$

and

$$(2/\beta)^{1/2} = -z - 2D'_{-1/2}(z)/D_{-1/2}(z)\ . \tag{6.17}$$

Note that as $z \to +\infty$, the right side of (6.18) is asymptotic to $1/z$. Thus

$$\sqrt{\beta} \sim z\sqrt{2} \quad (z \to +\infty)$$

and $z \to +\infty$ corresponds with $\beta \to +\infty$. Hence, as $\beta \to +\infty$, $\tilde{G}(\beta) \sim 2$; thus a once-subtracted dispersion integral representation for $\tilde{G}(\epsilon)$ is required.

To obtain the discontinuity in $\tilde{G}(\epsilon)$ we will make use of the property that as $\epsilon \to 0^+$, $\mathrm{Re}z \to -\infty$ but that z has a small imaginary part. In particular, we show that if we let $z = -R + i\eta$ then η can be calculated as a series in powers of $1/R$ and that to leading order $\eta = e^{-R^2/2}$, where $R \to +\infty$ as $\beta \to 0^+$.

We will calculate the first two corrections to the leading behavior of d_n. To do so we will need the first two corrections to the leading asymptotic behavior of $D_{-1/2}(z)$ as $z \to -\infty$:

$$D_{-1/2}(z) \sim z^{-1/2} e^{-z^2/4}\left[1 - \frac{3}{8z^2} + \frac{105}{128z^4} + \cdots\right] + i\sqrt{2}e^{z^2/4}z^{-1/2}\left[1 + \frac{3}{8z^2} + \frac{105}{128z^4} + \cdots\right] \quad (z \to -\infty)\ . \tag{6.18}$$

Note that we include the subdominant contribution to the asymptotic behavior here. Even though it is exponentially small, it is distinguished from the dominant term because it is real and not imaginary.

Next we compute $D'_{-1/2}(z)$, and use the result to determine the ratio $D'_{-1/2}(z)/D_{-1/2}(z)$. Substituting the result into (6.16) gives

$$(2/\beta)^{1/2} = -2z + \frac{1}{z} + \frac{3}{2z^3} + \frac{6}{z^5} + \cdots + \frac{e^{-z^2/2}}{i\sqrt{2}}\left[2z - \frac{3}{2z} - \frac{39}{16z^3} + \cdots\right]. \tag{6.19}$$

Now we substitute $z = -R + i\eta$, where η is assumed to be exponentially small, into (6.19) and take the real and imaginary parts of the resulting equation. The real part is an equation for R:

$$\frac{\sqrt{2}}{\sqrt{\beta}} = 2R - \frac{1}{R} - \frac{3}{2R^3} - \frac{6}{R^6} + \cdots\ . \tag{6.20}$$

The imaginary part gives η in terms of R:

$$\eta = \frac{R - 3/4R - 39/32R^5 + \cdots}{1 + 1/2R^2 + 9/4R^4 + \cdots}\,\frac{e^{-R^2/2}}{\sqrt{2}}\ . \tag{6.21}$$

The solution to (6.6) is

$$R = \frac{1}{\sqrt{2\beta}}(1 + \beta + 2\beta^2 + 14\beta^3 + \cdots)\ . \tag{6.22}$$

Note that as we have asserted, $R \to \infty$ as $\beta \to 0^+$.

Next we solve for η by substituting the result in (6.22) into (6.21):

$$\eta = \frac{e^{-1}/(4\beta)}{2\sqrt{e\beta}}(1 - \tfrac{5}{2}\beta - \tfrac{51}{8}\beta^2 + \cdots)\ . \tag{6.23}$$

Finally, we compute $\mathrm{Im}\tilde{G}$ from (6.16) by substituting $z = -R + i\eta$:

$$\mathrm{Im}G = \frac{1}{\sqrt{2e}}e^{-1/(4\beta)}(1 - \tfrac{11}{4}\beta - \tfrac{391}{32}\beta^2 + \cdots)\ . \tag{6.24}$$

Now we use a once-subtracted dispersion relation representation for $\tilde{G}(\epsilon)$ with the cut on the *positive* real axis instead of the negative real axis as in Ref. 4. (We do not prove here that the cut is the only singularity.) From this dispersion relation we have

$$d_n = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\tilde{G}(\beta)}{\beta^{n+1}}d\beta\ . \tag{6.25}$$

Substituting (6.24) into (6.25) and integrating gives the large-order behavior of d_n with two higher-order corrections:

TABLE III. The first 11 coefficients d_n in the renormalized strong-coupling perturbation expansion for $\tilde{G}$ in (2.14) compared with the asympotic approximation to d_n in (2.17).

Order n	Exact d_n	Asymptotic approximation to d_n in (2.17)
0	2	
1	-2	0.6218
2	-2	-0.6410
3	-14	-10.65
4	-166	-154.6
5	-2714	-2699
6	$-55\,866$	-5671×10
7	$-1\,377\,942$	-1405×10^3
8	$-39\,493\,518$	-4019×10^4
9	$-1\,288\,115\,570$	-1307×10^6
10	$-47\,086\,272\,754$	-4762×10^7
11	$-1\,906\,554\,619\,166$	-1923×10^9

$$d_n \sim \frac{-4^n}{\pi\sqrt{2e}}(n-1)!\left[1-\frac{11}{16n}-\frac{743}{512n^2}-\cdots\right]. \tag{6.26}$$

The first 11 coefficients d_n in the expansion (2.14) are listed in Table III.[5] The prediction in (6.20) compares well with the exact values of the coefficients. When $n=11$, the leading asymptotic behavior of d_n is -2.079×10^{12}. Including the $-11/(16n)$ term gives -1.948×10^{12}. The third term lowers this number to -1.923×10^{12}. The exact value of d_{11} from Table III is -1.9065×10^{12}. Thus, our best prediction is off by a relative error of 1%.

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[1]C. M. Bender, J. E. Mandula, and B. M. McCoy, Phys. Rev. Lett. 24, 681 (1970).

[2]See, for example, *Higher Transcendental Functions,* edited by A. Erdelyi (McGraw-Hill, New York, 1953), Vol. 2, p. 119.

[3]See Ref. 2, p. 123. There is a typographical error. Our formula is correct, as is easily verified by substituting the asymptotic series into the differential equation satisfied by the parabolic cylinder function.

[4]For a detailed discussion of the use of a dispersion relation to obtain the large-order behavior of a perturbation series see C. M. Bender, Adv. Math. 30, 250 (1978) or C. M. Bender and T. T. Wu, Phys. Rev. Lett. 27, 461 (1971).

[5]These coefficients were first given in C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. D 23, 2999 (1981).