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Coherent states for the time-dependent harmonic oscillator

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Exact coherent states for the time-dependent harmonic oscillator are constructed. These new coherent states have most, but not all, of the properties of the coherent states for the time-independent oscillator. For example, these coherent states give the exact classical motion, but they are not minimum-uncertainty states.

I. INTRODUCTION

Coherent states, for the harmonic oscillator, have been widely used for describing the radiation field of lasers since Glauber's investigations.[1–3] There has been recent work by Nieto and Simmons, constructing coherent states for general potentials.[4–9] These papers by Nieto and Simmons contain many references to the literature on coherent states. The generalization of coherent states to arbitrary potentials was suggested in a paper by Schrödinger[10] where he first constructed the harmonic-oscillator coherent states. Schrödinger was investigating quantum states, for the harmonic oscillator, such that the expectation value of the position and momentum operators were the same as the classical solutions. The states he found are the coherent states for the harmonic oscillator. These coherent states have several other novel properties including the following: (1) They are destruction operator eigenstates. (2) They are created from the ground state by a unitary operator. (3) They minimize the uncertainty relations and do not spread. (4) They are (over)complete, normalized but not orthogonal. These properties have been fully treated in the literature.[1–10]

In this paper we construct coherent states for the *time-dependent* harmonic oscillator

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2\ , \tag{1.1}$$

where $\omega^2(t)$ is an arbitrary function of time. The quantum theory of the time-dependent harmonic oscillator was first given by Lewis and Riesenfeld.[11] We use the Lewis-Riesenfeld theory to construct coherent states for (1.1). These new coherent states have most, but not all, of the properties of ordinary coherent states.

The coherent states found in this paper could be used to describe the radiation field outside a laser as the laser is being tuned.

First we shall briefly review the definition and some of the properties of ordinary coherent states, and then the Lewis-Riesenfeld theory for the time-dependent harmonic oscillator. Next we construct the new coherent states and discuss their properties. In the conclusion we consider some further generalizations and applications of coherent states for the time-dependent harmonic oscillator.

II. COHERENT STATES FOR THE TIME-DEPENDENT OSCILLATOR

For the time-independent harmonic oscillator

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega_0^2q^2\ , \tag{2.1}$$

where we have set the mass equal to one for simplicity, the coherent states at $t=0$ are defined by

$$|\,\alpha\,\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\,|\,n\,\rangle\ , \tag{2.2}$$

where $|\,n\,\rangle$ are normalized number operator eigenstates and $\alpha=u+iv$ is an arbitrary complex constant. Define the annihilation a and creation $a^\dagger$ operators

$$a=\left[\frac{1}{2\hbar\omega_0}\right]^{1/2}(\omega_0q+ip)\ , \tag{2.3}$$

$$a^\dagger=\left[\frac{1}{2\hbar\omega_0}\right]^{1/2}-(\omega_0q-ip)\ . \tag{2.4}$$

These operators have the properties

$$H=\hbar\omega_0(a^\dagger a+\tfrac{1}{2})\ , \tag{2.5}$$

$$[a,a^\dagger]=1\ , \tag{2.6}$$

$$a\,|\,n\,\rangle=n^{1/2}\,|\,n-1\,\rangle\ , \tag{2.7}$$

$$a^\dagger\,|\,n\,\rangle=(n+1)^{1/2}\,|\,n+1\,\rangle\ . \tag{2.8}$$

Now we give some properties of the coherent states (2.2).

Coherent states are eigenstates of the destruction operator a with eigenvalue α:

$$a\,|\,\alpha\rangle=\alpha\,|\,\alpha\rangle\ . \tag{2.9}$$

This can be proven easily by using (2.2) and (2.7).

Coherent states are generated by the unitary operator

$$D(\alpha)=e^{\alpha a^\dagger-\alpha^* a} \tag{2.10}$$

operating on the vacuum,

$$|\,\alpha\rangle=D(\alpha)\,|\,0\rangle\ . \tag{2.11}$$

The time dependence of the coherent states can be obtained using the evolution operator associated with the Schrödinger equation

$$H\,|\,\psi,t\,\rangle=i\hbar\frac{\partial\,|\,\psi,t\,\rangle}{\partial t}\ , \tag{2.12}$$

which is

$$U(t)=e^{-iHt/\hbar}\ , \tag{2.13}$$

since H does not depend explicitly upon time. The time dependence of an arbitrary state is then

$$|\,\psi,t\,\rangle=U(t)\,|\,\psi,0\,\rangle\ , \tag{2.14}$$

which for the coherent states (2.2) gives

$$|\,\alpha,t\,\rangle=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{(n!)^{1/2}}e^{-i\omega_0(n+1/2)t}\,|\,n\,\rangle\ . \tag{2.15}$$

If we now calculate the uncertainty in q and p in the state $|\,\alpha,t\,\rangle$ we obtain

$$(\Delta q)^2=\langle q^2-q^2\rangle=\frac{\hbar}{2\omega_0} \tag{2.16}$$

and

$$(\Delta p)^2=\langle p^2-p^2\rangle=\tfrac{1}{2}\hbar\omega_0\ . \tag{2.17}$$

From (2.16) and (2.17) we see that the coherent states are states of minimum uncertainty $\Delta q\,\Delta p=\hbar/2$; and the states do not "spread" in time, i.e., $d(\Delta q)/dt=0$.

Using (2.15) we calculate the average position for a coherent state $|\,\alpha,t\,\rangle$ and find

$$\langle\alpha,t\,|\,q\,|\,\alpha,t\,\rangle=\left[\frac{2\hbar\,|\,\alpha\,|^2}{\omega_0}\right]^{1/2}\sin(\omega_0 t+\delta)\ , \tag{2.18}$$

where δ is the phase of α, $\tan\delta=u/v$. Equation (2.18) is the classical solution for the motion of an oscillator with energy $\hbar\omega_0\,|\,\alpha\,|^2=\langle\alpha,t\,|\,H\,|\,\alpha,t\,\rangle-\frac{1}{2}\hbar\omega_0$. The classical solution (2.18) was Schrödinger's original result.

III. QUANTUM THEORY OF THE TIME-DEPENDENT OSCILLATOR

Consider an explicitly time-dependent Hamiltonian $H(t)$ along with a Hermitian invariant $I(t)$:

$$\frac{dI}{dt}=\frac{1}{i\hbar}[I,H]+\frac{\partial I}{\partial t}=0\ . \tag{3.1}$$

We assume, following Lewis and Riesenfeld, that the invariant I does not involve time differentiation. From (3.1) it follows that the eigenvalues of $I(t)$ are constants:

$$I(t)\,|\,\lambda_n,t\,\rangle=\lambda_n\,|\,\lambda_n,t\,\rangle,\quad \frac{\partial\lambda_n}{\partial t}=0\ . \tag{3.2}$$

The eigenstates $|\,\lambda_n,t\,\rangle$ do not satisfy the Schrödinger equation

$$H(t)\,|\,\psi,t\,\rangle_S=i\hbar\frac{\partial\,|\,\psi,t\,\rangle_S}{\partial t} \tag{3.3}$$

in general, however, if we modify the phases of the states $|\,\lambda_n,t\,\rangle$ via

$$|\,\lambda_n,t\,\rangle_S=e^{i\alpha_n(t)}\,|\,\lambda_n,t\,\rangle\ , \tag{3.4}$$

then the new states $|\,\lambda_n,t\,\rangle_S$ evolve in time by the Schrödinger equation if the phase functions $\alpha_n(t)$ satisfy

$$\hbar\,\frac{d\alpha_n(t)}{dt}=\left\langle\lambda_n,t\,\middle|\,i\hbar\frac{\partial}{\partial t}-H(t)\,\middle|\,\lambda_n,t\right\rangle\ . \tag{3.5}$$

The general solution to the Schrödinger equation can then be written in terms of the eigenstates of the invariants $I(t)$ by

$$|\,\psi,t\,\rangle_S=\sum_n c_n e^{i\alpha_n(t)}\,|\,\lambda_n,t\,\rangle\ , \tag{3.6}$$

where the c_n are constants and the S subscript indicates the states evolve in time according to the Schrödinger equation.

For the particular case of the harmonic oscillator

$$H(t)=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2\ , \tag{3.7}$$

Lewis and Riesenfeld constructed the invariant

$$I(t)=\tfrac{1}{2}[(xp-\dot{x}q)^2+(q/x)^2]\ , \tag{3.8}$$

where $x(t)$ is a c-number quantity satisfying the auxiliary equation

$$\ddot{x}+\omega^2(t)x=1/x^3 \ . \tag{3.9}$$

The invariant (3.8) can be factored into raising- and lowering-type operators $a(t)$, $a^\dagger(t)$:

$$I=\hbar[a(t)^\dagger a(t)+\tfrac{1}{2}] \ , \tag{3.10}$$

where

$$a(t)=\frac{1}{(2\hbar)^{1/2}}[q/x+i(xp-\dot{x}q)] \ , \tag{3.11}$$

$$a(t)^\dagger=\frac{1}{(2\hbar)^{1/2}}[q/x-i(xp-\dot{x}q)] \ , \tag{3.12}$$

and

$$[a(t),a(t)^\dagger]=1 \ . \tag{3.13}$$

Using (3.10) and (3.13) the eigenvalue problem for I can be solved exactly, just as for the Hamiltonian in the time-independent case. Thus, we have, writing $|\lambda_n,t\rangle=|n,t\rangle$,

$$I\,|n,t\rangle=\hbar(n+\tfrac{1}{2})\,|n,t\rangle \ , \tag{3.14}$$

$$a(t)\,|n,t\rangle=n^{1/2}\,|n-1,t\rangle \ , \tag{3.15}$$

$$a(t)^\dagger\,|n,t\rangle=(n+1)^{1/2}\,|n+1,t\rangle \ . \tag{3.16}$$

The phase functions $\alpha_n(t)$ are calculated from (3.5) which yields

$$\alpha_n(t)=-(n+\tfrac{1}{2})\int_0^t\frac{dt'}{x^2(t')} \ . \tag{3.17}$$

The general solution to the Schrödinger equation is then

$$|\psi,t\rangle_S=\sum_n c_n e^{i\alpha_n(t)}\,|n,t\rangle \ . \tag{3.18}$$

The time dependence of solutions to the Schrödinger equation can also be written in terms of an evolution operator $U(t)$ which satisfies

$$i\hbar\frac{\partial U}{\partial t}=H(t)U \ , \tag{3.19}$$

$$|\psi,t\rangle_S=U(t)\,|\psi,0\rangle_S \ . \tag{3.20}$$

Of course, for $\partial H(t)/\partial t=0$ we can solve (3.19) for $U(t)=e^{-iHt/\hbar}$, however, here $H(t)$ depends upon time. The time evolution of the eigenstates of I, $|n,t\rangle$, can be determined from (3.4) as

$$|n,t\rangle=e^{-i\alpha_n(t)}U(t)\,|n,0\rangle \ , \tag{3.21}$$

where at $t=0$, $U(0)=1$, $\alpha_n(0)=0$, and the states $|n,0\rangle_S$ and $|n,0\rangle$ are equal.

All of our discussion corresponding to the invariant $I(t)$ would hold for any other invariant corresponding to $H(t)$. Two different invariants will, in general, have different properties, but, of course, one must obtain the same physical results regardless of which invariant is employed. Lewis and Riesenfeld show in detail how, for example, a physical transition probability is independent of the choice of invariant, that is, the particular solution to (3.9) chosen.

All of our results for $H(t)$ reduce to the usual time-dependent oscillator in the limit $\omega(t)\to\omega_0$ $=$const if we take the particular solution

$$x=\frac{1}{\omega_0^{1/2}} \ , \quad \omega(t)=\omega_0 \ , \tag{3.22}$$

to the auxiliary equation (3.9). For example, $a(t)$ given by (3.11) becomes

$$a=\frac{1}{(2\omega_0\hbar)^{1/2}}(\omega_0 q+ip) \ , \tag{3.23}$$

which is exactly the same as (2.3)

IV. COHERENT STATES FOR THE TIME-DEPENDENT OSCILLATOR

We now use the Lewis-Riesenfeld theory of the previous section to construct coherent states for the time-dependent harmonic oscillator. We assume a coherent state at $t=0$ of the form

$$|\alpha,0\rangle_S=\sum_n c_n\,|n,0\rangle \ , \tag{4.1}$$

where $\alpha=u+iv$ is an arbitrary complex constant. Require $|\alpha,0\rangle_S$ to be an eigenstate of $a(0)$, (3.11), with eigenvalue α:

$$a(0)\,|\alpha,0\rangle_S=\alpha\,|\alpha,0\rangle_S \ . \tag{4.2}$$

Next using (4.1), (4.2), the equality of $|n,0\rangle_S$ with $|n,0\rangle$, (3.15), and the orthogonality of the state $|n,0\rangle$ we find

$$|\alpha,0\rangle_S=c_0\sum_n\frac{\alpha^n}{(n!)^{1/2}}\,|n,0\rangle \ . \tag{4.3}$$

Normalization of the states $|\alpha,0\rangle_S$ gives

$$|\alpha,0\rangle_S=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{(n!)^{1/2}}\,|n,0\rangle \ . \tag{4.4}$$

Next we let the state $|\alpha,0\rangle_S$ evolve in time with $U(t)$ in Eq. (3.20) to obtain

$$|\alpha,t\rangle_S=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{(n!)^{1/2}}\,|n,t\rangle_S \ , \tag{4.5}$$

or using (3.4) we obtain the form of our coherent states in terms of the eigenstates $|n,t\rangle$ of the invariant I:

$$|\alpha,t\rangle_S = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{(n!)^{1/2}} e^{i\alpha_n(t)}|n,t\rangle \ . \quad (4.6)$$

The states $|\alpha,t\rangle_S$ are coherent states for the time-dependent harmonic oscillator.

In the limit $\omega(t)\to\omega_0$ we can easily show using (3.22) that

$$\alpha_n = -(n+\tfrac{1}{2})\omega_0 t \ , \quad (4.7)$$

$$|n,t\rangle = |n,0\rangle \ , \quad (4.8)$$

the latter result following from (3.21). Hence, using (4.7) and (4.8) in (4.6) we find that in the limit $\omega(t)\to\omega_0$ our coherent states (4.6) go over exactly into the coherent states (2.15)

Next we examine some of the properties of the coherent states (4.6). Of course, by construction the states $|\alpha,t\rangle_S$ are eigenstates of the destruction operators $a(t)$, the eigenvalue being $\alpha e^{2i\alpha_0(t)}$ where $\alpha_0 = -\frac{1}{2}\int_0^t dt'/x^2(t')$.

Also these states can be created from the ground state by the unitary operator $D(\alpha)$, (2.10). We first create the state $|\alpha,0\rangle_S$ from the ground state using $D(\alpha)$ and then let $|\alpha,0\rangle_S$ evolve in time.

Calculation of the uncertainties (Δq), (Δp) for the coherent states $|\alpha,t\rangle_S$ yields

$$(\Delta q)^2 = \frac{\hbar}{2}x^2 \ , \quad (4.9)$$

$$(\Delta p)^2 = \frac{\hbar}{2}(\dot{x}^2 + 1/x^2) \quad (4.10)$$

with uncertainty relation

$$\Delta q \, \Delta p = \frac{\hbar}{2}(\dot{x}^2 x^2 + 1)^{1/2} \ . \quad (4.11)$$

Thus, our coherent states are not states of minimum uncertainty and the wave packet corresponding to $|\alpha,t\rangle_S$ "spreads" in general according to (4.9). These results (4.9) and (4.10) reduce to (2.16) and (2.17) in the limit $\omega(t)\to\omega_0$ using (3.22) for x.

Finally we calculate the motion of the wave packet corresponding to $|\alpha,t\rangle_S$; ${}_S\langle\alpha,t\,|\,q\,|\,\alpha,t\rangle_S$. Define a "generalized frequency" variable $\Omega(t)$ by

$$\Omega(t) = -2\alpha_0(t) = \int_0^t dt'/x^2(t') \ . \quad (4.12)$$

The calculation of the position of the wave packet yields

$${}_S\langle\alpha,t\,|\,q\,|\,\alpha,t\rangle_S = (2\hbar|\alpha|^2x^2)^{1/2} \times \sin[\Omega(t)+\delta] \ , \quad (4.13)$$

where $\tan\delta = u/v$. This result goes over into (2.15) as $\omega(t)\to\omega_0$. It can be proven[12,13] that the solution to the equation of motion for the classical time-dependent harmonic oscillator $q_{\rm cl}$,

$$\ddot{q}_{\rm cl} + \omega^2(t) q_{\rm cl} = 0 \ ,$$

can be written

$$q_{\rm cl} = (2Ix^2)^{1/2}\sin(\Omega+\delta) \ ,$$

where x satisfies the auxiliary equation (3.9) and I is defined in the same way as (3.8) but using the classical variables $p\to\dot{q}_{\rm cl}$, $q\to q_{\rm cl}$. Thus, (4.13) is exactly the solution for a classical oscillator with invariant $\hbar|\alpha|^2 = {}_S\langle\alpha,t\,|\,I\,|\,\alpha,t\rangle_S - \frac{1}{2}\hbar$. We see that Schrödinger's property of coherent states, that they give the classical motion, is satisfied for the coherent states (4.6).

V. CONCLUSIONS

We have constructed exact coherent states for the time-dependence harmonic oscillator. These coherent states were constructed using the Lewis-Riesenfeld theory of the time-dependent oscillator. The new coherent states give the exact classical motion for the oscillator, are eigenstates of the destruction operator, and can be generated from the ground state by a unitary operator. These three properties are all the same as for the time-independent case. The new coherent states in general spread and are not minimum-uncertainty states. This latter result is important since it shows that it is possible for a state to give the exact classical motion yet not be a minimum-uncertainty or nonspreading state.

There are many time-dependent systems that have Lewis-type invariants $I(t)$.[14] As an example we mention that for the Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2(t)q^2 + c/q^2 \ , \quad c = \text{constant} \ , \quad (5.1)$$

we have the invariant

$$I = \tfrac{1}{2}(xp - \dot{x}q)^2 + \tfrac{1}{2}(q/x)^2 + \frac{c}{2}(x/q)^2 \ , \quad (5.2)$$

where x satisfies the same auxiliary equation (3.9). Nieto and Simmons[8] have discussed coherent states for the time-independent system associated with (5.1). Thus, it seems it would not be any problem to construct coherent states for the time-dependent Hamiltonian (5.1) using the same technique as presented in this paper.

There are several possible physical applications of the coherent states (4.6). Since there are coherent states for arbitrary $\omega(t)$ they can be employed to describe the radiation field of a laser as it is tuned according to $\omega(t)$. The existence of the states (4.6) imply the radiation remains coherent as the laser is tuned. It is possible the states (4.6) could be useful in studying coherence properties of the radiation from lasers as the lasers are tuned.

Finally, we mention that Puri and Lawande have discussed coherent states for time-dependent systems.[15]

Note added. Coherent states for explicitly time-dependent systems have also been considered by Crosignani, Di Porto, and Solimeno,[16,17] and by Dodonov, Malkin, and Man'ko.[18] Reference 18 contains references to other papers dealing with this subject. These references do not overlap significantly with this paper since our goal is to express the coherent states in tems eigenstates of the Lewis invariant equation (4.6), and to relate the coherent states to the classical motion through the Lewis invariant equation (4.13). Neither of these results is discussed in any of the references we are familiar with.

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[1]R. J. Glauber, Phys. Rev. Lett. 10, 84 (1963).
[2]R. J. Glauber, Phys. Rev. 130, 2529 (1963).
[3]R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[4]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. Lett. 41, 207 (1978).
[5]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1321 (1979).
[6]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 19, 438 (1979).
[7]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1321 (1979).
[8]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1332 (1979).
[9]M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1342 (1979).
[10]E. Schrödinger, Naturwissenschaften 14, 664 (1926), an English translation of this paper appears in E. Schrödinger, *Collected Papers on Wave Mechanics* (Chelsea, New York, 1978), p. 41.
[11]H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).
[12]M. Lutzky, Phys. Lett. 68A, 3 (1978).
[13]J. L. Reid and J. R. Ray, J. Math. Phys. 21, 1583 (1980).
[14]J. R. Ray and J. L. Reid, J. Math. Phys. 22, 91 (1981).
[15]R. R. Puri and S. V. Lawande, Phys. Lett. 70A, 69 (1979).
[16]B. Crosignani, P. Di Proto, and S. Solimeno, Phys. Lett. 28A, 271 (1968).
[17]B. Crosignani, P. Di Porto, and S. Solimeno, J. Math, Phys. 10, 1922 (1969).
[18]V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Physica 59, 241 (1972) and references contained therein.