PHYSICAL REVIEW D VOLUME 25, NUMBER 2 15 JANUARY 1982

Simple approach to renormalization theory

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(Received 23 February 1981)

We present a simple argument which shows that all the ultraviolet divergences of quantum field theory may be subtracted by the use of purely local Hermitian counterterms. Using minimal subtraction and differentiation with respect to external momenta we are able to circumvent the problem of overlapping divergences, and many results of renormalization theory are easily derived. The subtraction procedure also allows arbitrary infrared well-defined rearrangements of the regularized integrand to be carried out.

I. INTRODUCTION

The fundamental result of renormalization theory states that to all orders in perturbation theory the ultraviolet (UV) divergences of a quantum field theory may be formally absorbed into the parameters defining the theory while locality, unitarity, and Lorentz invariance are maintained. If we want theories to have considerable predictive powers, they must be specified by only a finite number of parameters, and the insistence that this is consistent with the modification of parameters by renormalization leads us to consider the restrictive class of theories known as renormalizable theories. The concepts of renormalization and renormalizability are quite distinct, and there is no reason why the divergences of even nonrenormalizable theories cannot be absorbed into an albeit infinite family of local counterterms.

If one performs simple one-loop Feynman-diagram calculations, it soon becomes clear that all the divergences manifest themselves as polynomials in the external momenta, which means they are (quasi)local in configuration space. However, this does not continue to hold for unsubtracted Feynman diagrams with more complicated internal structure. It is the purpose of renormalization theory to show that all the divergences to all orders in the coupling-constant expansion may be removed by local subtractions, and further that the combinatorics is such that these subtractions may be implemented formally as counterterms in the Lagrangian.

There have been four main approaches to showing that quantum field theories may be renormalized: The first was initiated by Dyson[1] and is based on the use of skeleton expansions and integral equations for the Green's functions of the theory. This approach was developed further by Salam[2] and completed with the proof of Weinberg's theorem.[3] Unfortunately, the proof contains many pitfalls, and it was found that a particular simplification failed for a fourteenth-order diagram in quantum electrodynamics (QED)[4] (although the method could be modified so as to circumvent this difficulty). The second approach involves a recursive subtraction scheme motivated by the concept of counterterms. It was developed by Bogoliubov and Parasiuk,[5] based on the earlier work of Stückelberg and Green.[6] Their proof was also very complex, and included an intermediate theorem which was in fact not true. The first correct proof in this framework was given by Hepp,[7] and hence this renormalization scheme is known as BPH. The generality of this scheme has made it particularly appropriate for use in non-Abelian gauge theories. Furthermore, the BPH formalism is the most convenient for performing subtractions on individual diagrams, which is especially useful from a calculational viewpoint. The third approach, due to Epstein and Glaser,[8] is rigorously based on the postulates of constructive field theory and makes use of unitarity as expressed by cutting equations. The fourth method, suggested by Callan and proven by Blaer and Young,[9] is an inductive proof using the renormalization-group equations and skeletal expansions of nondivergent Green's functions. A good historical review has been given by Wightman.[10]

A necessary precursor of renormalization is the regularization of Feynman integrals. A cutoff parameter is introduced into the diagrams by modifying it in such a way as to make the divergences appear only as the cutoff parameter tends to some limiting value. By this artifice, it becomes possible to make mathematically respectable what would otherwise be purely formal manipulations of divergent quantities. A good regularization method maintains as many of the desirable features of the theory as it can. The simplest momentum-space cutoff regularization violates Poincaré invariance, and it becomes a nontrivial task to verify that renormalization preserves even this vital symmetry. The early proofs in renormalization theory therefore used Pauli-Villars regulators[11] which maintain manifest Poincaré symmetry. The role of

regularization in the BPH scheme is not immediately obvious, as this scheme is based on the idea of subtracting suitable parts of the integrand of a Feynman diagram so as to make the resulting integrals well defined (and in fact absolutely convergent in Euclidean space); nevertheless this does involve an implicit regularization insofar as the original theory really only defines the whole Feynman integral and not just the integrand on its own, even though this may be read from the Feynman rules for the theory. For this reason careful treatments of renormalization always do use some sort of regularization procedure (e.g., analytic regularization[12] in Hepp's proof).

The greater complexity of the symmetry of non-Abelian gauge theories made the introduction of a new and better regularization scheme vital for constructing a proof of their renormalizability, and the invention of dimensional regularization[13] indeed played a central role in this proof. The dimensional method involves calculating with an arbitrary space-time dimension n, "analytically continuing" the expressions so obtained into the complex n plane, and then isolating the divergences as poles in the Feynman integrals at $n=4$. The vital property of dimensional regularization is that it preserves gauge invariance, in the sense that the derivation of the Ward identities which express local gauge invariance in terms of relations between Feynman diagrams is valid for any n (except for the anomalies associated with γ_5).

For our purposes dimensional regularization has a further advantage, which was first realized by 't Hooft[14] who introduced the idea of minimal subtraction. In order to explain the significance of minimal subtraction, let us consider how we normally remove the divergent parts of a Green's function. We choose a renormalization point in the space of the external momenta and we expand the Green's function in a Taylor series about this point (this requires that the Green's function is regular at the renormalization point, so that the usual choice of on-shell renormalization is only possible for infrared well-behaved theories). If all the subdivergences of the Green's function have been removed, the remaining divergences can only appear in the leading coefficients of the Taylor expansion, and the conventional renormalization prescription replaces just these coefficients with their experimentally measured values. In a renormalizable theory this requires a finite set of experimental values which take the place of the parameters in the classical Lagrangian. Minimal subtraction, on the other hand, removes the pole terms in the Laurent expansion in the space-time dimension about four, rather than removing terms in the momentum-space Taylor expansion. The important difference is that the expansions are made in different spaces and is not related to the absence of finite renormalizations: The infinities are removed by subtractions in $\epsilon \equiv 2 - n/2$, and the Green's functions are normalized at particular points in momentum space by a subsequent finite renormalization. This implies one of the key points of our approach, that the operation of minimal subtraction commutes with that of differentiation with respect to the external momenta.

The essence of our proof is that we differentiate a diagram enough times with respect to external momenta for it to have no "overall overlapping divergences." Combining Weinberg's theorem (that a naively convergent integral with no subdivergences is absolutely convergent) with Bogoliubov's recursive R operation is enough to demonstrate that the differentiated diagram has only local pole terms. The commutativity of the subtraction procedure with the differentiations suffices to show that the original diagram also only has pole parts which are polynomial in the external momenta. Not all the overlapping divergences of a diagram present a problem: The ones that are contained within a subgraph are dealt with automatically by the recursive nature of the subtraction procedure.

Our aim is to present a simple proof that the renormalization procedure may be carried out to all orders in perturbation theory, on a level of rigor comparable with that of functional methods: A simple proof is important not only for the student, but also to clarify exactly what needs to be proved.[15] We know of no obstacles to constructing a mathematically rigorous proof along the lines of this paper.

To prove that a quantum field theory is renormalizable, we must show that we can remove the infinities of the theory while preserving unitarity, locality, and Lorentz invariance. The interest of Lagrangian field theory is precisely that a theory defined by a local, Lorentz-invariant, Hermitian Lagrangian has those properties in perturbation theory.[17] Unfortunately, in perturbation theory Green's functions and S-matrix elements have divergences, and these show up as poles in the space-time dimension at the physical dimension when the theory is dimensionally regularized. If we are able to show that the infinities may be consistently removed by a redefinition of the parameters of the Lagrangian (or equivalently by adding counterterms to the Lagrangian), we will have shown the theory to be an acceptable one (i.e., that it has the symmetries and properties necessary to be an acceptable theory: For it to be renormalizable as well, the number of counterterms necessary must be finite). We do this in two steps. We first show that the subtractions that must be made

for any diagram are local, which means polynomial in momentum space, and so are of the correct form to be added to the Lagrangian. It is at this stage that the overlapping divergences usually cause their worst complications. We then show that the local subtractions of the individual graphs of perturbation theory may be combined together to form counterterms (i.e., that the subtractions may be viewed as simply a redefinition of the Lagrangian).

The outline of the paper is as follows: We introduce the R operation in Sec. II A, and in II B we show that it indeed removes all of the divergences in perturbation theory. Section II C explains the relation between formal counterterms and the R operation. Section III shows how our methods may be used to prove some useful properties of this renormalization scheme. The final subsection (III D) presents an example to illustrate the practical simplicity of these methods, the vacuum polarization of massless Euclidean QED in two-loop order.

II. BOGOLIUBOV'S R OPERATION AND THE STRUCTURE OF FEYNMAN DIAGRAMS

A. Structure of Feynman diagrams

In this subsection we describe how to subtract the divergent parts of an arbitrary Feynman diagram G. We discuss the graphical structure of the subtraction procedure, but we leave the proof that it successfully removes all the divergences using only local subtractions to the next subsection.

We first introduce some terminology to describe the structure of a Feynman diagram. A Feynman diagram is a collection of lines and vertices which may be associated with a term in the perturbation expansion; we shall use the words graph and diagram interchangeably throughout. A 1PI *subgraph* of a Feynman diagram G is a connected collection of lines and vertices which is one-particle irreducible. That is, it cannot be disconnected by removing any one of the lines.[18] We write $\gamma \subset G$ to indicate that γ is a 1PI subgraph of G, and we represent such subgraphs diagrammatically by enclosing them in a box (Fig. 1). A *proper* 1PI *subgraph* $\overline{\gamma} \subset G$ is any 1PI subgraph except for G itself (Fig. 2). A (proper) *spinney*[19] $S(\overline{S})$ of G is a disjoint family of (proper) 1PI subgraphs, and we draw a spinney by drawing boxes around each member subgraph (Fig. 3). Formally we may define it as

$$S \equiv \{\{\ldots, \gamma_i, \ldots, \gamma_j, \ldots\} \mid \gamma_i, \gamma_j \subset G \text{ and } \gamma_i \cap \gamma_j = \emptyset\}.$$

A (proper) *wood*[20] $W(G)$ ($\overline{W}(G)$) is the set of all (proper) spinneys of G (Fig. 4). A *subtracted graph* $\tilde{\gamma} * G/\{\gamma\}$ is a graph G whose 1PI subgraph γ has been replaced by the expression $\tilde{\gamma}$ (which is usually some modification of the expression corresponding to γ itself). Note that this means the part of the Feynman integral corresponding to γ is replaced by $\tilde{\gamma}$ within momentum integrals if appropriate. More generally we may subtract several disjoint subgraphs at once,

$$\left(\prod_{\gamma \in S} \tilde{\gamma}\right) * G/S,$$

where each γ in S is replaced by its corresponding $\tilde{\gamma}$. Note that this means $1 * G/\{\gamma\} = G/\{\gamma\}$ corresponds to the subgraph γ being shrunk to a point, that $\gamma * G/\{\gamma\} = G$, and that if $\gamma \subset \gamma'$ and $\gamma' \subset G$, then

$$[\tilde{\gamma} * \gamma'/\{\gamma\}] * G/\{\gamma'\} = \tilde{\gamma} * G/\{\gamma\}.$$

We now define our minimal-subtraction operation: Let K be the operation of isolating the singular (pole) part in the dimensional regularization Laurent expansion of a function; for example, if

$$F = \sum_{j=-s}^{\infty} f_j \epsilon^j, \text{ then } KF = \sum_{j=-s}^{-1} f_j \epsilon^j.$$

We are assuming that the only singularities of dimensionally regularized Feynman integrals occurring at four dimensions are isolated poles. For

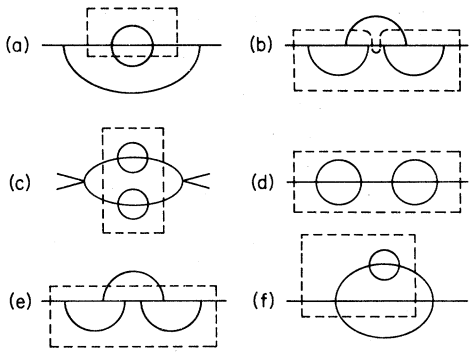

FIG. 1. In part (a) the dashed box encloses a 1PI subgraph. In the remaining diagrams the dashed boxes do not enclose 1PI subdiagrams. The enclosed subgraphs of parts (b) and (c) are not connected, and those of (d), (e), and (f) are not one-particle irreducible.

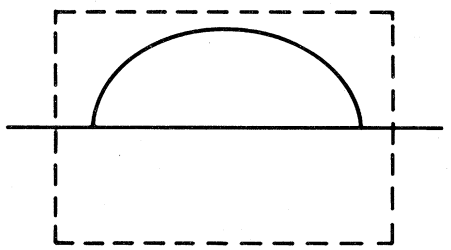

FIG. 2. The box encloses a 1PI subgraph which is not proper.

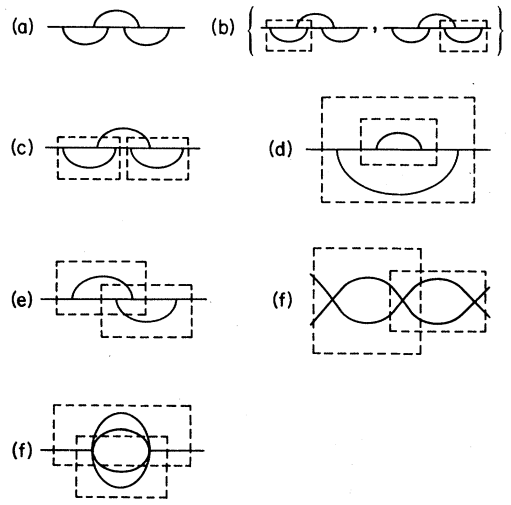

FIG. 3. If the graph G of (a) has the spinney S shown in (b), then we will draw S as in (c). The boxes of (d), (e), (f), and (g) do not represent spinneys. In (d) they are nested, and in the others they are overlapping.

two different approaches to proving this, see Speer[21] and 't Hooft and Veltman.[17] Our use of minimal subtraction means that we use K in place of the truncated Taylor series operation T of Bogoliubov and Parasiuk. We can now introduce an operation R which acts on the dimensionally regularized integral corresponding to the graph G; as no confusion is liable to arise we shall also call this integral G. We shall show in the next section that RG is rendered finite by *local* subtractions when analytically continued to four dimensions. The definition is

$$RG \equiv (1-K)\bar{R}G\,,$$

where $\bar{R}$ is

$$\bar{R}G \equiv \sum_{S\in\bar{W}(G)} \left(\prod_{\gamma\in S} -K\bar{R}\gamma\right) *G/S\,.$$

As a matter of interest, although we shall not need it in the following, we may also write RG as

$$RG \equiv \sum_{S\in W(G)} \left(\prod_{\gamma\in S} -K\bar{R}\gamma\right) *G/S\,.$$

B. Analytic behavior of the R operation

We have now set up enough graphical apparatus to turn to the proof of the main result of renormalization theory, the BPH theorem, which states that the R operation does in fact remove all the divergences of G using only local subtractions. We see at once that this requires showing two things, first that the $\bar{R}$ operation locally subtracts all subdivergences, and second that the remaining overall divergence of $\bar{R}G$ is purely local (in our minimal-subtraction approach this is the same as saying that $K\bar{R}G$ is a polynomial in all the external momenta). As may be surmized from the recursive nature of the definition of $\bar{R}G$, the proof of the first part will be by induction on the number of loops in G.

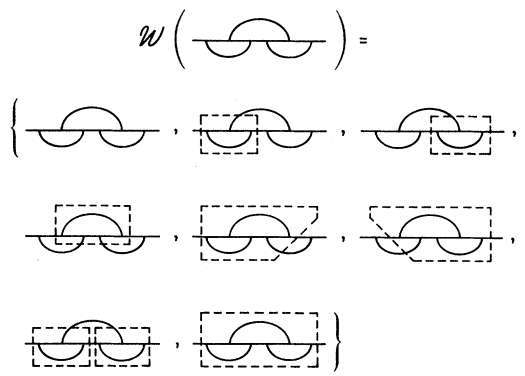

FIG. 4. An example of a wood. For a proper wood the final diagram is omitted. Notice that the wood contains the empty spinney as one of its elements.

Before starting the proof proper it is very helpful to make precise the ideas of an overall divergence and of a subdivergence. The *overall degree of divergence* deg(G) is the usual index obtained by counting the powers of loop momenta in a graph G. A graph has a *subdivergence* if it is naively divergent even when one of the loop momenta is held fixed, or to put it another way, if it has an overall divergent subgraph. It should be stressed that these are properties of graphs which are obtained by simple power counting, and are not defined in terms of the analytic properties of the associated integrals. The Weinberg-Dyson convergence theorem is: If an integral has no subdivergences and is overall convergent, $\deg(G)<0$, then it is absolutely convergent in Euclidean space.[22] This important statement is not obvious despite the suggestive terminology, nevertheless it is true. As the proof is somewhat technical,[3,25] we shall not reproduce it here, but we do want to indicate its general features.[25] Consider the simple Feynman diagram of Fig. 5, which gives rise to the Euclidean-space Feynman integral

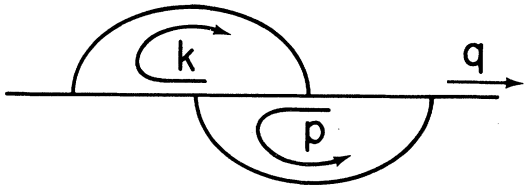

FIG. 5. ϕ^3 theory graph used as an example of the convergence theorem.

$$I=\int \frac{d^4k\,d^4p}{[k^2+m^2][(k+p-q)^2+m^2][(k-q)^2+m^2][(p-q)^2+m^2][p^2+m^2]}$$

of overall degree of divergence −2. The UV behavior of this integral is dominated by the expression

$$I'=\int_D \frac{d^4k\,d^4p}{k^4(k+p)^2p^4},$$

obtained by neglecting masses and external momenta, with D being a region excluding the infrared (IR) divergences near $k,p=0$, say $k^2,p^2\geq 1$. We now view the integral as being over an eight-dimensional space spanned by k and p, and we see that the integrand is sufficiently damped as we move to infinity in this space except in certain directions where the naive power-counting argument fails. In these exceptional directions certain combinations of momenta—in our case k, p, or $k+p$—are fixed as we go to infinity in the radial direction, and the radial integral seems to degenerate to a divergent one. The aim of the proof is to show that these would-be divergences must correspond to subdivergences which by assumption are not present. One way of showing this in the present case is to divide D into three regions in each of which one of the quadratic denominator factors is smaller than both the others. For our example, we shall only consider the region D_k for which $k^2 \leq \min[p^2,(p+k)^2]$:

$$I''=\int_U d^4k \int_V \frac{d^4p}{k^4(k+p)^2p^4},$$

where $U\equiv\{k\,|\,k^2\geq 1\}$, $V\equiv\{p\,|\,p^2\geq k^2,(p+k)^2\geq k^2\}$. A simple rescaling of the integration variables in the inner integral, namely $k\equiv k'\,|\,k\,|, p\equiv p'\,|\,k\,|$, then gives

$$I''=\int_U \frac{d^4k}{|k|^6}\int_{V'}\frac{d^4p'}{k'^4(k'+p')^2p'^4}$$

with

$$V'=\{p'\,|\,p'^2\geq 1,(p'+k')^2\geq 1\}.$$

For any given value of k', the inner integral is absolutely convergent to some finite value $J_{k'}$, for otherwise it would be a subdivergence, but k' lies within the unit four-sphere which is a compact subspace of R^4, and therefore $J_{k'}$ must be bounded by some value A. This means that

$$I''\leq A\int_U \frac{d^4k}{|k|^6},$$

which is convergent, showing that I is also absolutely convergent. We refrain from giving further details of the general proof which is simply a systematic extension of the above method to treat all possible momentum regions.[25]

When we renormalize a theory using minimal subtraction, the proof of the necessary convergence theorem is more subtle. The subtractions are made by removing pole terms rather than by subtracting momentum-space integrals. Therefore, unlike in our simple example, the subdivergences are not absent but are instead explicitly canceled by the subtraction of a pole term, and the proof of the convergence theorem must be generalized to include this case. The unsubtracted integral and the subtraction terms are both well-defined by dimensional regularization, and we expect that the theorem is true for this case also. We shall assume this in the following.

A more general statement on subdivergence-free integrals may be made, but to do so we must consider a few properties of the operation of differentiating a graph with respect to its external momenta (Fig. 6). As we shall consider each such momentum separately, it is adequate to write ∂G for $\partial G/\partial p_\mu$, where p is a typical external momentum; then $\partial^2 G=\partial^2 G/\partial p_\mu \partial p_\nu$ and so on. This operation has three extremely useful properties: (i) A function f vanishes after a finite number of differentiations if and only if it is a polynomial in p, (ii) the subtraction operator K and the derivative operator ∂ commute as they act on different spaces, and (iii) ∂ lowers the overall degree of divergence of a graph G, specifically $\deg(\partial^s G)\leq \deg(G)-s$. The first two properties are obvious, and the third follows from simple considerations of the form of the integrand of the Feynman diagram.[26] We require that the differentiation be with respect to an external momentum vector with a unique (unpaired) Lorentz index in the definition of property (ii), for otherwise there arise terms of the form $\delta_{\mu\mu}=n$ and the operator K will no longer commute with ∂.

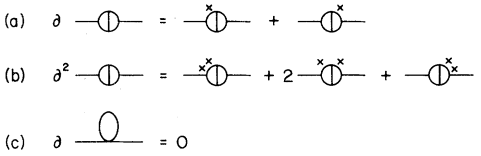

FIG. 6. (a) and (b) are examples of differentiating a graph with respect to an external momentum. A cross represents a differentiation. In this ϕ^3 graph the external momentum is routed through the top of the diagram. All the results in the text are independent of the routing of the momenta. (c) is a tadpole graph in ϕ^4 theory, and provides an example of how differentiation with respect to an external momentum can result in zero.

This subtle effect is also the source of the trace anomalies of quantum field theory, and has been fully analyzed by Speer.[27] For a local field theory the Feynman rules give an integrand whose numerator is a polynomial in p and whose denominator is a product of factors each of the form $[(p+q+k)^2+m^2]$, where k is a combination of loop momenta, q a combination of external momenta, and m some mass; for a 1PI diagram k is never identically zero. From these observations it is clear that ∂ gives a sum of terms each of which has either a numerator of lower degree or a denominator of higher one [of course, it might give zero, which for convenience we shall assign an overall degree of divergence of $-\infty$, hence the inequality in relation (iii) above].

We can now give the promised generalization of the Weinberg-Dyson convergence theorem in the case of minimal subtraction: The pole part of any diagram G which has no subdivergences is a polynomial in each external momentum. Suppose $\deg(G) \equiv \omega$, then by property (iii) above we have that $\deg(\partial^{\omega+1}G) < 0$. Furthermore, we will soon show that if G is subdivergence free, then $\partial^{\omega+1}G$ is also; $\partial^{\omega+1}G$ therefore satisfies the criteria of the convergence theorem and is absolutely convergent in four dimensions. Equivalently this means that $\partial^{\omega+1}G$ has no pole at four dimensions in its dimensionally regularized form, so $K\partial^{\omega+1}G = 0$. It follows from property (ii) that $\partial^{\omega+1}KG = 0$, and thus by property (i) KG is a polynomial in p. As this is true for every p our result is established.

We now have to prove that $\bar{R}G$ is indeed made subdivergence free by local subtractions, for if this is so then RG is made divergence free by local subtractions as well. For this a little more notation is called for. A *renormalization part* is a proper 1PI subgraph γ such that $\deg(\gamma) \geq 0$ (Fig. 7). A diagram has an *overlapping divergence* if two renormalization parts overlap. Formally this may be stated as

$$\gamma_i \cap \gamma_j \neq \emptyset, \quad \gamma_i \not\subset \gamma_j, \text{ and } \gamma_j \not\subset \gamma_i .$$

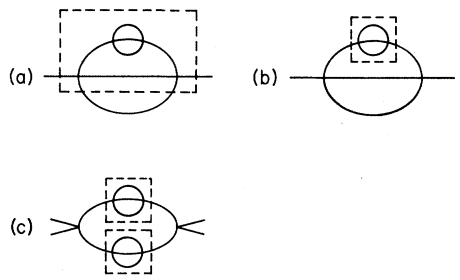

FIG. 7. Some examples of renormalization parts in four-dimensional ϕ^4 theory.

A diagram has an *overall overlapping divergence* if it is covered by a set of (non-nested) renormalization parts (Fig. 8).

We now proceed to prove our desired result by induction on the number of loops in G. We need only establish that the inductive step is true, for it is quite clear that for zero loops tree graphs are perfectly finite. Let us assume G to have l loops, and that $\bar{R}\gamma$ for any 1PI graph γ with less than l loops has no subdivergences and involves only local subtractions.

If we inspect the definition of $\bar{R}G$ we observe that as the summation runs over the proper wood $\bar{W}(G)$, all the subgraphs γ appearing have less than l loops, hence our inductive hypothesis combined with the convergence theorem tells us that only spinneys made solely of renormalization parts contribute to $\bar{R}G$, for otherwise there is a factor of $K\bar{R}\gamma = 0$ appearing for at least one overall convergent γ. This observation is useful because it enables us to reduce the problem to two cases: (a) G has a disjoint set of proper 1PI subgraphs $\{\Gamma_1, \ldots, \Gamma_t\}$ such that each renormalization part is contained in one of them[28] (Fig. 9), or (b) G has an overall overlapping divergence. We may show that these are the only possible cases by the following argument: If G does not have an overall overlapping divergence, then there is no set of

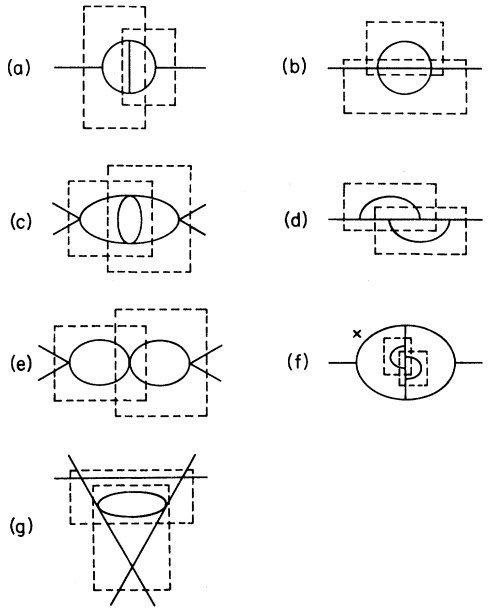

FIG. 8. (a) through (e) are examples of overall overlapping divergences in ϕ^4 theory in four dimensions, and in ϕ^3 theory in six dimensions. (f) has an overlapping divergence which is not an overall overlapping divergence. (g) has two overlapping "overall" subgraphs, one of which is not a renormalization part.

renormalization parts which covers it, and thus there is at least one line which is not in any renormalization part. Consider a subgraph θ which contains all of G except for one such line. θ is either 1PI or consists of two pieces connected by one or no lines. In the former case $\{\theta\}$ forms a set satisfying the criteria of case (a), while in the latter case we reapply the above argument to the subgraphs corresponding to each of the two pieces of θ until we obtain a set of 1PI subgraphs compatible with (a).

We consider case (a) first, and to simplify the presentation we shall first assume that $t=1$, in other words all the renormalization parts lie within one proper 1PI subgraph Γ. In this case

$$\overline{W}(G) \sim W(\Gamma) = \overline{W}(\Gamma) \cup \{\Gamma\},$$

where $\sim$ means equal except for some spinneys not made entirely of renormalization parts, so we may obtain our desired result by simple manipulations:

$$\begin{aligned}
\overline{R}G &\equiv \sum_{S \in \overline{W}(G)} \left(\prod_{\gamma \in S} - K\overline{R}\gamma \right) * G/S \\
&= (-K\overline{R}\Gamma) * G/\{\Gamma\} \\
&\quad + \sum_{S \in \overline{W}(\Gamma)} \left[\left(\prod_{\gamma \in S} - K\overline{R}\gamma \right) * \Gamma/S \right] * G/\{\Gamma\} \\
&= (-K\overline{R}\Gamma) * G/\{\Gamma\} + \overline{R}\Gamma * G/\{\Gamma\} \\
&= (1-K)\overline{R}\Gamma * G/\{\Gamma\} \\
&= R\Gamma * G/\{\Gamma\}.
\end{aligned}$$

This is manifestly free of subdivergences, and as Γ has less than l loops the subtraction $K\overline{R}\Gamma$ is a polynomial in all the external momenta. It is important not to let the cumbersome notation obscure the underlying triviality of the proof. The same is true for the general situation with $t \neq 1$, for which the proof is given in Appendix A.

We are left with case (b), where there is an overall overlapping divergence. In other approaches this is the hardest part of the problem, but we shall be able to circumvent it by reducing such cases by differentiation to those already treated. The method hinges on the following property of the ∂ operation: (iv) If a graph G is covered by a family $\{\Gamma_1, \ldots, \Gamma_t\}$ of renormalization parts with $\deg(\Gamma_i) \equiv \omega_i$, then $\partial^\lambda G$ has *not* got an overall overlapping divergence, where $\lambda \equiv 1 + \sum_{i=1}^{t} \omega_i$. To see this we argue as follows. From the definition of an overall overlapping divergence, every line and every vertex must be included in at least one renormalization part Γ_i. By property (iii) of ∂ given before and the way that ∂ distributes itself over the graph G (Fig. 6), in every term of ∂G at least one Γ_i has its overall degree of divergence lowered

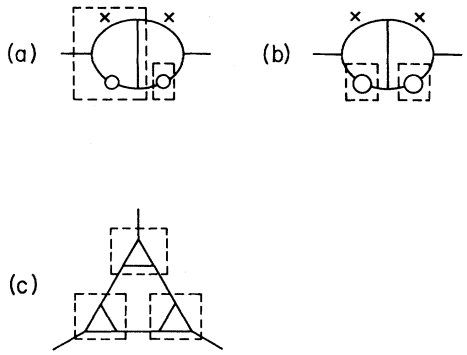

FIG. 9. Some examples of diagrams which do not have overall overlapping divergences. The diagrams are in ϕ^3 theory in six dimensions, and crosses represent differentiations with respect to the external momentum (which is routed through the top of the diagrams). The boxes represent a set of 1PI subgraphs which contain all the renormalization parts. Note that (a) and (b) clearly show that such a set of subgraphs is not unique.

(although which Γ_i this is depends on the term under consideration). It therefore follows that differentiating G λ times is enough to ensure that in every term at least one Γ_i has been made overall convergent, which in turn means that the differentiated graph has no overall overlapping divergence. We may apply the analysis of case (a) to show that $\overline{R}$ removes all the subdivergences of $\partial^\lambda G$ by local subtractions.[29] Provided that $\overline{R}$ commutes with ∂ this completes our proof, as then $\overline{R}G$ is made subdivergence free by local subtractions.

Finally, we must show that (v) $[\overline{R}, \partial] = 0$. The idea of the proof is quite simple: differentiation of a graph produces many terms (because each line and vertex must be differentiated), but each term has the same graphical structure as the original graph and so the $\overline{R}$ operation will act on each one in the same way. Combining the topological definition of $\overline{R}$ with the term-by-term nature of differentiation makes the actual proof rather tedious, and thus we relegate it to Appendix B.

We close this long and arduous subsection by observing that the Hermiticity of the local subtractions follows at once from the fact that dimensional regularization formally preserves the Hermiticity of the Lagrangian for arbitrary real dimension, and thus for each term in the Laurent expansion about four dimensions. The subtraction operation K simply picks out a necessarily Hermitian pole part.

C. Counterterminology

One of the basic ideas behind renormalization is that all the subtractions necessary to render a perturbative quantum field theory finite may be form-

ally absorbed into a set of extra vertices called counterterms. Once this has been shown to be true, the properties of locality and unitarity inherent in the Lagrangian approach are guaranteed to all orders in perturbation theory. To implement the counterterm approach we add to the renormalized Lagrangian—namely the zero-loop "classical" Lagrangian involving only finite parameters—a set of counterterm monomials (which are pure pole terms in the minimal-subtraction scheme). The resulting bare Lagrangian is then viewed as the "true" Lagrangian of the theory despite having divergent parameters in four dimensions, as in a formal sense (as the boundary value of the regularized theory) it nevertheless generates only finite results. In other treatments the meanings of the words "bare" and "renormalized" are sometimes the reverse of ours.

Despite the fact that dimensional regularization specifies uniquely the pole part of any counterterm, the finite part is still ambiguous and has to be fixed by some renormalization condition on the Green's functions in which it appears. If the theory is renormalizable then all the counterterms have the same structure as terms already present in the renormalized Lagrangian, and may be absorbed into a formal redefinition of the renormalized parameters into bare parameters. As the renormalized parameters need to be determined experimentally anyhow the ambiguities in the finite parts of the counterterms become unimportant. The observant reader will have noticed that the definition of renormalizability given above and that given in the Introduction differ slightly—in the Introduction we just insisted that only a finite number of counterterms were required to remove all the divergences. This difference is usually unimportant provided the theories considered conform to the rule that all possible terms consistent with the symmetries of the theory and of the appropriate dimension are included in the renormalized Lagrangian, as they will appear as counterterms anyhow.

The counterterms subtract out divergences of the Green's functions, but unlike the R operation they do not work on a graph-by-graph basis.[30] There are two reasons why the equivalence between counterterms and subtractions is not obvious. First one has to be certain that the symmetry factors which occur when there are identical particles are handled correctly, and second a given countergraph may correspond to subtractions for several different diagrams, arising from the different orientations in which the subgraph corresponding to the counterterm may be inserted. A proof that the R operation can be implemented with counterterms is therefore necessary.

Another source of confusion is the distinction between "multiplicative" renormalization, involving rescaling of fields and the like, and "additive" renormalization using counterterms. The distinction between the two is really rather artificial, and is totally unrelated to the subtraction method. In the renormalized Lagrangian it is conventional to make use of the arbitrary normalization of the fields to set the coefficient of their kinetic terms to some fixed value, say one. This coefficient will be renormalized by the addition of some counterterm δZ, giving a total value of $Z = 1 + \delta Z$, and it is conventional (though there is no particularly good practical reason for it) to rescale the fields to "bare" fields to restore this coefficient back to its canonical value for the bare Lagrangian written in terms of the bare fields. Of course, this means all the counterterms for the coupling constants, masses, and Green's functions of the theory will also have to be separated into pieces corresponding to the field rescaling and to "intrinsic" parameter renormalizations, but the whole procedure amounts to no more than a way of making the bare Lagrangian look the same as the renormalized one except for a change in the values of its parameters.

What we must show in order to justify this counterterm viewpoint is that the combinatorics of the BPH subtraction scheme is such that the subtractions needed are exactly those which are generated by some suitable set of counterterms. Although proofs of this exist for normal-ordered Lagrangians,[7,31] we do not know of any for the case where tadpole graphs are kept. It is straightforward to generalize the proof to this case too, however, as we shall show here.

We first define a *generalized vertex* U of a graph G as a subgraph containing some (nonempty) set of vertices together with *all* of the lines connecting them. [The boxes in Figs. 7(b) and 7(c) contain generalized vertices, while Fig. 7(a) is an example of a 1PI subgraph which is not a generalized vertex.] We allow generalized vertices to contain just one vertex, unlike Anikin and Zav'yalov,[31] as we are including tadpole graphs in our formalism. Next we introduce an operator Λ which acts on a generalized vertex U according to the following rules: if U is not connected and 1PI then ΛU is zero. Otherwise

$$\Lambda U = \kappa(U) + \sum_{\gamma} -K\bar{R}\gamma ,$$

where $\kappa(U)$ equals U if U is a single vertex, and is zero otherwise. The summation runs over all 1PI subgraphs γ which include all of the vertices in U and none other. Naturally none of these γ can be disjoint since they all share the same vertices.

This means that if we take the graph G and cover it with a set of disjoint generalized vertices $\{U_i\}$, then

$$RG = \sum_{\{U_i\}} \prod_i \Lambda U_i ,$$

where the summation is over all such coverings of G by U_i. The terms $\kappa(U_i)$ where U_i is a *single* vertex will generate the usual subtracted type of graph G/S. Also, G need not even be a connected or 1PI diagram as the subtractions will be made appropriately in any case.

This change in notation enables us to transform RG into a form in which the equivalence with counterterms is obvious. Our problem is a purely combinatorial one, and it is natural to use functional integral methods to organize the combinatorics.[32] To do this we consider the generating functional $W[J]$, which is defined as the vacuum-to-vacuum amplitude

$$W[J] \equiv \int [d\phi] \exp\left(-S[\phi] - \int d^n x \, \phi \cdot J\right),$$

where S is the renormalized action. To obtain a perturbation expansion we use the usual trick of splitting S into a quadratic part S_0 and an interaction part S_I, as then we obtain

$$W[J] = \left(\sum_{N=0}^{\infty} (-S_I[\delta/\delta J])^N / N!\right) W_0[J]$$

with

$$W_0[J] = \exp\left(-\frac{1}{2}\int d^n x \, d^n y J(x) \Delta(x,y) J(y)\right) W_0[0],$$

where $\Delta(x,y)$ is the propagator for the ϕ field. Each term in the expansion above may be viewed as a set of vertices with functional derivatives attached in such a way that when they act on $W_0[J]$ all Feynman diagrams are generated (as the combinatorial weight for each diagram is defined by the number of times it appears in this expansion these factors will be correctly handled automatically). The action of R on $W[J]$ is therefore

$$RW[J] = \left(\sum_{N=0}^{\infty} R(-S_I[\delta/\delta J])^N / N!\right) W_0[J],$$

where both sides may be interpreted in terms of the graphical expansion. We now express RS_I^N in terms of Λ as the sum over all coverings of (each term of) S_I^N by disjoint sets of vertices, which will become generalized vertices when the functional derivatives act on W_0. A given cover consists of N_j sets of j vertices, where $N_1 + 2N_2 + \cdots + kN_k = N$, corresponding to the term

$$(\Lambda S_I)^{N_1} (\Lambda S_I^2)^{N_2} \cdots (\Lambda S_I^k)^{N_k} .$$

Each such cover can occur in

$$\frac{N!}{N_1! N_2! \cdots N_k! (1!)^{N_1} (2!)^{N_2} \cdots (k!)^{N_k}}$$

distinct ways, so

$$RW[J] = \left(\sum_{N=0}^{\infty} \sum_{N_1 + \cdots + kN_k = N} \frac{[\Lambda(-S_I)/1!]^{N_1}}{N_1!} \cdots \frac{[\Lambda(-S_I)^k k!]^{N_k}}{N_k!}\right) W_0[J] .$$

If we rewrite the summations as independent summations over $N_1, \ldots, N_k, \ldots$, we obtain an exponential form for the right-hand side, namely

$$RW[J] = \exp[\Lambda(e^{-S_I[\delta/\delta J]} - 1)] W_0[J] .$$

Defining the interaction part of the bare action S_{BI} to be minus the expression inside the exponent above, $\Lambda(1 - \exp(-S_I))$, we see that we have obtained an explicit expression for the bare Lagrangian, as $RW[J]$ is the generating functional for physical (i.e., subtracted) graphs. The counterterm part of the action is just $S_{BI} - S_I$ which is in general an infinite sum of local monomials, but which has a finite number of nonvanishing terms for the class of renormalizable theories.

III. SIMPLICITY AND CONVENIENCE

The renormalization method used in the previous section has many convenient properties that make calculations much simpler than those using other renormalization methods. The structure of the counterterms found using minimal subtraction is quite simple. We shall show that they are a polynomial not only in the momenta, but also in other parameters of the theory. Further, the integrand may be reordered in many useful ways to simplify the evaluation of an integral *before* the integral is performed. Finally, this is the best scheme in which to perform renormalization-group calculations, as has been shown recently by the massive massless calculations of Tarasov *et al.*[33]

A. Polynomial pole terms and massless theories

The pole terms are simply polynomials in the momenta and the parameters of the Lagrangian.[34] The order of the polynomial is determined by the naive physical dimension of the graph in four dimensions. We have shown in the previous section that the pole terms are polynomials in the momenta. A similar proof, with differentiation by a mo-

mentum replaced by differentiation by a mass, allows us to conclude that the pole terms are also polynomial in the masses (or any other parameter which enters the Feynman rules and Lagrangian polynomially). It suffices to simply reinterpret ∂G as $\partial G/\partial m$, where m is an arbitrary mass parameter, and to note that this operation also improves the convergence of the integral when a propagator is differentiated in the same way as differentiation by a momentum. The pole terms are therefore polynomials in all the momenta and the masses. These are the only dimensionful parameters available, and so they must form a homogeneous polynomial of degree equal to the naive physical dimension of the integral in four dimensions. For a complicated diagram with many operator insertions, the listing of the possible contributions to this polynomial tells us what local subtractions must in fact be made in the Lagrangian.[35]

Our methods treat equally well massless theories (as long as the graphs are not infrared divergent). The subtle point is that, for a given graph, our differentiation procedure increases the IR divergence of the affected loops (equally as it decreases the UV divergence). If the diagram is differentiated enough times to create an IR divergence, that divergence will show up as a pole term in ϵ. The K operation does not know the difference between these IR and UV pole terms, and the simplest procedure for avoiding this problem is to get rid of all of the induced IR divergences. We may easily do this by putting an arbitrary mass into each of the propagators in the graph. The integral will necessarily be IR well behaved. Our results show that the counterterms of the *undifferentiated* graph are homogeneous polynomials in the masses and the momenta. At this point, we may take the masses to be equal to zero. By hypothesis the limit is smooth, and we have therefore successfully proven the BPH theorem for massless theories also. The ease of proof is a result of minimal subtraction being a soft renormalization, i.e., it does not oversubtract a graph and therefore does not suffer from the worse IR behavior of such oversubtracted theories. Technically, this is much simpler than the soft-mass insertions of Lowenstein and Zimmermann.[36] Note that the addition of a small mass is only a technical trick in demonstrating the polynomial character of the pole terms of a particular graph. It does not affect gauge invariance (which is in fact a relation among several graphs).

An example of an integral which differentiation makes IR divergent in four dimensions is

$$I_0 = \int \frac{d^n p\, p_\mu}{(k+p)^2 p^2} .$$

Differentiating twice, $(\partial/\partial k_\alpha)(\partial/\partial k_\beta)$, to remove the linear UV divergence produces the logarithmically IR-divergent integral

$$I_0'' = 2 \int \frac{d^n p\, p_\mu [4(k+p)_\alpha (k+p)_\beta - \delta_{\alpha\beta}(k+p)^2]}{[(k+p)^2]^3 p^2} .$$

This integral has a pole coming from the small $(k+p)^2$ region of the integrand. Clearly naive UV power counting to determine whether a pole exists does not work on an IR-divergent integral. Adding a mass before differentiation regulates this divergence,

$$I_m = \int \frac{d^n p\, p_\mu}{[(k+p)^2+m^2]p^2} ,$$

which when differentiated gives the obviously finite result

$$I_m'' = 2 \int \frac{d^n p\, p_\mu \{4(k+p)_\alpha (k+p)_\beta - \delta_{\alpha\beta}[(k+p)^2+m]\}}{[(k+p)^2+m^2]^3 p^2} .$$

We may Feynman parametrize the denominator of I_m and explicitly perform the resulting momentum-space integration. We obtain the result $(n \equiv 4-2\epsilon)$

$$I_m = -\pi^2 k_\mu \Gamma(\epsilon) \int_0^1 dx \, \frac{x^{1-\epsilon}}{[m^2+k^2(1-x)]^\epsilon} .$$

Expanding the denominator in a power series in ϵ, we find that the pole term is independent of m (and of k, as it must be for a logarithmically divergent integral) and that each term in the expansion is finite as $m \to 0$. I_m'' is divergent as $m \to 0$ when evaluated in four dimensions (this is where the IR divergence shows up), but that is of no concern to us.

B. Calculational convenience

The integral for a graph G is well defined in n dimensions, and therefore any rearrangement of the integrand that does not change the graphical structure [and hence the wood $W(G)$] will not change the value of RG. The most useful operations include shifting of subintegrations, breaking the graph into a sum of several separately treated terms, and completing the square of terms in the numerator. This last operation leads to cancellations of denominators [the wood $W(G)$ may be kept unchanged by giving lines a propagator of 1 where necessary], and the integrals which must be calculated are often technically simpler. This trick generally leads to the evaluation of integrals of a higher overall degree of divergence than that of the initial graph. Unlike the Taylor series subtraction T, the K operation is not changed by the greater overall degree of divergence of the graph. We may also add denominators which already ex-

ist [perhaps with different masses, which will not affect $W(G)$]. This procedure can be useful if we are interested in the divergent parts only, as the resultant terms can sometimes be separated into a hard finite integral and an easy divergent one.[33] We can even add in a denominator (as long as the resulting integral has a graphical interpretation) for the wood will be unchanged.

As a simple example of the utility of the rearrangements discussed above, consider the graph

$$G=\int \frac{d^n k (p\cdot k)^2}{(p+k)^2 (k^2)^2}\,.$$

Then, using $p\cdot k=[(p+k)^2-p^2-k^2]/2$ and symmetric integration, we find

$$G=\frac{p^2}{4}\int \frac{d^n k}{(p+k)^2 k^2}-\frac{p^2}{2}\int \frac{d^n k\, p\cdot k}{(p+k)^2 (k^2)^2}\,.$$

The R operation can be applied either to the initial form of the integral G, or separately to the two terms in the transformed version. The second term can be further simplified by the change of variable $k\to -p-k$ and by a rewriting of the resultant terms with the use of the identity

$$\frac{(p+k)\cdot p}{[(p+k)^2]^2}=-\frac{1}{2}p_\mu \frac{\partial}{\partial p_\mu}\frac{1}{(p+k)^2}\,.$$

We must then calculate

$$G=\frac{p^2}{4}\int \frac{d^n k}{(p+k)^2 k^2}-\frac{p^2}{4}p_\mu \frac{\partial}{\partial p_\mu}\int \frac{d^n k}{(p+k)^2 k^2}\,.$$

By dimensional analysis, the last integral must be a constant times $(p^2)^{(n-4)/2}$, so $p^\mu \partial/\partial p^\mu$ recovers the naive dimension $n-4$ (Euler's theorem), i.e., it simply multiplies the integral by $(n-4)$. The result is

$$G=\frac{p^2}{4}(5-n)\int \frac{d^n k}{(p+k)^2 k^2}\,,$$

an easier integral to evaluate, and the explicit factor of p^2 can be useful in canceling an adjacent propagator when G occurs as a propagator correction in a larger graph. The power of these techniques only really manifests itself for higher-order calculations, where it is not possible to evaluate the initial integral trivially (as we can in this case).

C. Renormalization-group parameters

The most striking applications of the methods of this paper come in the calculation of the renormalization-group equation parameters, as has been amply demonstrated in the calculations of Tarasov *et al.* The renormalization-group parameters are related to the single-pole terms of the perturbation theory graphs in a simple fashion.[33] The basic trick is to differentiate a graph G until it is logarithmically divergent. Then the pole terms $-K\bar{R}\partial' G$, which are selected by the differentiation, are a homogeneous polynomial of degree zero, i.e., they are simply constants. But we may now change the masses and the external momenta in a totally arbitrary fashion, as long as no IR divergence is introduced. Typically all but one or two of the masses and external momenta are set to zero, greatly simplifying the necessary integrals. For example, the pole part of the logarithmically divergent ϕ^4 theory integral

$$I=\int \frac{d^n k}{[k^2+m^2][(p+k)^2+m^2]}$$

is unchanged if we look at

$$K\bar{R}I=KI=K\int \frac{d^n k}{[k^2+m^2][k^2+m^2]}$$

$$=K\int \frac{d^n k}{k^2(p+k)^2}\,.$$

The freedom to treat different graphs independently is also useful in tailoring the calculational approach to the graph involved. Chetyrkin and Tkachov[37] have developed very useful formulas for the necessary massless propagator integrals through two-loop order, allowing the calculation of arbitrary three-loop renormalization-group parameters.

D. Example: Two-loop vacuum polarization in massless QED

We wish to present an example that shows the simplicity and utility of the renormalization method studied here. We choose to study the standard example of an overlapping divergence in a gauge theory, the two-loop vacuum polarization in QED. We will be most interested in the behavior of the vacuum polarization as the momentum gets large, and we will therefore study only massless, Euclidean QED (we choose to Wick rotate the time component so that we have a negative metric Euclidean space, and hence anti-Hermitian γ matrices, etc.). Our approach is similar to that of Itzykson and Zuber[32]: The main difference from their work is that we will look directly at $\Pi(k^2)$ rather than at the vacuum polarization tensor $\Pi_{\mu\nu}(k)=(k_\mu k_\nu - \delta_{\mu\nu}k^2)\Pi(k^2)$. This will allow us to rearrange the integrand into a quite simple form, and we will be able to obtain easily even the finite part of the vacuum polarization.

For the explicit evaluation of integrals, we use the results of Chetyrkin *et al.*[33] Although the leading pole part which is removed by minimal subtraction is uniquely specified, the remaining finite part depends upon the exact way in which we define dimensional regularization. The usual measure

$d^np/(2\pi)^n$ leads to messy factors of $\ln(4\pi)$ and γ (Euler's constant), so we shall choose to include an extra factor of $[\Gamma(1+\epsilon)B(1-\epsilon,1-\epsilon)(4\pi)^\epsilon]^{-1}$ in the measure in order to remove them. Notice that we are free to do this only insofar as we are only interested in regularizing the four-dimensional theory, but not if we want the canonical results for arbitrary integer values of the space-time dimension n. At the one-loop level this redefinition just corresponds to a finite renormalization of the theory. The simplest one-loop massless integral becomes, with this normalization,

$$\int \frac{d^np}{(2\pi)^n}\frac{1}{p^{2\alpha}(p-k)^{2\beta}}=\frac{G(\alpha,\beta)}{16\pi^2(k^2)^{\alpha+\beta-n/2}},$$

where

$$G(\alpha,\beta)=\frac{\Gamma(\alpha+\beta-2+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\epsilon)}\frac{B(2-\alpha-\epsilon,2-\beta-\epsilon)}{B(1-\epsilon,1-\epsilon)}.$$

and the space-time dimension $n\equiv 4-2\epsilon$. This fixes the volume of the unit sphere (when it is continued from four space-time dimensions) in a way which is particularly convenient in practice, for example $G(1,1)=1/\epsilon$. It is often convenient to insert a mass scale μ by multiplying the integration volume d^np by $\mu^{2\epsilon}$; the integrals then have their correct naive dimension for all ϵ. This is necessary only if we wish to discuss the behavior of the theory under the renormalization group (a change of the mass scale μ). We will not explicitly include μ, but it may be inserted by letting $\Pi(k^2)\to\Pi(k^2/\mu^2)$ at the end of the calculation.

We set the stage for the two-loop calculation by taking a slightly indirect approach: We calculate $\Pi_{\mu\mu}(k)$ first, whereas the usual approach would be to look at the coefficient of $k_\mu k_\nu$. Taking the trace of

$$\Pi_{\mu\nu}(k)=(k_\mu k_\nu-\delta_{\mu\nu}k^2)\Pi(k^2),$$

we find

$$\Pi(k^2)=-\Pi_{\mu\mu}(k)/(n-1)k^2.$$

It is easy to calculate $\Pi_{\mu\mu}(k)$. We first calculate the one-loop vacuum polarization, shown in Fig. 10(a). Using the usual Euclidean QED Feynman rules, we find

$$\Pi^{(1)}_{\mu\mu}(k)=-e^2\int\frac{d^np}{(2\pi)^n}\frac{\mathrm{tr}\,[\not p\gamma_\mu(\not p-\not k)\gamma_\mu]}{p^2(p-k)^2}$$

$$=4e^2(n-2)\int\frac{d^np}{(2\pi)^n}\frac{p\cdot(p-k)}{p^2(p-k)^2}.$$

When we complete the square in the numerator and set massless tadpoles to zero, we obtain the result

$$\Pi^{(1)}_{\mu\mu}(k)=2e^2(2-n)k^2\int\frac{d^np}{(2\pi)^n}\frac{1}{p^2(p-k)^2}$$

$$=-\frac{\alpha}{\pi}\frac{1-\epsilon}{\epsilon}(k^2)^{1-\epsilon}.$$

In the second line we simply used the one-loop integration formula quoted above and set $\alpha\equiv e^2/4\pi$. $\Pi^{(1)}(k^2)$ is then calculated to be

$$\Pi^{(1)}(k^2)=\frac{1}{3}\frac{\alpha}{\pi}\left(\frac{1}{\epsilon}-\ln(k^2)-\frac{1}{3}+O(\epsilon)\right).$$

We find the renormalized vacuum polarization by first noting that the one-loop vacuum polarization has no subdiagrams and therefore $\overline{R}$ has no effect on it. The renormalized vacuum polarization is then found to be

$$\begin{aligned}R\Pi^{(1)}_{\mu\nu}(k)&=(1-K)\overline{R}\Pi^{(1)}_{\mu\nu}(k)\\&=(1-K)\Pi^{(1)}_{\mu\nu}(k)\\&=(k_\mu k_\nu-\delta_{\mu\nu}k^2)(1-K)\Pi^{(1)}(k^2)\\&=(k_\mu k_\nu-\delta_{\mu\nu}k^2)\frac{\alpha}{3\pi}[-\ln(k^2)-\tfrac{1}{3}+O(\epsilon)].\end{aligned}$$

Note that we must calculate $R\Pi_{\mu\nu}$ before μ and ν are contracted. Calculating $R\Pi_{\mu\mu}$ would give a different result because of the dependence on ϵ of

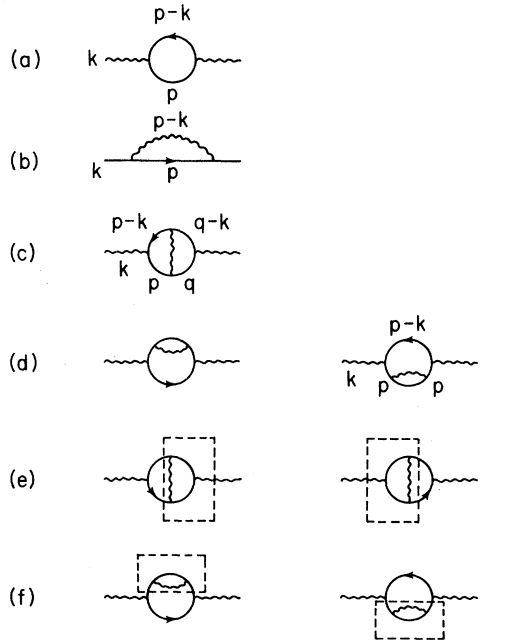

FIG. 10. The Feynman graphs contributing to the vacuum polarization in QED through two loops. (a) is the one-loop contribution and (b) is the one-loop electron propagator radiative correction, which occurs as a part of (d). (c) and (d) are the two-loop diagrams, and (e) and (f), respectively, are the subtractions of (c) and (d) that are made by $\overline{R}$.

the factor $n-1$ relating $\Pi_{\mu\nu}$ to $\Pi_{\mu\mu}$: This difference is an example of a trace anomaly.[27] The photon renormalization factor Z_3 is identified by the requirement that $Z_3[1+\Pi(k^2)]$ be finite as $\epsilon \to 0$, or equivalently that the pole part of $\Pi_{\mu\nu}$ must be canceled by the counterterm involving $\delta Z_3 \equiv Z_3 - 1$. This allows us to find

$$Z_3 = 1 - \frac{1}{3}\frac{\alpha}{\pi}\frac{1}{\epsilon} + O(\alpha^2) .$$

The β function is then determined by

$$\beta(\alpha) \equiv -2\alpha \frac{\partial}{\partial \alpha} Z_3^{[1]} = \frac{2}{3}\frac{\alpha}{\pi} + O(\alpha^2) ,$$

where $Z_3^{[1]}$ is the coefficient of the single-pole term in the Laurent expansion of Z_3 about $\epsilon = 0$.[38]

The one-loop fermion self-energy [Fig. 10(b)] is necessary for the two-loop vacuum polarization, and it is convenient to calculate it separately. In the Feynman gauge we obtain

$$\Sigma^{(1)}(k) = -e^2 \int \frac{d^n p}{(2\pi)^n} \frac{\gamma_\mu \not{p} \gamma_\mu}{p^2 (p-k)^2} .$$

$\Sigma(k)$ must be proportional to $\not{k}$, and so it is most easily found by writing $\Sigma(k) = A(k^2)\not{k}$ and forming the projection $\mathrm{tr}[\Sigma(k)\not{k}] = -4A(k^2)k^2$ with $\mathrm{tr}[1] = 4$. This allows us to calculate $A(k^2)$ using only scalar integrals, and it becomes (once more we drop massless integrals which are regulated to zero)

$$A^{(1)}(k^2) = e^2 \frac{(2-n)}{2} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 (p-k)^2} .$$

The unrenormalized electron propagator in one-loop order is then

$$\Sigma^{(1)}(k) = e^2 \frac{(2-n)}{2} \not{k} \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 (p-k)^2} ,$$

where the integral is the same scalar one that we had to do previously. The renormalized propagator to this order is obtained simply by subtracting the pole term.

We calculate the two-loop vacuum polarization in a similar fashion, adding together the three diagrams of Figs. 10(c) and 10(d). In forming $R\Pi_{\mu\nu}^{(2)}$ we are instructed to make, e.g., the subtractions shown in Figs. 10(e) and 10(f). For massless QED these terms exactly cancel as a result of the Ward identity $Z_1 = Z_2$.[39] We make these cancellations immediately, reducing the algebra to the evaluation of the graphs of Figs. 10(c) and 10(d).[40]

The algebra is carried out in the same way as in the one-loop case. To reduce the number of different integrals that must be studied, we use the symmetries of the diagrams in momentum space. For example, the value of the diagram of Fig. 10(c) is unchanged if we invert it (which is equivalent to the change of variables $p \to k - p$ and $q \to k - q$) or if we exchange the two ends ($p \leftrightarrow q$). Again completing squares in the numerator and dropping zero terms such as the generalized tadpole

$$\int \frac{d^n p\, d^n q}{p^2 (p-q)^2 q^2} = 0 ,$$

we obtain for the unrenormalized two-loop vertex correction to the vacuum polarization [Fig. 10(c)]

$$\Pi_{\mu\mu}^{(10c)}(k) = -e^4 \int \frac{d^n p\, d^n q}{(2\pi)^{2n}} \frac{\mathrm{tr}[(\not{p}-\not{k})\gamma_\nu (\not{q}-\not{k})\gamma_\mu \not{q} \gamma_\nu \not{p} \gamma_\mu]}{p^2 (p-k)^2 (p-q)^2 q^2 (q-k)^2}$$
$$= e^4 \int \frac{d^n p\, d^n q}{(2\pi)^{2n}} \left(\frac{4(n-2)(k^2)^2}{p^2 (p-k)^2 (p-q)^2 q^2 (q-k)^2} + \frac{4(n-2)(n-4)}{p^2 (p-q)^2 (q-k)^2} - \frac{2k^2 (n-7)(n-2)}{p^2 (p-k)^2 q^2 (q-k)^2} - \frac{16(n-2)k^2}{p^2 (p-k)^2 (p-q)^2 (q-k)^2} \right) .$$

Similarly, the propagator corrections of Fig. 10(d) are

$$\Pi_{\mu\mu}^{(10d)}(k) = -2e^2 \int \frac{d^n p}{(2\pi)^n} \frac{\mathrm{tr}[(\not{p}-\not{k})\gamma_\mu \not{p} \Sigma(p) \not{p} \gamma_\mu]}{p^2 (p-k)^2} = -2e^4 (n-2)^2 \int \frac{d^n p\, d^n q}{(2\pi)^{2n}} \left(\frac{p^2 - k^2}{p^2 (p-k)^2 (p-q)^2 q^2} \right) .$$

The above integrals [for the graphs of Figs. 10(c) and 10(d)] are of the same form, and we add them together before we perform the integrations. We calculate the vacuum polarization $\Pi(k^2)[= -\Pi_{\mu\mu}(k)/(n-1)k^2]$ to be

$$\Pi^{(2)}(k^2) = -e^4 \frac{(n-2)}{(n-1)} \int \frac{d^n p\, d^n q}{(2\pi)^{2n}} \left(\frac{4k^2}{p^2 (p-k)^2 (p-q)^2 q^2 (q-k)^2} + \frac{2(n-6)}{p^2 (p-q)^2 (q-k)^2 k^2} + \frac{2(7-n)}{p^2 (p-k)^2 q^2 (q-k)^2} + \frac{2(n-10)}{p^2 (p-k)^2 (p-q)^2 (q-k)^2} \right) .$$

Were we interested in only the pole part of this graph, we would need to calculate only *one-loop* integrals. The first term, the only one that cannot be reduced to analytically known one-loop integrals, is finite. In fact, all the two-loop massless propagator integrals are also known[33] and the analytic results are

$$\Pi^{(2)}(k^2) = -\frac{e^4}{(4\pi)^4}\frac{(n-2)}{(n-1)}(k^2)^{-2\epsilon}\left(24\zeta(3)+O(\epsilon)-(4+4\epsilon)\frac{[1+\frac{5}{2}\epsilon+O(\epsilon^2)]}{-4\epsilon}+\frac{2[3+2\epsilon]}{\epsilon^2}-(12+4\epsilon)\frac{[1+\epsilon+3\epsilon^2+O(\epsilon^3)]}{2\epsilon^2}\right)$$

$$=\frac{\alpha^2}{16\pi^2}\left(\frac{2}{\epsilon}-4\ln(k^2)-16\zeta(3)+\tfrac{31}{3}+O(\epsilon)\right).$$

The behavior found above for $\Pi^{(2)}(k^2)$ is even correct for massive QED in the limit of large k^2. Once again the renormalized vacuum polarization is found by simply canceling off the pole term (in fact, the double-pole term canceled on its own, a result specific to QED and due to the cancellation of all the subtraction graphs). The photon renormalization factor may be obtained from the above,

$$Z_3 = 1-\left(\frac{1}{3}\frac{\alpha}{\pi}+\frac{1}{8}\frac{\alpha^2}{\pi^2}\right)\frac{1}{\epsilon}+O(\alpha^3),$$

the β function is then determined to be

$$\beta(\alpha) = \frac{2}{3}\frac{\alpha}{\pi}+\frac{1}{2}\frac{\alpha^2}{\pi^2}+O(\alpha^3).$$

The β function is in fact renormalization-scheme independent through two-loop order, and this result agrees with the β function calculated by the usual on-shell renormalization method. The renormalization techniques discussed here are also applicable to massive theories, although the results for the finite parts of diagrams are in general much more complicated than those treated above.

We leave further (finite) renormalizations to the discretion of the reader. The infinities have now all been dealt with: A finite renormalization simply adds a finite part onto the pole counterterms, but the pole parts of the counterterms are not changed under such a renormalization.

ACKNOWLEDGMENT

We wish to thank Eugene R. Speer for a helpful conversation about overlapping divergences and multiple ice-cream graphs [Fig. 8(c) is a double ice-cream graph]. This work was supported in part by the National Science Foundation.

APPENDIX A

In this appendix we prove that the $\bar{R}$ operation locally removes all subdivergences of a diagram G for the general non-overall overlapping case. Here we have a set of t disjoint 1PI subgraphs $\{\Gamma_1,\ldots,\Gamma_t\}$ whose elements contain all the renormalization parts of G. These may be reduced one at a time as follows. First note that

$$\bar{W}(G) \sim \left\{\bigcup_{i=1}^{t} S_i \,|\, S_i \in W(\Gamma_i)\right\}.$$

Thus

$$\bar{R}G = \sum_{S\in\left\{\bigcup_{i=1}^{t} S_i \,|\, S_i\in W(\Gamma_i)\right\}}\left(\prod_{\gamma\in S}-K\bar{R}\gamma\right)*G/S$$

$$= \sum_{S'\in\left\{\bigcup_{i=2}^{t} S_i \,|\, S_i\in W(\Gamma_i)\right\}}\left(\prod_{\gamma'\in S'}-K\bar{R}\gamma'\right)\sum_{S_1\in W(\Gamma_1)}\left(\prod_{\gamma_1\in S_1}-K\bar{R}\gamma_1\right)*G/(S'\cup S_1).$$

As before we use the decomposition $W(\Gamma_1)=\bar{W}(\Gamma_1)\cup\{\Gamma_1\}$ to obtain

$$\bar{R}G = \sum_{S'\in\left\{\bigcup_{i=2}^{t} S_i \,|\, S_i\in W(\Gamma_i)\right\}}\left(\prod_{\gamma'\in S'}-K\bar{R}\gamma'\right)\left[(-K\bar{R}\Gamma_1)+\sum_{S_1\in\bar{W}(\Gamma_1)}\left(\prod_{\gamma_1\in S_1}-K\bar{R}\gamma_1\right)*\Gamma_1/S_1\right]*G/(\{\Gamma_1\}\cup S'),$$

where we have made use of the disjointness of the Γ_i. We may rewrite the last equation as

$$\bar{R}G = \sum_{S' \in \{\bigcup_{i=2}^{t} S_i \mid S_i \in W(\Gamma_i)\}} \left(\prod_{\gamma' \in S'} -K\bar{R}\gamma'\right)(1-K)\bar{R}\Gamma_1 * G/(\{\Gamma_1\} \cup S'),$$

$$= (1-K)\bar{R}\Gamma_1 \sum_{S' \in \{\bigcup_{i=2}^{t} S_i \mid S_i \in W(\Gamma_i)\}} \left(\prod_{\gamma' \in S'} -K\bar{R}\gamma'\right) * G/(\{\Gamma_1\} \cup S'),$$

and repeating the procedure for $\Gamma_2, \ldots, \Gamma_t$ we finally obtain

$$\bar{R}G = \left(\prod_{i=1}^{t}(1-K)\bar{R}\Gamma_i\right) * G/\{\Gamma_1, \ldots, \Gamma_t\} = \left(\prod_{i=1}^{t} R\Gamma_i\right) * G/\{\Gamma_1, \ldots, \Gamma_t\},$$

which states that $\bar{R}G$ has the effect of replacing each Γ_i by its subtracted form.

APPENDIX B

We must show that $[\bar{R}, \partial] = 0$. The idea of the following proof is quite simple. The $\bar{R}$ operation is topological in nature and so it should commute with differentiation. The actual proof requires that we distribute the differentiation operation down until it acts on the component lines and vertices of the graph, then collect together terms that act on the same component of the graph, summed over all components of the graph. It is only here that we can identify the $\bar{R}$ operation on the derivative of a particular component of G. The reason for the complication of the proof is that differentiation is naturally defined in terms of the lines and vertices of the graph, while the $\bar{R}$ operation is defined in terms of spinneys.

To deal with components of a graph G, we introduce the symbol ξ to stand for a vertex or line of the graph G. Then the operator ∂_ξ will simply differentiate that line or vertex inside the graph. If the named line or vertex is not in G, then $\partial_\xi G$ will be defined to be G (i.e., not zero). Then clearly

$$\partial G = \sum_{\xi \in G} \partial_\xi G$$

and ∂_ξ is distributive over $*$, e.g.,

$$\partial_\xi[\gamma * G/\gamma] = \partial_\xi \gamma * \partial_\xi(G/\gamma).$$

This allows us to prove that $[\partial, \bar{R}] = 0$ reasonably simply. We will as usual do a proof by induction on the number of loops in the graph, noting that for tree graphs there are no subtractions and so $\bar{R}$ is trivial on such graphs, before or after differentiation. From the definition of $\bar{R}$ we find

$$\partial\bar{R}G = \partial \sum_{S \in \bar{W}(G)} \left(\prod_{\gamma \in S} -K\bar{R}\gamma\right) * G/S$$

which, when we differentiate the terms in the product, becomes

$$\partial\bar{R}G = \sum_{S \in \bar{W}(G)} \left[\left(\partial \prod_{\gamma \in S} -K\bar{R}\gamma\right) * G/S + \left(\prod_{\gamma \in S} -K\bar{R}\gamma\right) \partial(G/S)\right].$$

When we differentiate the product of $-K\bar{R}\gamma$, we find a sum of terms of the type $\partial K\bar{R}\gamma$. However, we know that ∂ commutes with K, and by our induction hypothesis it commutes with $\bar{R}$ when applied to the proper subgraph γ, so in each term we use $\partial K\bar{R}\gamma = K\bar{R}\partial\gamma$ and obtain

$$\partial\bar{R}G = \sum_{S \in \bar{W}(G)} \left\{ \sum_{\gamma' \in S} \left[\left(\prod_{\substack{\gamma \in S \\ \gamma \neq \gamma'}} -K\bar{R}\gamma\right)(-K\bar{R}\partial\gamma')\right] * G/S + \left(\prod_{\gamma \in S} -K\bar{R}\gamma\right) * \partial(G/S)\right\}.$$

We must now distribute the differentiation over each propagator and vertex of the subgraphs using $\partial G = \sum_{\xi \in G} \partial_\xi G$:

$$\partial\bar{R}G = \sum_{S \in \bar{W}(G)} \left\{ \sum_{\gamma' \in S} \sum_{\xi \in \gamma'} \left[\left(\prod_{\substack{\gamma \in S \\ \gamma \neq \gamma'}} -K\bar{R}\gamma\right)(-K\bar{R}\partial_\xi\gamma')\right] * G/S + \sum_{\xi \in G/S} \left(\prod_{\gamma \in S} -K\bar{R}\gamma\right) * \partial_\xi(G/S)\right\}.$$

We now put ∂_ξ on all the other terms (on which it has no effect), and obtain

$$\partial\bar{R}G = \sum_{S \in \bar{W}(G)} \left\{ \sum_{\xi \text{ in } S} \left[\left(\prod_{\gamma \in S} -K\bar{R}\partial_\xi\gamma\right)\right] * \partial_\xi(G/S) + \sum_{\xi \in G/S} \left(\prod_{\gamma \in S} -K\bar{R}\partial_\xi\gamma\right) * \partial_\xi(G/S)\right\}.$$

Any particular vertex or propagator is either in some $\gamma \in S$ or in G/S:

$$\partial\bar{R}G = \sum_{S \in \bar{W}(G)} \sum_{\xi \in G} \left(\prod_{\gamma \in S} -K\bar{R}\partial_\xi\gamma\right) * \partial_\xi(G/S).$$

The ξ sum is now over all parts of G, and so may be exchanged with the operation of summing over spinneys,

$$\partial\bar{R}G = \sum_{\xi \in G} \sum_{S \in \bar{W}(G)} \left(\prod_{\gamma \in S} -K\bar{R}\partial_\xi\gamma\right) * \partial_\xi(G/S).$$

To each wood $\bar{W}(G)$ there corresponds a unique wood $\bar{W}(\partial_\xi G)$ with the same topology, but with the appropriate vertex or propagator differentiated. We may rewrite our sum to be over this new wood,

$$\partial\bar{R}G=\sum_{\ell\in G}\sum_{S\in\bar{W}(\partial_\ell G)}\left(\prod_{\gamma\in S}-K\bar{R}\gamma\right)*(\partial_\ell G/S)\,.$$

Finally, the inner sum is simply the definition of $\bar{R}\partial_\ell G$, and when we use the linearity of $\bar{R}$ we obtain

$$\partial\bar{R}G=\sum_{\ell\in G}\bar{R}\partial_\ell G=\bar{R}\sum_{\ell\in G}\partial_\ell G=\bar{R}\partial G\,.$$

As a corollary of this result we have shown that if G is subdivergence free, then $\partial^{\omega+1}G$ is also, as promised previously. In this case $G=\bar{R}G$ and thus $\bar{R}\partial^{\omega+1}G=\partial^{\omega+1}\bar{R}G=\partial^{\omega+1}G$.

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Although in this paper we stress the desirable properties of minimal subtraction, which follow from property (iii), our proof can also be carried out using the truncated Taylor series operation T. The only additional subtlety is that while ∂ and T do not commute as operations, nevertheless it is always true that $\partial T^\gamma=T^{\gamma-1}\partial$ for any γ, where $T^\gamma f$ is the Taylor series for f truncated after γ terms. Therefore property (iii) of Feynman graphs ensures that $\partial T^{\deg(G)}G=T^{\deg(\partial G)}\partial G$, which is all that is needed for our proof. Furthermore for such a momentum space subtraction we do not need the postulated extension of the Weinberg-Dyson convergence theorem.

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[40]If we wish to find only the pole parts of Feynman diagrams, it is sometimes simpler to differentiate them until they are logarithmically divergent, and then change the dimensional parameters in the integral to simplify the calculation. In that case, we would not want to use the subtraction cancellation, because it is only $\bar{R}G$ that has polynomial pole terms (and is a pure number for a logarithmically divergent integral).